

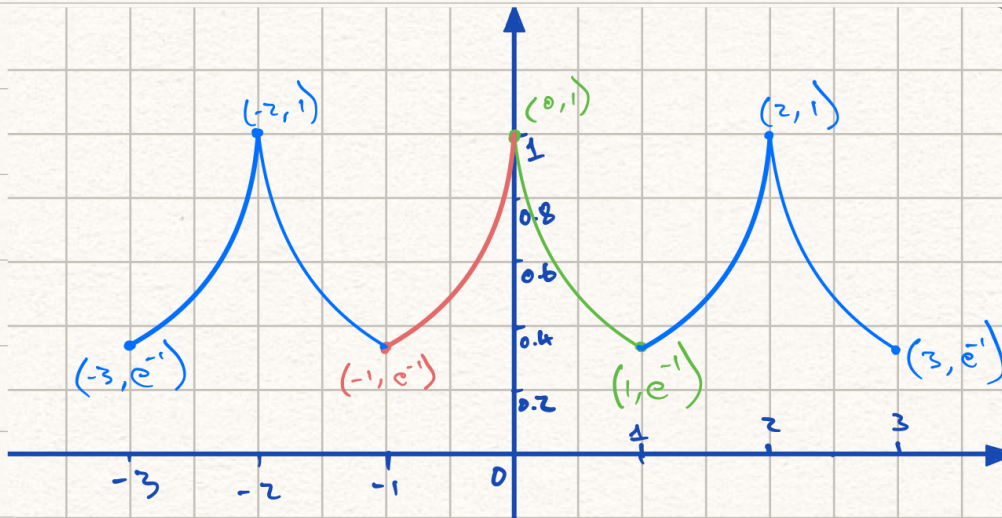
ASSIGNMENT 3

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APPLIED CLASS DAY & TIME: FRIDAY 8AM - 10AM

APPLIED CLASS TUTOR: STEPHEN

1) i.



$$\text{for even extension: } f(x) = \begin{cases} f(x) = e^{-x}, & 0 \leq x \leq 1 \\ f(-x) = e^x, & -1 \leq x \leq 0 \end{cases}$$

$$\text{Fourier even extension (cosine series): } f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), L=1$$

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{2}{2L} \int_0^L f(x) dx = \frac{1}{(1)} \int_0^1 e^{-x} dx = \frac{e-1}{e}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{(1)} \int_0^1 e^{-x} \cos(n\pi x) dx$$

Through integration by parts:

$$\int h(x) g'(x) dx = h(x) g(x) - \int h'(x) g(x) dx$$

$$a_n = 2 \int_0^1 e^{-x} \cos(n\pi x) dx, \quad h(x) = e^{-x} \Rightarrow h'(x) = -e^{-x}$$

$$g'(x) = \cos(n\pi x) \Rightarrow g(x) = \frac{1}{n\pi} \sin(n\pi x)$$

$$\therefore 2 \int_0^1 e^{-x} \cos(n\pi x) dx = \frac{2}{n\pi} \left[e^{-x} \sin(n\pi x) \right]_0^1 - 2 \int_0^1 (-e^{-x}) \left(\frac{\sin(n\pi x)}{n\pi} \right) dx$$

$$2 \int_0^1 e^{-x} \cos(n\pi x) dx = \frac{2e^{-1} \sin(n\pi)}{n\pi} + \frac{2}{n\pi} \int_0^1 e^{-x} \sin(n\pi x) dx$$

Through integration by parts:

$$\frac{2}{n\pi} \int_0^1 e^{-x} \sin(n\pi x) dx, \quad h(x) = e^{-x} \Rightarrow h'(x) = -e^{-x}$$

$$g'(x) = \sin(n\pi x) \Rightarrow g(x) = -\frac{1}{n\pi} \cos(n\pi x)$$

$$\frac{2}{n\pi} \int_0^1 e^{-x} \sin(n\pi x) dx = -\frac{2}{n^2 \pi^2} \left[e^{-x} \cos(n\pi x) \right]_0^1 - \frac{2}{n\pi} \int_0^1 (-e^{-x}) \left(\frac{-\cos(n\pi x)}{n\pi} \right) dx$$

$$\frac{2}{n\pi} \int_0^1 e^{-x} \sin(n\pi x) dx = \frac{-2e^{-1} \cos(n\pi)}{n^2 \pi^2} + \frac{2}{n^2 \pi^2} - \frac{2}{n^2 \pi^2} \int_0^1 e^{-x} \cos(n\pi x) dx$$

$$\therefore 2 \int_0^1 e^{-x} \cos(n\pi x) dx = \frac{2e^{-1} \sin(n\pi)}{n\pi} - \frac{2e^{-1} \cos(n\pi)}{n^2 \pi^2} - \frac{2}{n^2 \pi^2} - \frac{2}{n^2 \pi^2} \int_0^1 e^{-x} \cos(n\pi x) dx$$

$$2 \frac{(n^2 \pi^2 + 1)}{n^2 \pi^2} \int_0^1 e^{-x} \cos(n\pi x) dx = \frac{2n\pi e^{-1} \sin(n\pi) - 2e^{-1} \cos(n\pi) - 2}{n^2 \pi^2}$$

$$\therefore a_n = 2 \int_0^1 e^{-x} \cos(n\pi x) dx = \frac{n^2 \pi^2}{(1 + n^2 \pi^2)} \cdot \frac{2n\pi e^{-1} \sin(n\pi) - 2e^{-1} \cos(n\pi) - 2}{n^2 \pi^2}$$

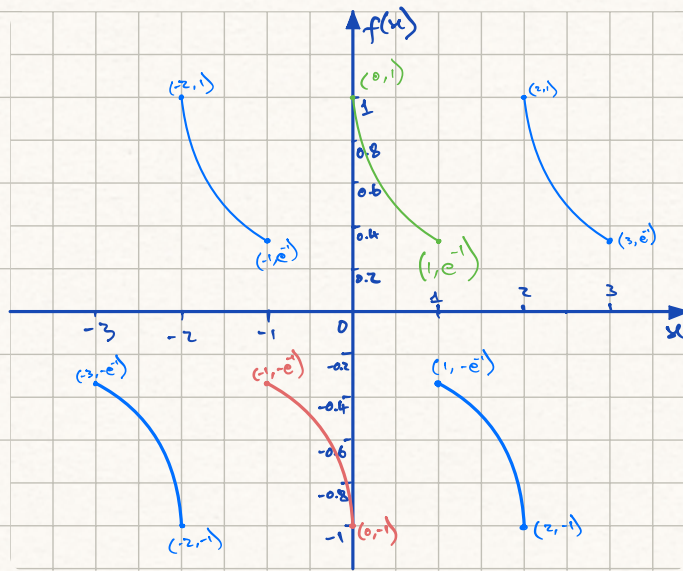
$$\therefore a_n = \frac{2n\pi e^{-1} \sin(n\pi) - 2e^{-1} \cos(n\pi) - 2}{1 + n^2 \pi^2}$$

$$\text{Hence } f(x) = \frac{e-1}{e} + \sum_{n=1}^{\infty} \frac{2n\pi e^{-1} \sin(n\pi) - 2e^{-1} \cos(n\pi) - 2}{1 + n^2 \pi^2} \cdot \cos(n\pi x)$$

$$f(x) = \frac{e-1}{e} + \sum_{n=1}^{\infty} \frac{-2e^{-1}(-1)^n - 2}{1 + n^2 \pi^2} \cdot \cos(n\pi x)$$

Therefore, $A = \frac{e-1}{e}, B = -2e^{-1}(-1)^n - 2, C = \cos(n\pi x)$

1) ii.



for odd extension:
$$f(x) = \begin{cases} f(x) = e^{-x}, & 0 \leq x \leq 1 \\ -f(-x) = -e^{-x}, & -1 \leq x \leq 0 \end{cases}$$

Fourier odd extension (sine series):
$$f(x) = a_0/2 + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), L=1$$

Because of odd function and symmetry across x & y axis: $a_0 = \text{average value} = 0$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, L=1 \Rightarrow b_n = 2 \int_0^1 e^{-x} \sin(n\pi x) dx$$

Through integration by parts,

$$2 \int_0^1 e^{-x} \sin(n\pi x) dx, \text{ Let: } h(x) = e^{-x} \Rightarrow h'(x) = -e^{-x}$$

$$g'(x) = \sin(n\pi x) \Rightarrow g(x) = -\frac{1}{n\pi} \cos(n\pi x)$$

$$\therefore 2 \int_0^1 e^{-x} \sin(n\pi x) dx = 2 \left[\frac{-e^{-x} \cos(n\pi x)}{n\pi} \right]_0^1 - 2 \int_0^1 (-e^{-x}) \left(-\frac{1}{n\pi} \cos(n\pi x) \right) dx$$

$$2 \int_0^1 e^{-x} \sin(n\pi x) dx = \frac{-2e^{-x} \cos(n\pi) + 2}{n\pi} - \frac{2}{n\pi} \int_0^1 e^{-x} \cos(n\pi x) dx$$

$$\left\{ \begin{array}{l} h(x) = e^{-x}, h'(x) = -e^{-x} \\ g'(x) = \cos(n\pi x), g(x) = \frac{\sin(n\pi x)}{n\pi} \end{array} \right.$$

$$= \frac{-2e^{-1} \cos(n\pi) + 2}{n\pi} - \frac{2}{n\pi} \left[\frac{e^{-x} \sin(n\pi x)}{n\pi} \right]_0^1 - \frac{2}{n^2 \pi^2} \int_0^1 e^{-x} \sin(n\pi x) dx$$

$$\therefore \frac{2(1+n^2\pi^2)}{n^2\pi^2} \int_0^1 e^{-x} \sin(n\pi x) dx = \frac{-2n\pi e^{-1} \cos(n\pi) + 2n\pi - 2e^{-1} \sin(n\pi)}{n^2\pi^2}$$

$$b_n = 2 \int_0^1 e^{-x} \sin(n\pi x) dx = \frac{n^2\pi^2}{1+n^2\pi^2} \cdot \frac{-2n\pi e^{-1} \cos(n\pi) + 2n\pi - 2e^{-1} \sin(n\pi)}{n^2\pi^2}$$

$$= \frac{-2n\pi e^{-1} \cos(n\pi) + 2n\pi - 2e^{-1} \sin(n\pi)}{1+n^2\pi^2}$$

$$\text{Hence } f(x) = 0 + \sum_{n=1}^{\infty} \frac{-2n\pi e^{-1} \cos(n\pi) + 2n\pi - 2e^{-1} \sin(n\pi)}{1+n^2\pi^2} \cdot \sin(n\pi x)$$

$$f(x) = 0 + \sum_{n=1}^{\infty} \frac{-2n\pi e^{-1} (-1)^n + 2n\pi}{1+n^2\pi^2} \cdot \sin(n\pi x)$$

$$\text{Therefore, } D = 0, E = -2n\pi e^{-1} (-1)^n + 2n\pi, F = \sin(n\pi x)$$

i) iii Plotted using Matlab, see attached matlab publication below.

1) iv Points of continuity over domain $x \in [-1, 1]$: $(0, 1)$ (even extension)

Points of discontinuity over domain $x \in [-1, 1]$: when $x=0$ (odd extension)

1) v. Fourier cosine series:

$$f(x) = \frac{e^{-1}}{e} + \sum_{n=1}^{\infty} \frac{-2e^{-1}(-1)^n - 2}{1+n^2\pi^2} \cdot \cos(n\pi x) = e^{-x}$$

Assuming Fourier series converges to 1, at $x=0$:

$$\frac{e^{-1}}{e} - 2 \sum_{n=1}^{\infty} \frac{e^{-1}(-1)^n - 2}{1+n^2\pi^2} \cdot \cos(0) = 1$$

$$\sum_{n=1}^{\infty} \frac{e^{-1}(-1)^n - 2}{1+n^2\pi^2} = \frac{e^{-1} - e + 1}{2e} = \frac{1}{2e}$$

$$\text{For all } n=\text{odd}, \text{ Sum term} = \frac{-e^{-1} - 2}{1+n^2\pi^2}$$

$$n=\text{even}, \text{ sum term} = \frac{e^{-1} - 2}{1+n^2\pi^2}$$

$$\therefore \sum_{n=1}^{\infty} \frac{e^{-1}(-1)^n - 2}{1+n^2\pi^2} = \sum_{n=1}^{\infty} \frac{\cancel{e^{-1} - 2} + \cancel{e^{-1} - 2}}{1+n^2\pi^2} = \frac{1}{2e}$$

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2} = -\frac{1}{8e} \quad \left(\text{Note: I know this is wrong but I'm not sure how } \text{☹} \right)$$

$$2) \quad y'' - 4y' + 5y = 2e^x - \sin(x), \quad y(0) = 1 \quad \& \quad y'(0) = -1$$

i. By using Laplace transforms: $\mathcal{L}\{y''\} + \mathcal{L}\{-4y'\} + \mathcal{L}\{5y\} = \mathcal{L}\{2e^x - \sin(x)\}$

From addition and scalar multiplication of Laplace transforms,

$$\begin{aligned} \text{Let, } \mathcal{L}\{y\} &= Y(s) \\ \mathcal{L}^{-1}\{Y(s)\} &= y(x) \end{aligned} \quad \mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = 2\mathcal{L}\{e^x\} - \mathcal{L}\{\sin(x)\}$$

From common Laplace transforms, $\mathcal{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0)$

$$\mathcal{L}\{y'\} = sY(s) - y(0)$$

$$\mathcal{L}\{e^x\} = \frac{1}{s-1}, \quad \mathcal{L}\{\sin(x)\} = \frac{1}{s^2+1}$$

$$\text{Therefore, } s^2 Y(s) - sy(0) - y'(0) - 4(sY(s) - y(0)) + 5Y(s) = \frac{2}{s-1} - \frac{1}{s^2+1}$$

$$s^2 Y(s) - s + 1 - 4sY(s) + 4 + 5Y(s) = \frac{2}{s-1} - \frac{1}{s^2+1}$$

$$s^2 Y(s) - 4sY(s) + 5Y(s) = \frac{2}{s-1} - \frac{1}{s^2+1} + s - 5$$

$$Y(s) = \frac{2}{(s-1)(s^2-4s+5)} - \frac{1}{(s^2+1)(s^2-4s+5)} + \frac{s-5}{(s^2-4s+5)}$$

By ① Declaring 's' as a symbolic variable
② Using Matlab's 'partfrac' function

$$Y(s) = \frac{1}{s-1} - \frac{1}{8} \cdot \frac{s+1}{s^2+1} + \frac{1}{8} \cdot \frac{s-19}{s^2-4s+5}$$

$$Y(s) = \frac{1}{s-1} - \frac{1}{8} \cdot \frac{s}{s^2+1} - \frac{1}{8} \cdot \frac{1}{s^2+1} + \frac{1}{8} \frac{s-2-17}{(s-2)^2+1}$$

$$Y(s) = \frac{1}{s-1} - \frac{1}{8} \cdot \frac{s}{s^2+1} - \frac{1}{8} \cdot \frac{1}{s^2+1} + \frac{1}{8} \cdot \frac{(s-2)}{(s-2)^2+1} - \frac{17}{8} \frac{1}{(s-2)^2+1}$$

$$y(x) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{8} \cdot \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{8} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{1}{8} \cdot \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+1}\right\} - \frac{17}{8} \cdot \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+1}\right\}$$

$$\therefore y(x) = e^x - \frac{1}{8} \cos(x) - \frac{1}{8} \sin(x) + \frac{1}{8} e^{2x} \cos(x) - \frac{17}{8} e^{2x} \sin(x)$$

2) iii., iv. See attached Matlab publication below for code & results.

$$2) \underline{v.} \quad \lim_{x \rightarrow \infty} e^x - \frac{1}{8} \cos(x) - \frac{1}{8} \sin(x) + \frac{1}{8} e^{2x} \cos(x) - \frac{17}{8} e^{2x} \sin(x)$$

As $x \rightarrow \infty$, $\sin(x)$ and $\cos(x)$ will oscillate between it's amplitude

$$-1 \leq \sin(x), \cos(x) \leq 1$$

For the purpose of this question, take the maximum and let

$$\sin(x) = \cos(x) = 1, \quad x \rightarrow \infty$$

$$\therefore \lim_{x \rightarrow \infty} y(x) = \lim_{x \rightarrow \infty} \left(e^x - \frac{1}{8} - \frac{1}{8} + \frac{1}{8} e^{2x} - \frac{17}{8} e^{2x} \right)$$

$$\text{As observed above, } |e^x| < \left| \frac{1}{8} e^{2x} \right| < \left| -\frac{17}{8} e^{2x} \right|$$

$$\text{Therefore, the most dominant term} = \frac{-17}{8} e^{2x} \sin(x)$$

2) vi. See Matlab publication below.