

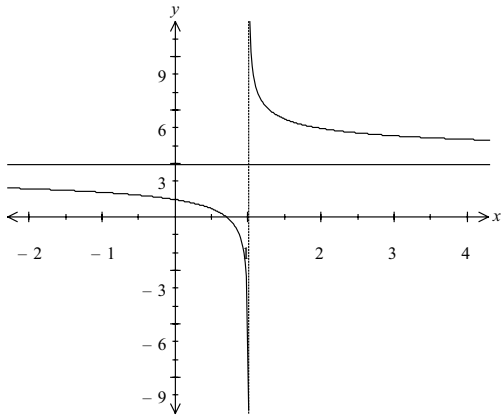
Mathematical Methods (CAS) Exam 2: SOLUTIONS

Section 1: Multiple Choice

Question 1

Answer B

$$y = \frac{2}{\sqrt[3]{(x-1)}} + 3$$



$$x - 1 \neq 0$$

$x = 1$ is a vertical asymptote

$y = 3$ is a horizontal asymptote

Question 2

Answer C

$$(x-a)^3(x+b)^2(x^2-c) = 0$$

Using the Null Factor Law

$$x - a = 0$$

$$x = a$$

or

$$x + b = 0$$

$$x = -b$$

or

$$x^2 - c = 0$$

$$x^2 = c$$

no real solution as $c < 0$

There are two distinct solutions.

Question 3

Answer E

The graph could be of the form

$y = A(x-1)^9 + 3$ where A is a positive real constant.

The y -intercept is negative

$$-A + 3 < 0$$

$$A > 3$$

$$\text{Hence } y = 4(x-1)^9 + 3$$

Alternatively,

The y -intercept is negative. Substituting $x = 0$ into each of the equations gives:

A. $-1 + 3 = 2$ No!

B. $+1 + 3 = 4$ No!

C. $-(-1) + 3 = 4$ No!

D. $-4 + 3$ Possibility

E. $-4 + 3$ Possibility

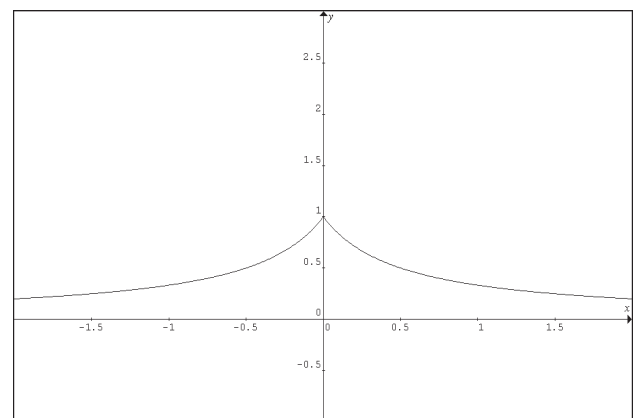
Try the graphs of these last two on the graphical calculator.

$$\text{Hence } y = 4(x-1)^9 + 3$$

Question 4

Answer A

$$h(k(x)) = \frac{1}{|2x|+1} \text{ with domain } R.$$



The range is $(0, 1]$.

Question 5

Answer C

$$f(g(x)) = 2e^{-x+1} = 2e^{-(x-1)}$$

reflection in the y -axis

translation 1 unit parallel to x -axis

Question 6**Answer E**

$f(x) = 3e^{-2x} + 1$ with domain R and range $(1, \infty)$

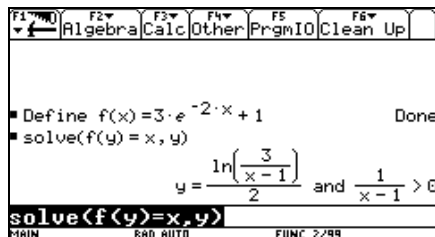
Let $y = 3e^{-2x} + 1$

Inverse: swap x and y

$x = 3e^{-2y} + 1$ and solve for y

$$y = \log_e \left(\sqrt{\frac{3}{x-1}} \right)$$

Alternatively, using CAS



$$y = \frac{1}{2} \log_e \left(\frac{3}{x-1} \right) = \log \left(\sqrt{\frac{3}{x-1}} \right)$$

Question 7**Answer D**

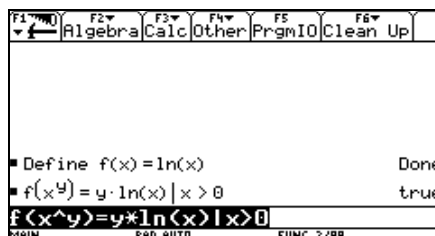
This is a property of logarithms

$$f(x) = \log_e(x)$$

$$\therefore f(x^y) = \log_e(x^y)$$

$$= y \log_e(x)$$

$$= yf(x)$$



This may be verified with a CAS

Question 8**Answer C**

Method 1: From the matrix, identify the sequence of transformations represented by T . Reflection in the y -axis and dilation by a scale factor of 2 from the x -axis, followed by a translation of two units to the right and a translation of 2 units up.

Hence the image has the equation

$$y = 2e^{-(x+2)} + 2.$$

This is an equivalent expression to

$$y = 2e^{-x-2} + 2$$

Question 9**Answer B**

$$\sqrt{3} \tan(2\theta) + 1 = 0$$

$$\tan(2\theta) = -\frac{1}{\sqrt{3}}$$

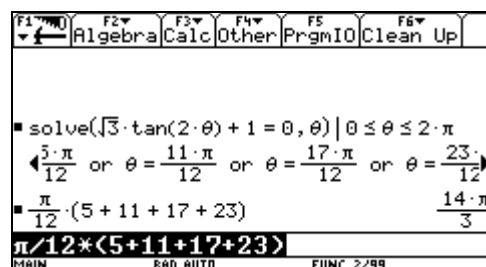
$$2\theta = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{5\pi}{6} + 2\pi, \frac{11\pi}{6} + 2\pi$$

$$= \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}$$

$$\theta = \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$$

Sum of the solutions

$$\frac{5\pi}{12} + \frac{11\pi}{12} + \frac{17\pi}{12} + \frac{23\pi}{12} = \frac{56\pi}{12} = \frac{14\pi}{3}$$



This may be verified with a CAS

Question 10**Answer E**

$$\text{Period} = \frac{2\pi}{\pi/12} = 24$$

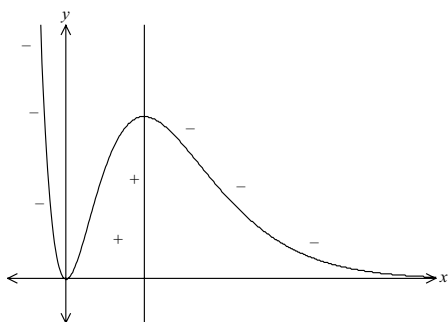
Maximum value of f is $1 + 5 = 6$

Minimum value of f is $1 - 5 = -4$

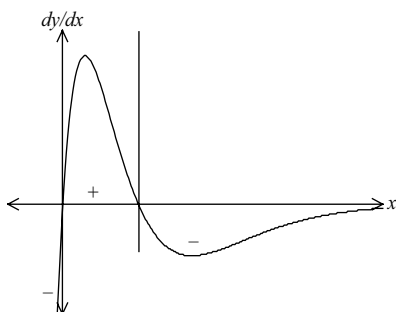
$$\text{Range} = [-4, 6]$$

Question 11

Consider the sign of the gradient



x intercepts of $\frac{dy}{dx}$ are the stationary points of y
 Gradient $\rightarrow 0$ (from the negative, i.e. from below) as $x \rightarrow \infty$

**Question 12****Answer C**

$$f: [0, \pi] \rightarrow \mathbb{R}, f(x) = 4 \cos\left(\frac{x}{2}\right)$$

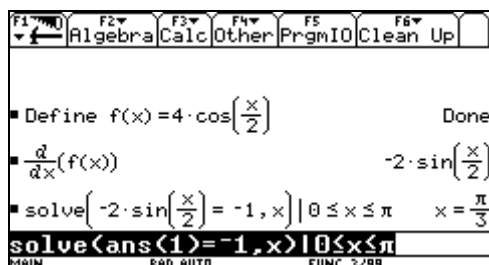
$$f'(x) = -1$$

$$-2 \sin\left(\frac{x}{2}\right) = -1$$

$$\sin\left(\frac{x}{2}\right) = \frac{1}{2}$$

$$\frac{x}{2} = \frac{\pi}{6}$$

$$x = \frac{\pi}{3}$$



This may be verified with a CAS

Answer B**Question 13****Answer D**

$$f(x) = \frac{1}{\sqrt[3]{x}}, h = 0.1, x = 1$$

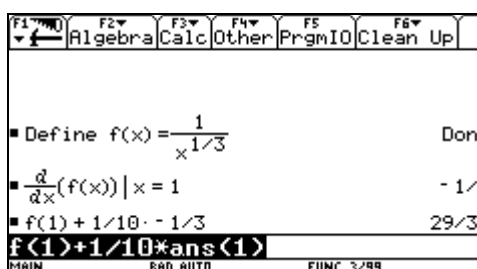
$$f(1) = 1$$

$$f'(x) = -\frac{1}{3x^{4/3}}, f'(1) = -\frac{1}{3}$$

$$f(x+h) \approx f(x) + hf'(x)$$

$$\frac{1}{\sqrt[3]{1.1}} \approx 1 + 0.1 \times -\frac{1}{3}$$

$$\approx \frac{29}{30}$$



This may be verified with a CAS

Question 14

$$f(x) = |x|^2 - |2x| + 5$$

Now, $|x|^2 = x^2$ for $x \in R$

$$f(x) = \begin{cases} x^2 - 2x + 5 & \text{for } x \geq 0 \\ x^2 + 2x + 5 & \text{for } x < 0 \end{cases}$$

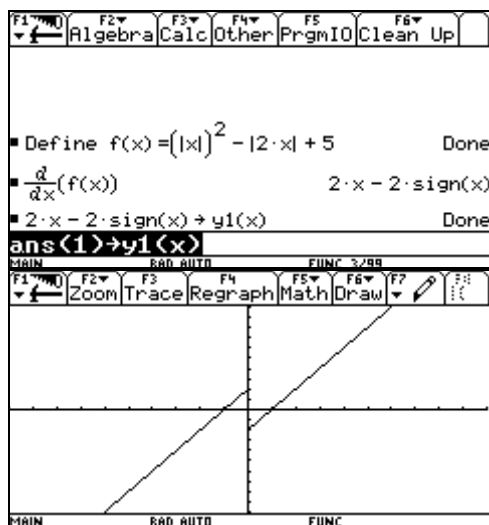
Therefore

$$f'(x) = \begin{cases} 2x - 2 & \text{for } x > 0 \\ 2x + 2 & \text{for } x < 0 \end{cases}$$

If using a CAS, students need to be familiar with the *signum* function:

$$\text{sign}(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -1 & \text{when } x < 0 \end{cases}$$

or graph the output.



Confirms $f'(x) = \begin{cases} 2x - 2 & \text{for } x > 0 \\ 2x + 2 & \text{for } x < 0 \end{cases}$

Question 15**Answer A**

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$y = \left(\frac{x}{2} - 3\right)^8$$

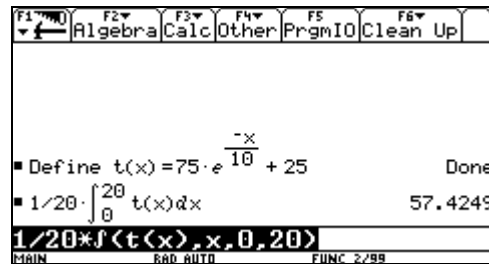
$$\frac{dy}{dx} = 4\left(\frac{x}{2} - 3\right)^7 \text{ and } \frac{dx}{dt} = 3$$

$$\begin{aligned} \frac{dy}{dt} &= 4\left(\frac{x}{2} - 3\right)^7 \times 3 \\ &= 12\left(\frac{x}{2} - 3\right)^7 \end{aligned}$$

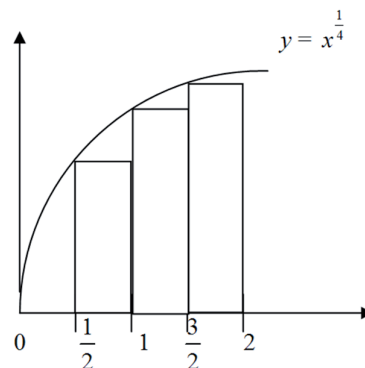
Answer D**Question 16****Answer D**

$$\begin{aligned} \text{Average value} &= \frac{1}{b-a} \int_a^b (f(x)) dx \\ &= \frac{1}{20} \int_0^{20} (75e^{-t/10} + 25) dx \\ &= 57 \end{aligned}$$

(correct to the nearest integer)

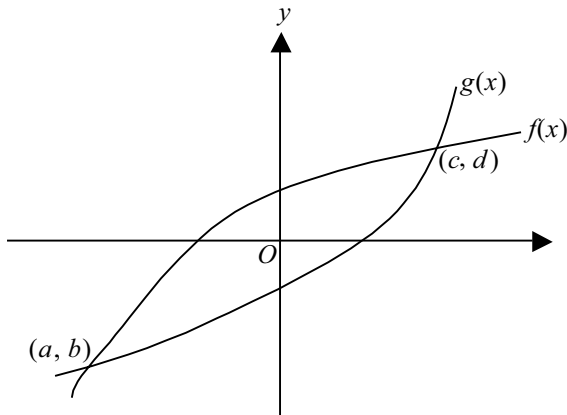


This may be verified with a CAS

Question 17**Answer A**

The area of the rectangles

$$\begin{aligned} &= \frac{1}{2} \left(f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) \right) \\ &= \frac{1}{2} \left(\left(\frac{1}{2}\right)^{\frac{1}{4}} + 1 + \left(\frac{3}{2}\right)^{\frac{1}{4}} \right) \text{ square units} \end{aligned}$$

Question 18**Answer D**

The area between the curves is

$$\int_a^c (\text{top curve} - \text{bottom curve}) dx$$

$$= \int_a^c (f(x) - g(x)) dx$$

Question 19**Answer A**

$$2t^2 + t = 1$$

$$2t^2 + t - 1 = 0$$

$$(2t - 1)(t + 1) = 0$$

$$2t = 1 \text{ (reject } t = -1 \text{ because } 0 \leq p(x) \leq 1)$$

$$t = \frac{1}{2}$$

Question 20**Answer D**

$$k \int_{-\infty}^{\infty} e^{-(x^2/2)} dx = 1$$

$$k \times \sqrt{2\pi} = 1$$

$$k = \frac{1}{\sqrt{2\pi}}$$

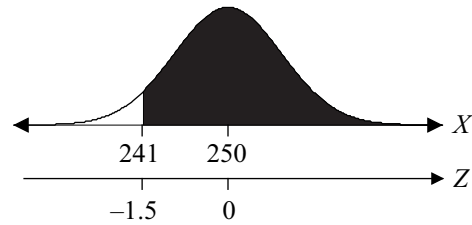
$$E(X) = \mu$$

$$= k \int_{-\infty}^{\infty} x e^{-(x^2/2)} dx$$

$$= k \times 0$$

$$= 0$$

Hence, $k = \frac{1}{\sqrt{2\pi}}$ and $E(X) = 0$

Question 21**Answer C**

$$250 - 241 = 9$$

$$1.5\sigma = 9$$

$$\sigma = \frac{9}{1.5}$$

$$\sigma = 6$$

Question 22**Answer B**

$$X \sim Bi\left(3, \frac{1}{3}\right)$$

$$\Pr(X = 2 | X \geq 1) = \frac{\Pr(X = 2)}{1 - \Pr(X = 0)}$$

$$= \frac{3 \times \left(\frac{1}{3}\right)^2 \times \frac{2}{3}}{1 - \left(\frac{2}{3}\right)^3}$$

$$= \frac{\frac{2}{9}}{\frac{19}{27}}$$

$$= \frac{6}{19}$$

F1	F2	F3	F4	F5	F6
Algebra	Calc	Other	PrmIO	Clean Up	
$3 \cdot \left(\frac{1}{3}\right)^2 \cdot \frac{2}{3} \quad \frac{2}{9}$ $1 - \left(\frac{2}{3}\right)^3 \quad \frac{19}{27}$ $\frac{2/9}{19/27} \quad \frac{6}{19}$					
ans(2)/ans(1)					
MAIN RAD AUTO FUNC 2/99					

Mathematical Methods (CAS) Exam 2: SOLUTIONS

Section 2: Extended answers

Question 1

a. i. $a = \frac{6}{3} = 2$

1A

$b = -6$

1A

ii. $x^2 + y^2 = 9$

$y^2 = 9 - x^2$

$y = \sqrt{9 - x^2}$

$c = 9$

1A

b. i. $-2 \int_0^3 (2x - 6) dx$ or 3×6

1M

$= 18 \text{ cm}^2$

1A

ii. $2 \int_0^3 \sqrt{9 - x^2} dx$

1M

$= \frac{9\pi}{2} \text{ cm}^2$

1A

c. i. $\frac{dr}{dt} = \frac{dv}{dt} \times \frac{dr}{dv}$

$v = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi r^2 2r = \frac{2}{3} \pi r^3$ 1M

$\frac{dv}{dr} = 2\pi r^2, \frac{dr}{dv} = \frac{1}{2\pi r^2}$ 1M

$\frac{dr}{dt} = -\pi \times \frac{1}{2\pi r^2} = -\frac{1}{2r^2}$ 1M

At $r = 2 \text{ cm}$

$\frac{dr}{dt} = -\frac{1}{8} \text{ cm/s}$

Decreasing at $\frac{1}{8} \text{ cm/s}$ 1A

ii. $v = 18\pi \text{ cm}^3, \frac{dv}{dt} = -\pi \text{ cm}^3/\text{s}$

$t = \frac{18\pi}{\pi} = 18 \text{ s}$

$t \geq 18 \text{ s}$ 1A

iii. $s = \sqrt{\frac{5}{8t}}$

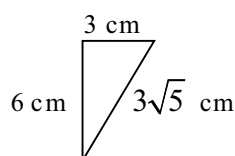
$d = \int \sqrt{\frac{5}{8t}} dt,$

where d is the distance in cm 1M

$= \sqrt{45}$ when $t = 18 \text{ s}$

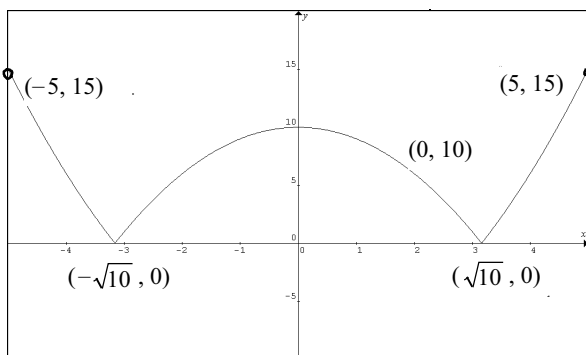
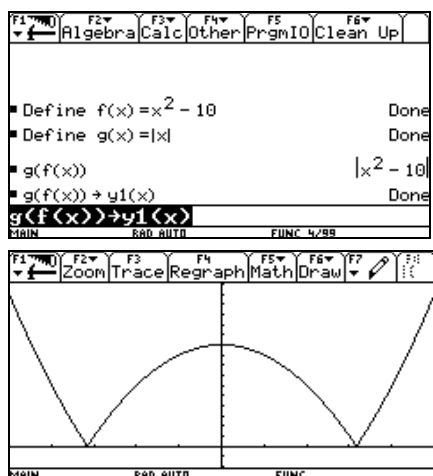
$= 3\sqrt{5} \text{ cm}$ 1A

The ant will be at the top of the cone. 1A



Question 2

a.



Correct endpoints

1A

Correct cusps and local maximum

1A

Correct x intercepts

1A

b. i. $0 < k < 10$

1A

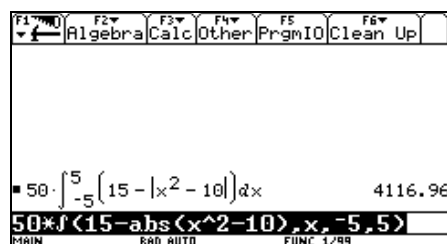
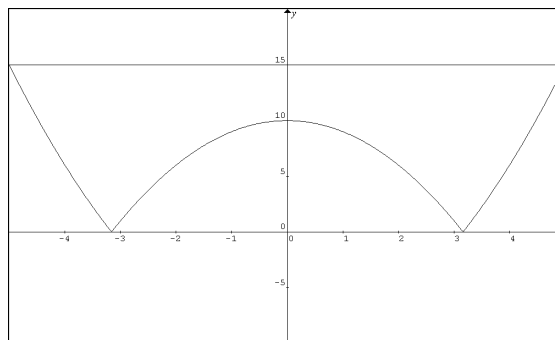
ii. $k = 0$ or

1A

 $10 < k \leq 15$

1A

c.



$$\text{Cost} = 50 \times \int_{-5}^5 (15 - |x^2 - 10|) dx \quad 1M$$

$$\approx \$4117 \quad 1A$$

d. i. Substitute $(0, 0)$ into the equation.

$$0 = 5e^0 + C$$

1M

$$C = -5$$

ii. Substitute $(14, 100)$ into the equation.

$$100 = 5e^{14B} - 5$$

$$e^{14B} = \frac{105}{5}$$

1M

$$= 21$$

$$B = \frac{\log_e(21)}{14}$$

e. Solve $200 = 5e^{\frac{\log_e(21)}{14}t} - 5$

$$t \approx 17.1$$

During the 18th day.

1A

f. Let $y = 5e^{\frac{\log_e(21)}{14}t} - 5$ Inverse: swap t and y .

$$t = 5e^{\frac{\log_e(21)}{14}y} - 5$$

1M

$$\frac{t+5}{5} = e^{\frac{\log_e(21)}{14}y}$$

$$y = \frac{14}{\log_e(21)} \log_e\left(\frac{t+5}{5}\right)$$

$$r^{-1}(t) = \frac{14}{\log_e(21)} \log_e\left(\frac{t+5}{5}\right), t \geq 0$$

1A

Alternatively, use CAS

Inverse: swap t and y .

$$t = 5e^{\frac{\log_e(21)}{14}} - 5$$

Define $f(t) = 5 \cdot e^{\frac{\ln(21) \cdot t}{14}} - 5$ Done
 solve($t = f(y), y$)
 $y = \frac{14 \cdot \ln\left(\frac{t+5}{5}\right)}{\ln(21)}$ and $t+5 > 0$
 solve($t=f(y), y$)

$$r^{-1}(t) = \frac{14}{\log_e(21)} \log_e\left(\frac{t+5}{5}\right), t \geq 0$$

$$g. r^{-1}(t) = \frac{14}{\log_e(21)} \log_e\left(\frac{20+5}{5}\right) \approx 7.4$$

7 foxes

$$h. r'(t) - (r^{-1})'(t)$$

$$= \frac{5 \log_e(21)}{14} e^{\log_e(21) \frac{t}{14}} - \frac{14}{\log_e(21)(t+5)}$$

or

$$= \frac{5 \log_e(21)}{14} \times 21^{\frac{t}{14}} - \frac{14}{\log_e(21)(t+5)}$$

Alternatively, use CAS

$$r'(t) - (r^{-1})'(t)$$

Define $\text{invr}(t) = \frac{14 \cdot \ln\left(\frac{t+5}{5}\right)}{\ln(21)}$ Done
 $\frac{d}{dt}(r(t)) - \frac{d}{dt}(\text{invr}(t))$
 $\frac{5 \cdot \ln(21) \cdot 21^{\frac{t}{14}}}{14} - \frac{14}{\ln(21) \cdot (t+5)}$
 $d(r(t), t) - d(\text{invr}(t), t)$

$$r'(t) - (r^{-1})'(t)$$

$$= \frac{5 \log_e(21)}{14} \times 21^{\frac{t}{14}} - \frac{14}{\log_e(21)(t+5)}$$

Question 3

$$a. x_{\max} = 30 + 20 = 50 \text{ cm}$$

1A

$$x_{\min} = 30 - 20 = 10 \text{ cm}$$

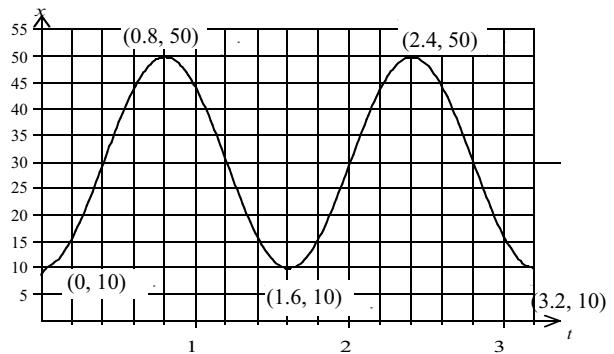
1A

$$b. \text{Period} = \frac{2\pi}{5\pi/4} = \frac{2\pi}{1} \times \frac{4}{5\pi} = 1.6$$

The period is 1.6 seconds.

1A

c.



Shape

1 mark

Endpoints

1 mark

Correct coordinates for maximum and minimum values

1 mark

$$d. \text{Time} = \frac{4}{3} - \frac{4}{15}$$

1M

$$= \frac{16}{15} \text{ seconds}$$

1A

Define $s(t) = 30 - 20 \cdot \sin\left(\frac{5\pi}{4} \cdot (t + 2/5)\right)$ Done
 solve($s(t) = 20, t$) | $t \geq 0$ and $t \leq 1.6$
 $t = 4/15$ or $t = 4/3$
 solve($s(t) = 20, t$) | $t \geq 0$ and $t \leq 1.6$

$$e. \text{Average rate of change} = \frac{s(2.4) - s(0.4)}{2.4 - 0.4}$$

1M

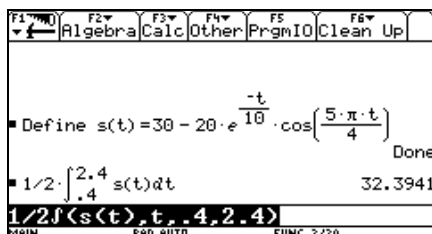
$$= 7.87$$

Define $s(t) = 30 - 20 \cdot e^{\frac{-t}{10}} \cdot \cos\left(\frac{5\pi}{4} \cdot t\right)$ Done
 $\frac{s(2.4) - s(0.4)}{2}$ 7.86628
 $(s(2.4) - s(0.4)) / 2$

The average rate of change is 7.87 cm/s.

1M

$$\begin{aligned} \text{f. Average value} &= \frac{1}{b-a} \int_a^b (f(x)) dx \\ &= \frac{1}{2.4 - 0.4} \int_{0.4}^{2.4} \left(30 - 20e^{-t/10} \cos\left(\frac{5\pi}{4}t\right) \right) dt \quad 1\text{M} \\ &= 32.39 \end{aligned}$$

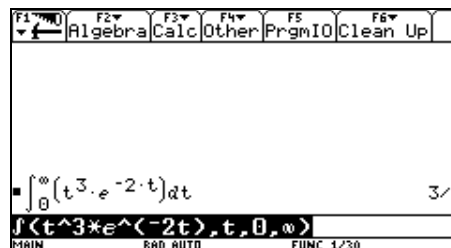


The average value of s is 32.39 cm.

1A

Question 4

a. i. $k \int_0^{\infty} t^3 e^{-2t} dt = 1$



1M

$$k \times \frac{3}{8} = 1$$

$$k = \frac{8}{3}$$

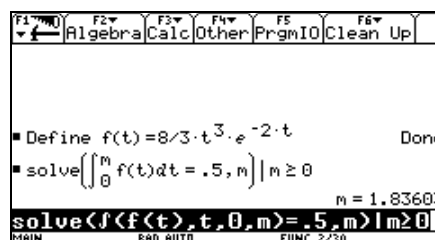
1M

ii. For the median, the area = 0.5
Solve for m , where $m > 0$

1M

$$\frac{8}{3} \int_0^m t^3 e^{-2t} dt = 0.5$$

1M



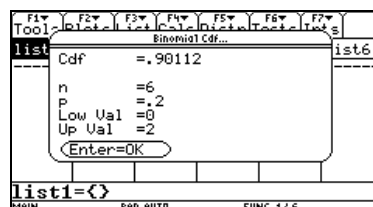
$$\therefore m = 1.84$$

The median time spent is 1.84 hours. 1A

b. Let X be the number of people from area A.

$$X \sim \text{Bi}(6, 0.2) \quad 1\text{M}$$

$$\begin{aligned} \Pr(X \leq 2) &= \sum_{x=0}^2 {}^6C_x (0.2)^x (0.8)^{6-x} \\ &= 0.9011 \end{aligned} \quad 1\text{A}$$



c. $X \sim N(6, 1.5^2)$

Conditional probability $\Pr(X < 8 | X > 5)$ **1M**

$$\Pr(X < 8 | X > 5) = \frac{\Pr(5 < X < 8)}{\Pr(X > 5)} \quad \mathbf{1M}$$

$$= \frac{0.656296}{0.747508}$$

$$\Pr(X < 8 | X > 5) = 0.8780 \quad \mathbf{1A}$$

d. i.

	Fast Today	Healthy Today
Fast Next day	0.6	0.2
Healthy Next day	0.4	0.8

The probability of *Healthy* menu one day and *Fast* menu the following day is 0.2. **1A**

ii.

$$T^4 = \begin{bmatrix} 0.35 & 0.32 \\ 0.65 & 0.68 \end{bmatrix}$$

	Fast Monday	Healthy Monday
Fast Friday	0.35	0.32
Healthy Friday	0.65	0.68

The probability of *Fast* menu on Monday and *Healthy* menu on Friday 0.65. **1A**

Alternatively,

$$T = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \text{ and } S_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{matrix} \leftarrow \text{Fast} \\ \leftarrow \text{Healthy} \end{matrix}$$

$$S_4 = T^4 S_0$$

$$= \begin{bmatrix} 0.35 \\ 0.65 \end{bmatrix} \begin{matrix} \leftarrow \text{Fast} \\ \leftarrow \text{Healthy} \end{matrix}$$

The probability of *Fast* menu on Monday and *Healthy* menu on Friday 0.65. **1A**

iii. $T = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 65 \\ 35 \end{bmatrix} \begin{matrix} \leftarrow \text{Fast} \\ \leftarrow \text{Healthy} \end{matrix}$

For the long term consider, say, the 100th state

$$S_{100} = T^{100} S_0$$

$$= \begin{bmatrix} 33 \\ 67 \end{bmatrix} \begin{matrix} \leftarrow \text{Fast} \\ \leftarrow \text{Healthy} \end{matrix} \quad \mathbf{1M}$$

In the long term, each day, 33 will choose from the *Fast* menu and 67 will choose from the *Healthy* menu. **1A**