

Mathematical Methods (CAS) Exam 2: SOLUTIONS

Section 1: Multiple Choice Questions

1. B 2. B 3. A 4. E 5. E 6. B 7. A 8. A 9. C
 10. D 11. D 12. C 13. D 14. A 15. D 16. D
 17. A 18. B 19. D 20. E 21. C 22. C

Question 1

$f(g(x)) = \frac{2}{x^2} + 1$. The range is $(1, \infty)$.

Question 2

$d_g = d_h \cap d_f$ which is $(1, 2)$.

$$g(x) = (x-1)^2(x+2)(1-x) \\ = -(x-1)^3(x+2)$$

Question 3

Translation 5 units up:

$$y \rightarrow y + 5$$

Reflection in the y -axis

$$x \rightarrow -x$$

Dilation by a factor of 3 from the x -axis

$$y \rightarrow 3(y + 5)$$

Dilation by a factor of 2 from the y -axis

$$x \rightarrow 2x$$

Combining these:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2x \\ 3y+5 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix}\right)$$

Question 4

The graph has the shape of the graph with rule $y = (ax)^{\frac{4}{3}} + 1$ or $y = (bx)^2 + 1$, where a and b are real constants. When $x = 2$, $y = 2$.

Thus, $y = x^2 + 1$ and $y = x^{\frac{4}{3}} + 1$ are

not possible. $y = \left(\frac{x}{2}\right)^{\frac{4}{3}} + 1$.

Question 5

Solving the matrix equation gives

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3k+4}{k^2-2} + 5 \\ \frac{3k+4}{k^2-2} \\ -\frac{4k+6}{k^2-2} \end{bmatrix}$$

B

The image shows a CAS calculator screen with the matrix equation $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3k+4}{k^2-2} + 5 \\ \frac{3k+4}{k^2-2} \\ -\frac{4k+6}{k^2-2} \end{bmatrix}$ displayed. The bottom of the screen shows the command `ref([0,-k,1,3][1,1,0,5][1,-`.

B

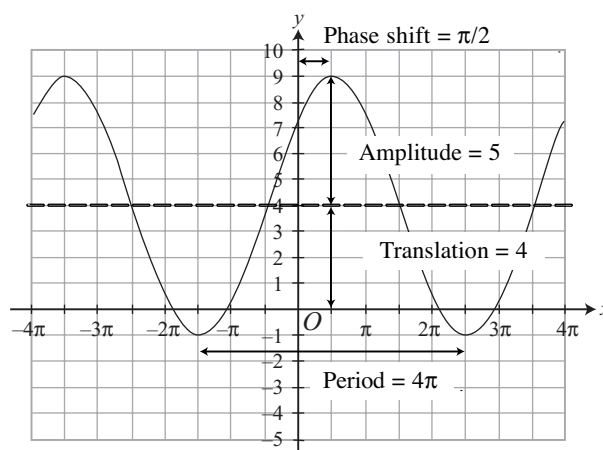
There will be no solution when

$$k^2 - 2 = 0, \text{ i.e. } k = \pm\sqrt{2}.$$

A unique solution for $k \in \mathbb{R} \setminus \{\pm\sqrt{2}\}$

E

Question 6



A

Amplitude = 5

Period = 4π

Rule is of the form $y = a \cos(n(x - \epsilon)) + k$

$$a = 5, \epsilon = \frac{\pi}{2}, k = 4$$

$$\text{Period} = \frac{2\pi}{n}, \text{ therefore } n = \frac{2\pi}{4\pi} = \frac{1}{2}$$

$$\text{Rule is } y = 5 \cos\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right) + 4$$

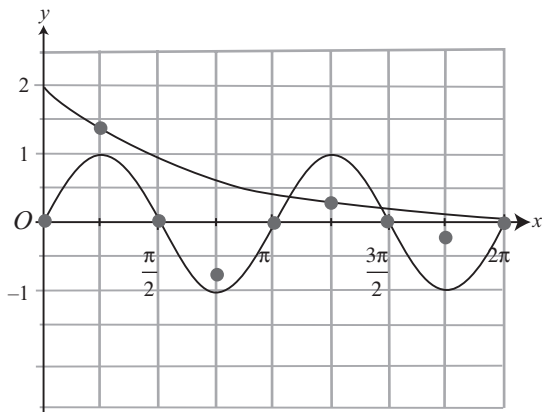
$$\text{or } y = 5 \cos\left(\frac{x}{2} - \frac{\pi}{4}\right) + 4$$

E

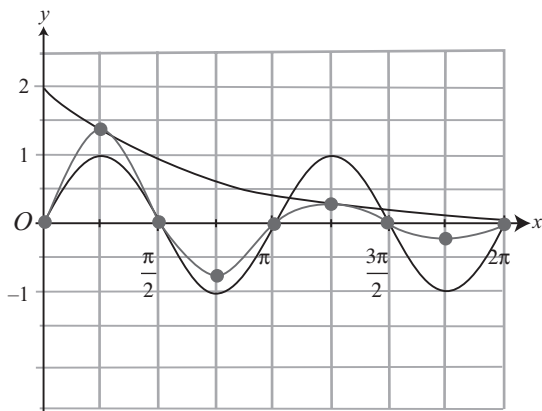
B

Question 7

When graphing product of functions, the key points occur when either function has the value of ± 1 or 0 , as illustrated in the diagram below. Mark these key points.



Then “join the dots”.



Question 8

$$g(x) = 3(e^{2(x-1)} + 3)$$

$$= 3e^{2(x-1)} + 9$$

The graph of g has an asymptote at $y = 9$. Thus the graph of the inverse of g has an asymptote at $x = 9$.

Question 9

The graph of $y = e^x$ has been reflected in both x and y axes, and then translated up.

Reject option B because it does not show the translation up. None of the other options indicate the possibility of a translation parallel to the x -axis.

Ignoring the magnitude of any dilations (because there is no scale on the axes), the transformed graph will be of the form $y = -e^{-x} + k$. Since $b < 0$, require $y = -ae^{bx} + k$.

C

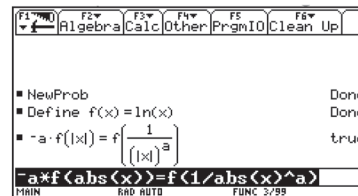
Question 10

This is a property of logarithms.

$$-a \log_e(|x|) = \log_e(|x|^{-a}) = \log_e\left(\frac{1}{|x|^a}\right)$$

This may be confirmed using CAS.

D



Question 11

A

The coordinates of the turning point of $y = 2(x-3)^6 + 4$ are $(3, 4)$. The tangent is $y = 4$.

D

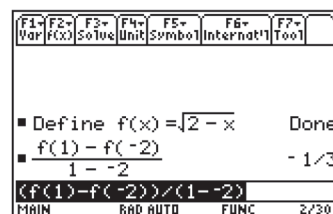
Question 12

The average rate of change = $\frac{f(1) - f(-2)}{1 - (-2)}$

$$= \frac{\sqrt{1} - \sqrt{4}}{3}$$

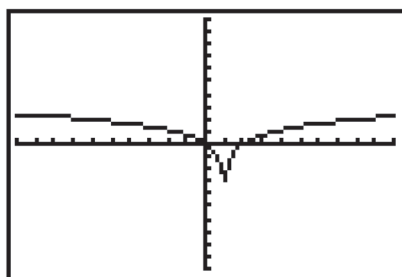
$$= -\frac{1}{3}$$

C

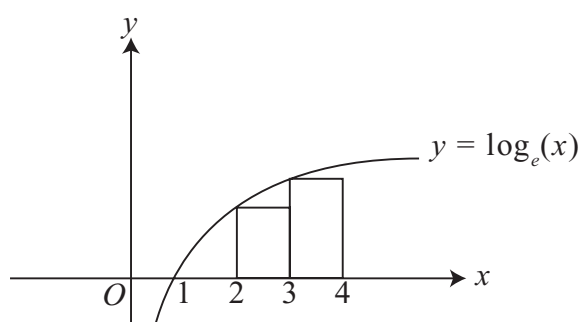


Question 13

The graph of $y = \log_e |x - 1|$ is shown below.



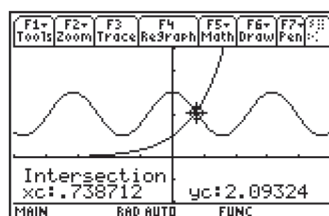
$$\frac{d}{dx} \log_e |x - 1| = \frac{1}{x - 1} \text{ for } x \in \mathbb{R} \setminus \{1\}$$

Question 14

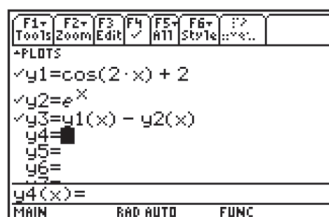
The area of the rectangles = $\log_e(2) + \log_e(3)$
 $= \log_e(6)$

A**Question 15**

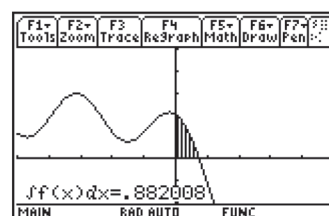
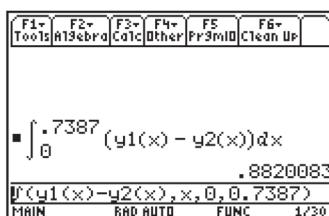
Sketch both curves and find the intersection.



Graph $f(x) - g(x)$

D

Find $\int_0^{0.7387} (f(x) - g(x)) dx$

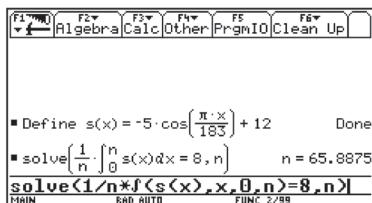
**OR****D**

Question 16

$$\text{Average value} = \frac{1}{b-a} \int_a^b s(x) dx$$

$$\text{Therefore, } \frac{1}{n} \int_0^n s(x) dx = 8$$

Solving for n gives the result $n \approx 66$, days.

**Question 17**

f has a minimum turning point at $x = -1$ and a stationary point of inflection at $x = 2$.

Question 18

Let X denote the number of children with

blue eyes. $X \sim Bi\left(n=3, p=\frac{1}{4}\right)$

$$\Pr(X \geq 1) = 1 - \Pr(X = 0)$$

$$= 1 - \left(\frac{3}{4}\right)^3$$

$$= 1 - \left(\frac{27}{64}\right)$$

$$= \frac{37}{64}$$

Question 19

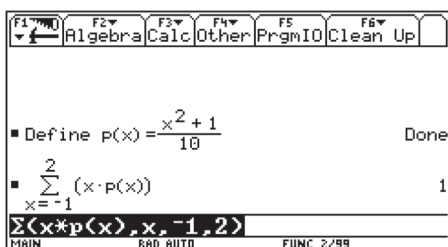
$$p(x) = \frac{x^2 + 1}{10}, \text{ for } x \in \{-1, 0, 1, 2\}$$

$$p(-1) = 0.2, p(0) = 0.1, p(1) = 0.2, p(2) = 0.5$$

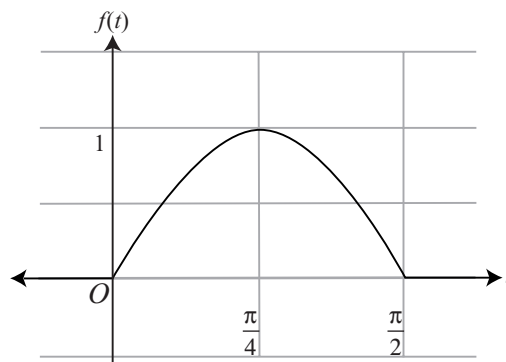
$$E(X) = \sum x p(x)$$

$$= -1 \times 0.2 + 0 \times 0.1 + 1 \times 0.2 + 2 \times 0.5$$

$$= 1$$

**Question 20**

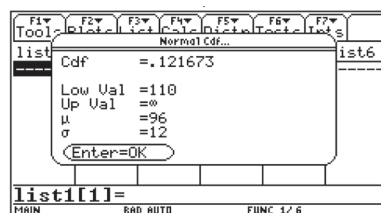
The mode is the value of t that gives the maximum value of $f(t)$.



By symmetry, this occurs at $\frac{\pi}{4}$.

Question 21

$$T \sim N(\mu = 96, \sigma^2 = 12^2)$$



$$\Pr(X > 110) = 0.1217$$

Approximately 12% exceeded the limit.

Question 22

The transition matrix must be of the form,

$$\begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix} \text{ where } 0 \leq a \leq 1 \text{ and } 0 \leq b \leq 1.$$

An element of the initial state matrix may be a cardinal number or a proportion: that is,

$$S_0 = \begin{bmatrix} p \\ q \end{bmatrix}, \text{ where } p + q = n(\xi), p, q \in \mathbb{Z}^+ \cup \{0\},$$

$$\text{or } S_0 = \begin{bmatrix} r \\ 1-r \end{bmatrix}, \text{ where } 0 \leq r \leq 1.$$

$$\begin{bmatrix} 0.35 & 0.55 \\ 0.65 & 0.45 \end{bmatrix} \text{ and } \begin{bmatrix} 27 \\ 33 \end{bmatrix} \text{ are the only pair of}$$

matrices that meet these criteria.

Mathematical Methods (CAS) Exam 2: SOLUTIONS

Section 2: Extended answers

Question 1

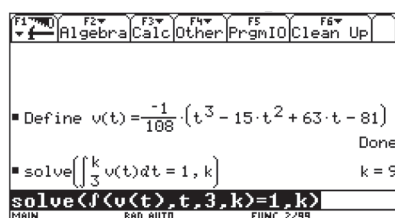
- a. For a probability density function,

$$\int_{-\infty}^{\infty} v(t) dt = 1. \text{ Therefore,}$$

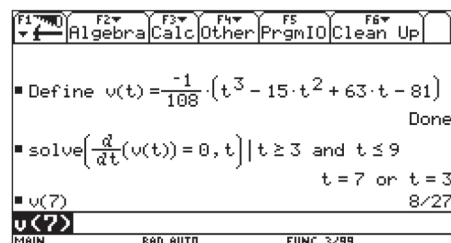
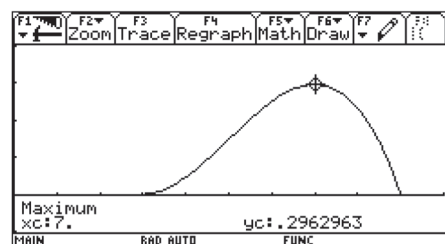
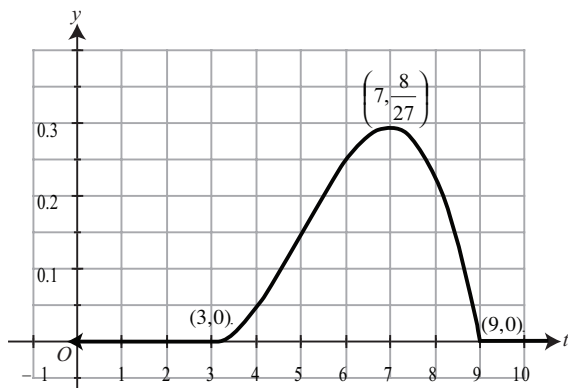
$$0 + \int_3^k -\frac{1}{108}(t^3 - 15t^2 + 63t - 81) dt = 1 \quad 1M$$

(dt must be shown to get the mark)

Solving this equation for k gives the result that $k = 9$, as required.



- b.

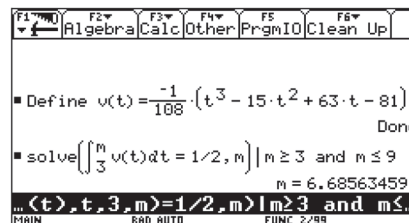


Correct shape and skew 1A
 Correct domain and axes intercepts 1A
 Correct exact stationary point 1A

- c. For the median, $\int_3^m v(t) dt = \frac{1}{2}$. 1M

Solving for m gives $m = 6.69$
 (correct to two decimal places)

The median incubation period is
 6.69 days. 1A

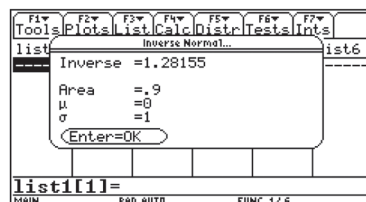


- d. $X \sim N(5, \sigma^2)$, $Z \sim N(0, 1)$

6 hours and 15 minutes = $6\frac{1}{4}$ hours

$$\Pr(Z < z) = 0.9$$

$$z = 1.28155$$



$$z = \frac{x - \mu}{\sigma}$$

$$1.28155 = \frac{6\frac{1}{4} - 5}{\sigma}$$

$$\sigma = 0.98$$

The standard deviation is 0.98 hours 1A

- e. Let Y be the number of patients for whom the response time to the drug is more than 6 hours 15 minutes.

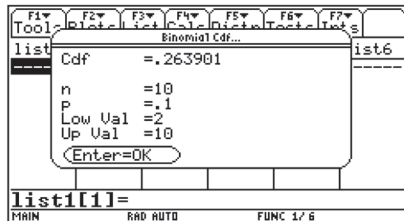
$$Y \sim \text{Bi}(10, 0.1)$$

1M

$$\Pr(Y \geq 2) = \sum_{n=2}^{10} {}^{10}C_n (0.9)^n (0.1)^{10-n}$$

$$= 0.264$$

1A



- f. $Y \sim \text{Bi}(k, 0.1)$

Consider the case where there is 95% chance.

$$\Pr(Y \geq 1) = 0.95$$

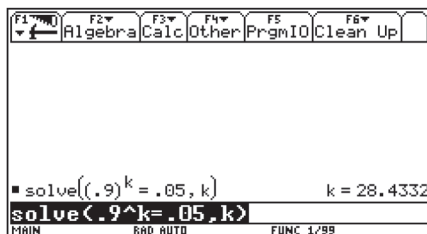
$$1 - \Pr(Y = 0) = 0.95$$

1M

$$1 - 0.9^k = 0.95$$

$$0.9^k = 0.05$$

$$k \approx 28.43 \quad (\text{using numerical solve/ solver})$$



Round up because $k \in \mathbb{Z}^+$ and the probability must be at least 95%.

The least value of k is 29 patients.

1A

g.

	Fluaway Yesterday	Blugone Yesterday
Fluaway Today	0.8	0.3
Blugone Today	0.2	0.7

i. $\Pr(B_{\text{tomorrow}} | B_{\text{today}}) = 0.7$

1A

ii. $S_4 = T^4 \times S_0$

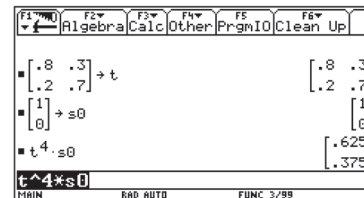
1M

$$= \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}^4 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.625 \\ 0.375 \end{bmatrix}$$

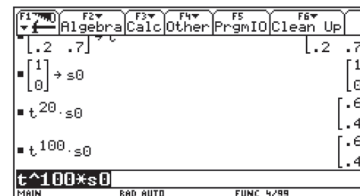
$$\Pr(B_{\text{Friday}} | F_{\text{Monday}}) = 0.375$$

1A



- iii. Consider, say, the 20th and 100th states.

$$T^{20} \times S_0 = T^{100} \times S_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$



In the long term, the proportion of days where Blugone is prescribed is 0.4

1A

Alternatively, if $T = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix}$,

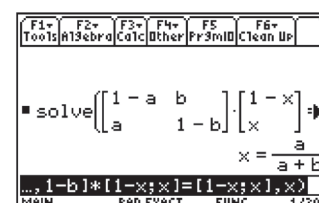
then as $n \rightarrow \infty$, $S_n \rightarrow \begin{bmatrix} 1 - \frac{a}{a+b} \\ \frac{a}{a+b} \end{bmatrix}$

$$\frac{a}{a+b} = \frac{0.2}{0.2+0.3} = \frac{2}{5} = 0.4$$

This formula arises from the fact that if the steady state, $\begin{bmatrix} 1-x \\ x \end{bmatrix}$, has been reached then the next state will also be $\begin{bmatrix} 1-x \\ x \end{bmatrix}$.

Solving the matrix equation

$$T \begin{bmatrix} 1-x \\ x \end{bmatrix} = \begin{bmatrix} 1-x \\ x \end{bmatrix} \text{ gives } x = \frac{a}{a+b}.$$



Question 2

a. $f(x) = A(x^{\frac{4}{3}} - 5)$

$-2 = A(0 - 5)$ **1M**

$A = \frac{2}{5}$

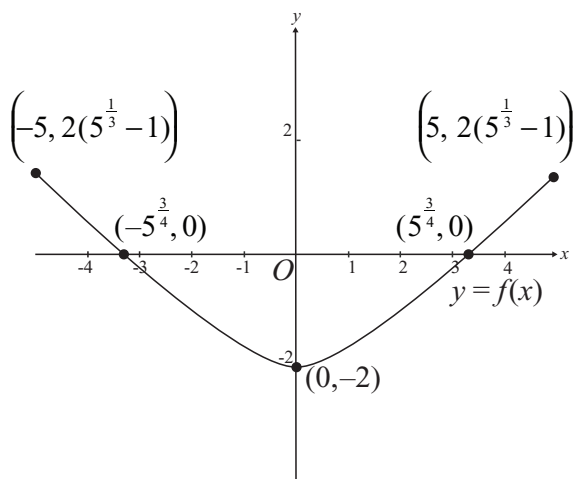
b. endpoints $(-5, 2(5^{\frac{1}{3}} - 1))$ and $(5, 2(5^{\frac{1}{3}} - 1))$ **2A**

turning point $(0, -2)$ and correct shape **1A**

x-intercepts $(-5^{\frac{3}{4}}, 0)$ and $(5^{\frac{3}{4}}, 0)$ **2A**

TI-84 Plus calculator screen showing:
 Define f(x) = 2/5 * (x^(4/3) - 5)
 f(-5) = 2 * (5^(1/3) - 1)
 f(5) = 2 * (5^(1/3) - 1)
 f(5)

TI-84 Plus calculator screen showing:
 Define f(x) = 2/5 * (x^(4/3) - 5)
 f(-5) = 2 * (5^(1/3) - 1)
 f(5) = 2 * (5^(1/3) - 1)
 solve(f(x) = 0, x)
 x = -5^(3/4) or x = 5^(3/4)
 solve(f(x) = 0, x)



c. Max depth = $2 + 2(5^{\frac{1}{3}} - 1)$
 $= 2(5^{\frac{1}{3}})$ **1A**

d. i. $A = -2 \times \frac{2}{5} \int_0^{5^{\frac{3}{4}}} (x^{\frac{4}{3}} - 5) dx$ **1M**
 $= \frac{16(5^{\frac{3}{4}})}{7} \text{ m}^2$ **1A**

TI-84 Plus calculator screen showing:
 Define f(x) = 2/5 * (x^(4/3) - 5)
 -2 * (2/5) * f(x) dx
 16 * 5^(3/4) / 7

ii. $V = 1000A$
 $= \frac{16000(5^{\frac{3}{4}})}{7} \text{ m}^3$ **1A**

e. i. $V = 1000A$ **1M**
 $= 1000(\frac{1}{2} \text{ base} \times \text{height})$
 $= 2000 \times 5^{\frac{3}{4}}$
 $= 6687.40 \text{ m}^3$ correct to two dp **1A**

ii. $\frac{dV}{dt} = \frac{dh}{dt} \times \frac{dV}{dh}$ **1A**
 $\frac{dV}{dt} = -0.02 \times \frac{dV}{dh}$
 $V = 1000 \frac{1}{2} bh$

Using similar triangles:

$\frac{\frac{1}{2}b}{h} = \frac{5^{\frac{3}{4}}}{2}$

$\frac{1}{2}b = \frac{5^{\frac{3}{4}}}{2} h$ **1M**

$V = 500 \times 5^{\frac{3}{4}} h^2$ **1A**

$\frac{dV}{dh} = 1000 \times 5^{\frac{3}{4}} h$

$\frac{dV}{dt} = -0.02 \times 1000 \times 5^{\frac{3}{4}} h$ **1M**
 $= -20 \times 5^{\frac{3}{4}} h \text{ m}^3/\text{h}$

The rate at which water is decreasing

is $20 \times 5^{\frac{3}{4}} h \text{ m}^3/\text{h}$

Question 3

a. $12 = A \times B^2 \quad \dots(1)$

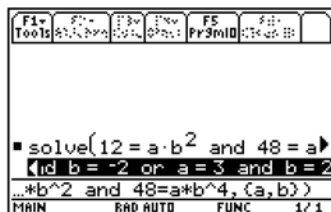
$48 = A \times B^4 \quad \dots(2)$

$B = 2, \text{ as } B > 0$

1A

$A = 3$

1A



b. Dilation by a factor of $\frac{1}{3}$ from the y -axis:

$y_1 = 3 \times 2^{3x} \quad 1A$

Translation of $\frac{4}{3}$ units parallel to the x -axis:

$y_2 = 3 \times 2^{(3x-4)} = 3 \times 2^{\left(3\left(x-\frac{4}{3}\right)\right)}$

Translation of 1 unit parallel to the y -axis:

$g_2(x) = 3 \times 2^{(3x-4)} + 1 = 3 \times 2^{\left(3\left(x-\frac{4}{3}\right)\right)} + 1 \quad 1A$

Range $(1, \infty)$ 1A

c. i. $g_3(x) = 3 \times 2^{(2x+1)} - 1$

Let $y = 3 \times 2^{(2x+1)} - 1$

Inverse: Swap x and y

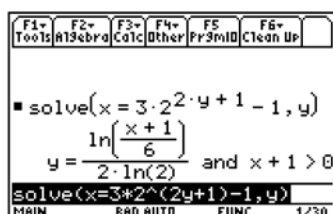
$x = 3 \times 2^{(2y+1)} - 1 \quad 1M$

$2^{2y+1} = \frac{x+1}{3}$

$2y+1 = \log_2\left(\frac{x+1}{3}\right)$

$g_3^{-1}(x) = \frac{1}{2} \log_2\left(\frac{x+1}{3}\right) - \frac{1}{2} \quad 1M$

$= \frac{1}{2} \log_2\left(\frac{x+1}{6}\right)$



ii. Dilation by a factor of $\frac{1}{2}$ from the x -axis

Dilation by a factor of 6 from the y -axis.

1A

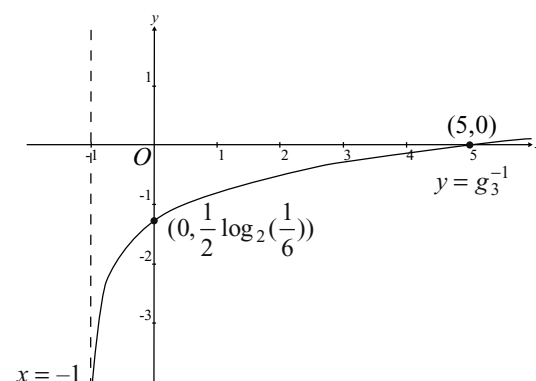
Translation of 1 unit to the left 1A

iii. Asymptote: $x = -1$ and shape 1A

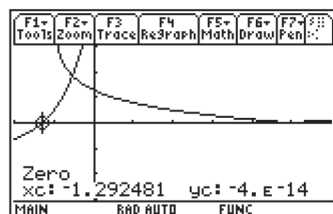
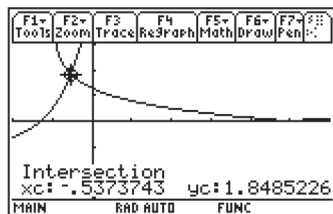
Coordinates of the x -intercept: $(5, 0)$

Coordinates of the y -intercept:

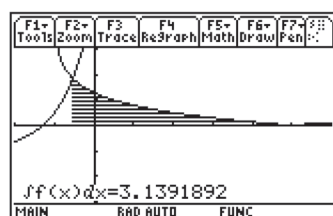
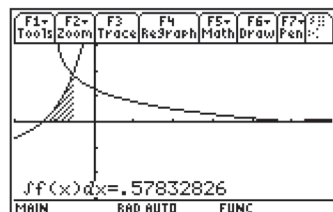
$\left(0, \frac{1}{2} \log_2\left(\frac{1}{6}\right)\right) \quad 1A$



- iv. Sketch both graphs and find the intersection and x-intercepts.

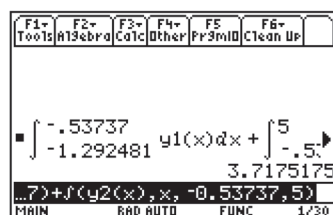


$$A = \int_{-1.292481}^{-0.53737} g_3(x) dx + \int_{-0.53737}^5 |g_3^{-1}(x)| dx \quad 1M$$



$$A \approx 0.578328 + 3.139189 \\ = 3.7 \text{ units}^2 \text{ correct to one decimal place}$$

OR



Question 4

- a. By Pythagoras' theorem

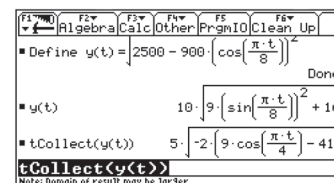
$$x^2 + y^2 = 50^2$$

$$y = \sqrt{2500 - x^2} \text{ (reject negative solution)} \quad 1A$$

- b. i. $y = \sqrt{2500 - x^2}$ and $x(t) = 30 \cos\left(\frac{\pi}{8}t\right)$

$$y(t) = \sqrt{2500 - 900 \cos^2\left(\frac{\pi}{8}t\right)}$$

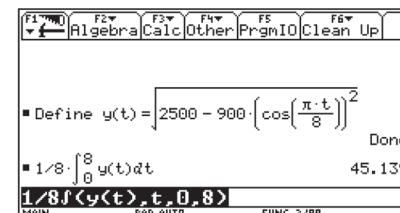
Other equivalent forms are possible. 1A



- ii. Average value = $\frac{1}{8} \int_0^8 y(t) dt$ 1M

$$= 45.139$$

(3 decimal places) 1A



- c. Highest position when $x = 0$.
Endpoint will be $(0, 50)$. $y = 50$ cm 1A

Lowest position when $x = \pm 30$. The corresponding y-axis value is

$$y = \sqrt{2500 - (30)^2} = 40 \text{ or}$$

$$y = \sqrt{2500 - (-30)^2} = 40$$

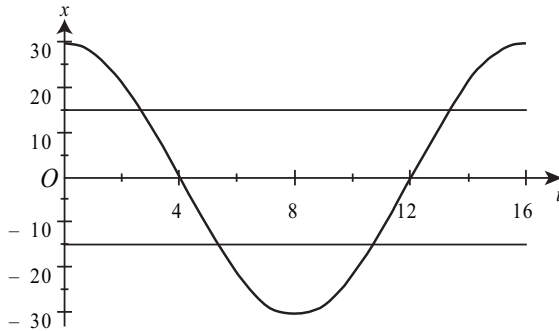
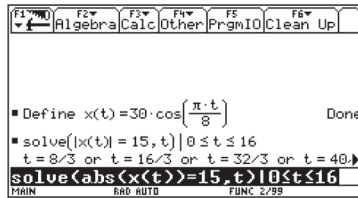
$$y = 40 \text{ cm}$$

1A

- d. Solve for t , $x(t) = \pm 15$ or $|x(t)| = 15$ such that $t \in [0, 16]$

$$t = \frac{8}{3}, \frac{16}{3}, \frac{32}{3}, \frac{40}{3}$$

1A



Time for which $-15 \leq x \leq 15$ is given by

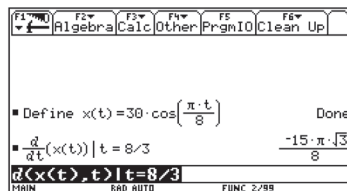
$$\left(\frac{16}{3} - \frac{8}{3}\right) + \left(\frac{40}{3} - \frac{32}{3}\right) = \frac{16}{3} \text{ seconds}$$

1A

- e. When $x = 15$, $t = \frac{8}{3}, \frac{40}{3}, \dots$ (from part d.)

$$x'\left(\frac{8}{3}\right) = -\frac{15\sqrt{3}\pi}{8}$$

1M



Speed is the magnitude of $x'(t)$

$$\text{Speed} = \frac{15\sqrt{3}\pi}{8} \text{ cm/s}$$

1A

f. $x'(t) = -\frac{15\pi}{4} \sin\left(\frac{\pi}{8}t\right)$

The maximum speed occurs where

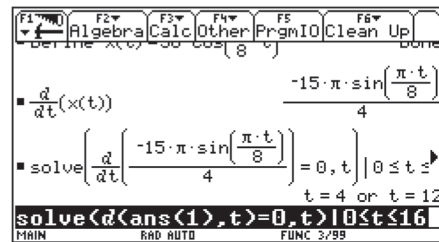
$$\frac{d}{dt}(x'(t)) = 0$$

1M

$$\frac{d}{dt}\left(-\frac{15\pi}{4} \sin\left(\frac{\pi}{8}t\right)\right) = 0$$

$$t = 4, 12, \dots$$

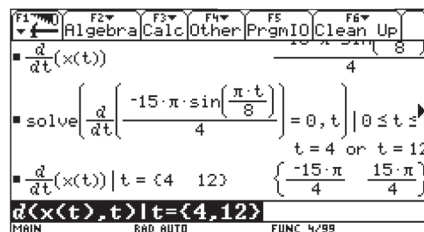
1M



(This is also evident from the graph of x v. t , which show the maximum gradient at $t = 4, 12, \dots$)

$$x'(4) = -\frac{15\pi}{4} \sin\left(\frac{\pi}{2}\right) = -\frac{15\pi}{4}$$

$$x'(12) = -\frac{15\pi}{4} \sin\left(\frac{3\pi}{2}\right) = \frac{15\pi}{4}$$



The speed is the magnitude of $x'(t)$.

Hence the maximum speed is $\frac{15\pi}{4}$ cm/s. 1A