

Mathematical Methods Exam 1: Solutions

Question 1

$$\begin{aligned} \text{a. } \frac{x+2}{x-3} &= 1 + \frac{5}{x-3} \\ \frac{x+2}{x-3} &= \frac{(x-3)+5}{x-3} \\ &= \frac{(x-3)}{x-3} + \frac{5}{x-3} \\ &= 1 + \frac{5}{x-3} \end{aligned}$$

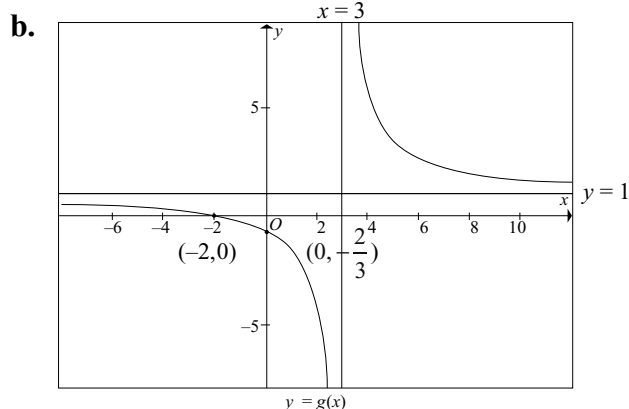
1M

Alternatively, use the long division algorithm.

$$\begin{array}{r} 1 \\ x-3 \overline{) x+2} \\ \underline{x-3} \\ 5 \end{array}$$

1M

$$\frac{x+2}{x-3} = 1 + \frac{5}{x-3}$$



Shape

1A

Asymptotes $y = 1$ and $x = 3$

1A

Intercepts $\left(0, -\frac{2}{3}\right)$ and $(-2, 0)$

1A

c. $(-\infty, -2] \cup (3, \infty)$

1A

Question 2

a. $a = \frac{1}{2}$

1A

b. Let $y = \log_e |2x-1|$, where $x < \frac{1}{2}$

For the inverse swap x and y

$$x = \log_e |2y-1|, \text{ where } y < \frac{1}{2}$$

1A

$$x = \begin{cases} \log_e (2y-1), & y > \frac{1}{2} \\ \log_e (1-2y), & y < \frac{1}{2} \end{cases}$$

$$e^x = 1 - 2y$$

$$y = \frac{1-e^x}{2}$$

$$f^{-1}(x) = \frac{1-e^x}{2}$$

1A

Question 3

$$4^x - 5(2^x) = k$$

Let $a = 2^x, a > 0$

$$a^2 - 5a - k = 0$$

1M

$$a = \frac{5 \pm \sqrt{25+4k}}{2}$$

$$0 < \Delta < 25 \text{ as } a > 0$$

$$0 < 25 + 4k < 25$$

1M for discriminant

1A for restriction

$$-\frac{25}{4} < k < 0$$

1A

Question 4

a. $f(g(x)) = (1 - \log_e(2x))^{\frac{1}{3}}$ **1A**

b. By the chain rule,

$$f'(g(x)) \times g'(x) = \frac{1}{3}(1 - \log_e(2x))^{\frac{-2}{3}} \times \frac{-1}{x} \quad \mathbf{1M}$$

Substitute $x = \frac{1}{2}$ into the derivative to find m .

$$m = \frac{1}{3}(1 - \log_e(1))^{\frac{-2}{3}} \times -2$$

$$= \frac{-2}{3}$$

1M

$$f\left(g\left(\frac{1}{2}\right)\right) = (1 - \log_e(1))^{\frac{1}{3}} = 1$$

1M

The equation of the tangent is

$$y - 1 = \frac{-2}{3}\left(x - \frac{1}{2}\right)$$

$$y = -\frac{2}{3}x + \frac{4}{3}$$

1A**Question 5**

$$\text{Area} = -\int_{-1}^0 (x(x+1)^2) dx = \int_0^{-1} (x(x+1)^2) dx \quad \mathbf{1A}$$

$$= \int_0^{-1} (x^3 + 2x^2 + x) dx$$

$$= \left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} \right]_0^{-1}$$

1M

$$= \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2} \right) - 0$$

$$= \frac{3 - 8 + 6}{12}$$

$$= \frac{1}{12} \text{ units}^2$$

1A**Question 6**

a. i. Range: $[-4 - 2, 4 - 2] = [-6, 2]$ **1A**

ii. Period $= \frac{2\pi}{\pi/6} = \frac{2\pi}{1} \times \frac{6}{\pi} = 12$ **1A**

b. Solve $4\sin\left(\frac{\pi}{6}t\right) - 2 = 0$.

$$\sin\left(\frac{\pi}{6}t\right) = \frac{1}{2}$$

1M

$$\frac{\pi}{6}t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \dots$$

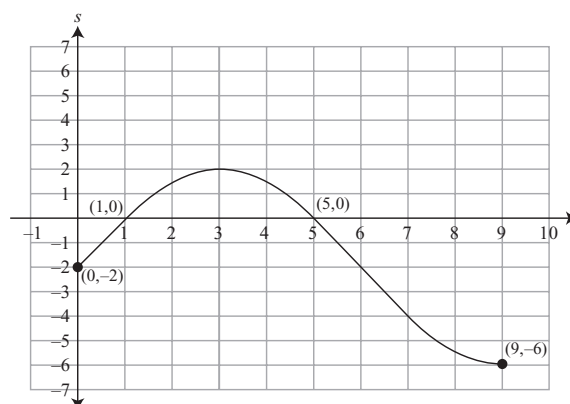
$$t = \frac{\pi}{6} \times \frac{6}{\pi}, \frac{5\pi}{6} \times \frac{6}{\pi}, \frac{13\pi}{6} \times \frac{6}{\pi}, \dots$$

Since $t \in [0, 9]$,

$$t = 1 \text{ or } t = 5$$

1A

c.



Correct shape:

1A

Coordinates of x -axis intercepts labelled:

1A

Endpoints labelled:

1A

Question 7

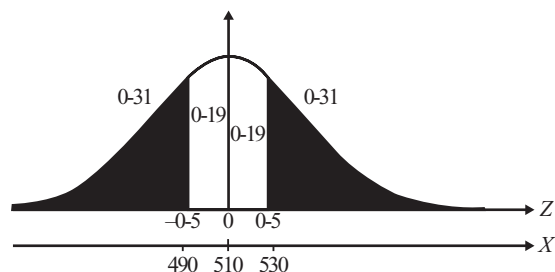
a. $\Pr(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95$

Therefore $\Pr(430 < X < 590) \approx 0.95$

$k = 430$

1A

b.



$\Pr(X < 530) = 1 - \Pr(Z < -0.5)$ 1M

$= 1 - 0.31$

$= 0.69$ 1A

c. $\Pr(X < 530 | X > 510) = \frac{\Pr(X > 510 \cap X < 530)}{\Pr(X > 510)}$

$= \frac{\Pr(510 < X < 530)}{\Pr(X > 510)}$ 1M

$= \frac{0.19}{0.5} = 0.19 \times 2$

$= 0.38$ 1A

Question 8

a. For f to be a probability density function,

$\int_{-\infty}^{\infty} f(x) dx = 1$. Therefore,

$0 + a \int_{-1}^3 (x+1) dx = 1$

$a \left[\frac{x^2}{2} + x \right]_{-1}^3 = 1$ 1M

$a \left[\left(\frac{9}{2} + 3 \right) - \left(\frac{1}{2} - 1 \right) \right] = 1$

$8a = 1$

$a = \frac{1}{8}$, as required 1M

b.

$\Pr(X < 0) = \frac{1}{8} \int_{-1}^0 (x+1) dx$

$= \frac{1}{8} \left[\frac{x^2}{2} + x \right]_{-1}^0$

$= \frac{1}{8} \left[0 - \left(\frac{1}{2} - 1 \right) \right]$

$= \frac{1}{16}$

1A

c.

$E(X) = \int_{-\infty}^{\infty} x f(x) dx$

$= 0 + \frac{1}{8} \int_{-1}^3 (x^2 + x) dx$

$= \frac{1}{8} \left[\frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^3$ 1M

$= \frac{1}{8} \left[\left(9 + \frac{9}{2} \right) - \left(-\frac{1}{3} + \frac{1}{2} \right) \right]$

$= \frac{1}{8} \left[\frac{26}{2} + \frac{1}{3} \right]$

$= \frac{1}{8} \times \frac{40}{3}$

$= \frac{5}{3}$

1A

d.

$$\frac{1}{8} \int_{-1}^m (x+1) dx = \frac{1}{2}$$

$$\int_{-1}^m (x+1) dx = 4$$

$$\left[\frac{x^2}{2} + x \right]_{-1}^m = 4 \quad \mathbf{1M}$$

$$\left[\left(\frac{m^2}{2} + m \right) - \left(\frac{1}{2} - 1 \right) \right] = 4$$

$$\frac{m^2}{2} + m + \frac{1}{2} = 4$$

$$m^2 + 2m - 7 = 0$$

Use quadratic formula or complete the square

$$m = \frac{-2 \pm \sqrt{4 + 28}}{2}$$

$$= \frac{-2 \pm \sqrt{32}}{2}$$

$$= \frac{-2 \pm 4\sqrt{2}}{2}$$

$$= -1 \pm 2\sqrt{2} \quad \mathbf{1M}$$

Note that $-1 - 2\sqrt{2}$ is outside the domain because $m > -1$.

$$\text{Median value is } -1 + 2\sqrt{2} \quad \mathbf{1A}$$