

Year 2012
VCE
Specialist Mathematics
Solutions
Trial Examination 1



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Question 1

$$z^3 + 8i = 0$$

$$z^3 = -8i$$

$$z^3 = 8 \operatorname{cis}\left(-\frac{\pi}{2} + 2k\pi\right)$$

$$z = \sqrt[3]{8} \operatorname{cis}\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right) \quad \text{M1}$$

when $k = 0$

$$z = 2 \operatorname{cis}\left(-\frac{\pi}{6}\right) = 2\left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right) = \sqrt{3} - i \quad \text{A1}$$

when $k = 1$

$$z = 2 \operatorname{cis}\left(\frac{3\pi}{6}\right) = 2\left(\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right)\right) = 2i \quad \text{A1}$$

when $k = -1$

$$z = 2 \operatorname{cis}\left(-\frac{5\pi}{6}\right) = 2\left(\cos\left(-\frac{5\pi}{6}\right) + i \sin\left(-\frac{5\pi}{6}\right)\right) = -\sqrt{3} - i \quad \text{A1}$$

Question 2

Let $\underline{a} = 4\underline{i} - 6\underline{j} - 2\underline{k}$ and $\underline{b} = 2\underline{i} - 2\underline{j} + \underline{k}$

$$\underline{a} \cdot \underline{b} = 8 + 12 - 2 = 18$$

$$|\underline{b}| = \sqrt{4 + 4 + 1} = 3 \quad \text{so that} \quad \hat{\underline{b}} = \frac{1}{3}(2\underline{i} - 2\underline{j} + \underline{k})$$

the component of $\underline{a}$ parallel to $\underline{b}$ is

$$(\underline{a} \cdot \hat{\underline{b}})\hat{\underline{b}} = 2(2\underline{i} - 2\underline{j} + \underline{k}) \quad \text{A1}$$

the component of $\underline{a}$ perpendicular to $\underline{b}$ is

$$\underline{a} - (\underline{a} \cdot \hat{\underline{b}})\hat{\underline{b}} = (4\underline{i} - 6\underline{j} - 2\underline{k}) - 2(2\underline{i} - 2\underline{j} + \underline{k}) = -2\underline{j} - 4\underline{k} \quad \text{A1}$$

Question 3

$$v(t) = \frac{36}{\sqrt{64-3t^2}}$$

$$d = \int_0^4 \frac{36}{\sqrt{64-3t^2}} dt \quad \text{A1}$$

$$d = \frac{36}{\sqrt{3}} \left[\sin^{-1} \left(\frac{\sqrt{3}t}{8} \right) \right]_0^4 \quad \text{A1}$$

$$d = \frac{36\sqrt{3}}{3} \left(\sin^{-1} \left(\frac{\sqrt{3}}{2} \right) - \sin^{-1}(0) \right) = 12\sqrt{3} \left(\frac{\pi}{3} - 0 \right)$$

$$d = 4\sqrt{3}\pi \text{ metres} \quad \text{A1}$$

Question 4

$$x^4 = x^2y - y^3 \quad \text{when } y = -2$$

$$x^4 = -2x^2 + 8$$

$$x^4 + 2x^2 - 8 = 0$$

$$(x^2 + 4)(x^2 - 2) = 0$$

$$\text{since } x \in \mathbb{R} \text{ and } x < 0 \quad x = -\sqrt{2} \quad \text{A1}$$

using implicit differentiation

$$4x^3 = 2xy + x^2 \frac{dy}{dx} - 3y^2 \frac{dy}{dx} \quad \text{M1}$$

$$(3y^2 - x^2) \frac{dy}{dx} = 2xy - 4x^3$$

$$\frac{dy}{dx} = \frac{2xy - 4x^3}{3y^2 - x^2} \quad \text{now when } x = -\sqrt{2} \text{ and } y = -2$$

$$\left. \frac{dy}{dx} \right|_{(-\sqrt{2}, -2)} = \frac{4\sqrt{2} + 8\sqrt{2}}{12 - 2}$$

$$\left. \frac{dy}{dx} \right|_{(-\sqrt{2}, -2)} = \frac{6\sqrt{2}}{5} \quad \text{A1}$$

Question 5

$$y = 3 \tan(2x)$$

$$V = \pi \int_0^{\frac{\pi}{8}} 9 \tan^2(2x) dx \quad \text{A1}$$

$$V = 9\pi \int_0^{\frac{\pi}{8}} (\sec^2(2x) - 1) dx \quad \text{M1}$$

$$V = 9\pi \left[\frac{1}{2} \tan(2x) - x \right]_0^{\frac{\pi}{8}}$$

$$V = 9\pi \left(\frac{1}{2} \tan\left(\frac{\pi}{4}\right) - \frac{\pi}{8} - (0) \right)$$

$$V = 9\pi \left(\frac{1}{2} - \frac{\pi}{8} \right) = \frac{9\pi}{8} (4 - \pi) \quad \text{A1}$$

Question 6**a.**

by Newtons law of motion $1200a = 4800te^{-2t} - 3600v$ using $a = \frac{dv}{dt}$ A1

$$\frac{dv}{dt} + 3v = 4te^{-2t}$$

b. $v(t) = (At + B)e^{-2t} + 4e^{-3t}$

since $v(0) = 0 \Rightarrow B + 4 = 0 \quad B = -4$ A1

so $v(t) = (At - 4)e^{-2t} + 4e^{-3t}$ differentiating using the product rule

$$\frac{dv}{dt} = Ae^{-2t} - 2(At - 4)e^{-2t} - 12e^{-3t} \quad \text{M1}$$

$$\frac{dv}{dt} = (8 + A - 2At)e^{-2t} - 12e^{-3t}$$

substituting into the *LHS* and equating to the *RHS* M1

$$\frac{dv}{dt} + 3v = (8 + A - 2At)e^{-2t} - 12e^{-3t} + 3(At - 4)e^{-2t} + 12e^{-3t}$$

$$\frac{dv}{dt} + 3v = (At + A - 4)e^{-2t} = 4te^{-2t}$$

so that $A = 4$ A1

Question 7

a. Consider $\frac{36}{4x^2+9}$ let $x = \frac{3}{2}\tan(t)$

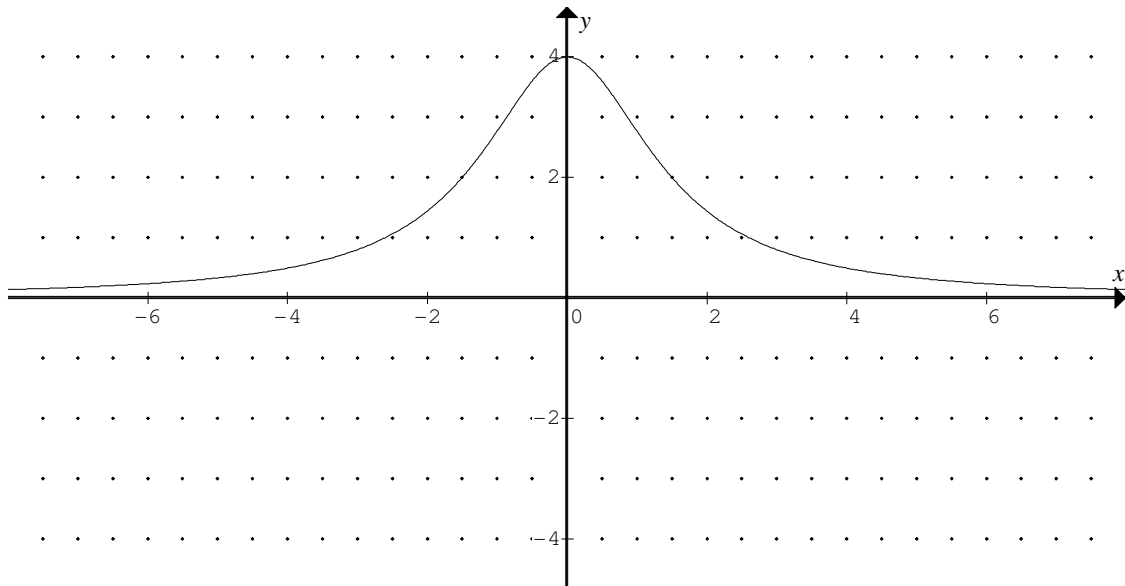
$$\frac{36}{4x^2+9} = \frac{36}{9\tan^2(t)+9} \quad \text{M1}$$

$$= \frac{36}{9(1+\tan^2(t))} = \frac{4}{\sec^2(t)}$$

$$= 4\cos^2(t) = 2(2\cos^2(t)) = 2(1+\cos(2t)) = y \quad \text{A1}$$

b. crosses the y-axis at $y = 4$ (0,4)

the x-axis $y = 0$ is a horizontal asymptote. graph correct shape A2



c. $A = \int_0^{\frac{3}{2}} \frac{36}{4x^2+9} dx$

$$A = \frac{36}{2 \times 3} \left[\tan^{-1} \left(\frac{2x}{3} \right) \right]_0^{\frac{3}{2}} \quad \text{A1}$$

$$A = 6(\tan^{-1}(1) - \tan^{-1}(0)) = 6 \times \frac{\pi}{4}$$

$$A = \frac{3\pi}{2} \quad \text{A1}$$

Question 8

Converting to polar $1+i = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)$ and $1-i = \sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)$, now

$$(1+i)^m + (1-i)^m = 0$$

$$\left[\sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)\right]^m + \left[\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)\right]^m = 0 \quad \text{using De'Moivre's Theorem} \quad \text{M1}$$

$$(\sqrt{2})^m \operatorname{cis}\left(\frac{m\pi}{4}\right) + (\sqrt{2})^m \operatorname{cis}\left(-\frac{m\pi}{4}\right) = (\sqrt{2})^m \left(\operatorname{cis}\left(\frac{m\pi}{4}\right) + \operatorname{cis}\left(-\frac{m\pi}{4}\right)\right) = 0$$

expanding since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$

$$\cos\left(\frac{m\pi}{4}\right) + i \sin\left(\frac{m\pi}{4}\right) + \cos\left(\frac{m\pi}{4}\right) - i \sin\left(\frac{m\pi}{4}\right) = 0 \quad \text{M1}$$

$$2 \cos\left(\frac{m\pi}{4}\right) = 0 \quad \text{A1}$$

$$\frac{m\pi}{4} = \frac{(2k+1)\pi}{2} \quad \text{where } k \text{ is an integer}$$

$$m = 2(2k+1) \quad \text{where } k \text{ is an integer} \quad \text{A1}$$

Question 9

resolving downwards around the M kg mass

$$(1) \quad Ma = Mg - T$$

resolving upwards around the 2 kg mass M1

$$(2) \quad T - 2g = 2a$$

adding to eliminate the tension T in the string, to find the acceleration a , of the system

$$(M+2)a = (M-2)g \quad \text{A1}$$

$$\text{Now using } s = ut + \frac{1}{2}at^2 \text{ with } s = \frac{1}{2}, u = 0, t = \frac{3}{7} \text{ and } a = \frac{(M-2)g}{M+2} \quad \text{M1}$$

$$\frac{1}{2} = 0 + \frac{1}{2} \times \frac{(M-2)g}{M+2} \times \frac{9}{49}$$

$$\frac{M-2}{M+2} = \frac{49}{9 \times 9.8} = \frac{5}{9} \quad \text{M1}$$

$$9(M-2) = 5(M+2)$$

$$9M - 18 = 5M + 10$$

$$4M = 28$$

$$M = 7 \text{ kg} \quad \text{A1}$$

Question 10

a.
$$I = \int_1^5 \frac{\sin^2\left(\frac{\pi x}{12}\right)}{x(6-x)} dx$$

let $u = 6 - x$ then $x = 6 - u$ and $\frac{du}{dx} = -1$ M1

now $\sin^2\left(\frac{\pi x}{12}\right) = \sin^2\left(\frac{\pi(6-u)}{12}\right) = \sin^2\left(\frac{\pi}{2} - \frac{\pi u}{12}\right) = \cos^2\left(\frac{\pi u}{12}\right)$

terminals when $x = 5 \Rightarrow u = 1$ and $x = 1 \Rightarrow u = 5$ M1

so
$$I = - \int_5^1 \frac{\cos^2\left(\frac{\pi u}{12}\right)}{(6-u)u} du = \int_1^5 \frac{\cos^2\left(\frac{\pi u}{12}\right)}{u(6-u)} du = \int_1^5 \frac{\cos^2\left(\frac{\pi x}{12}\right)}{x(6-x)} dx$$

b. consider
$$2I = \int_1^5 \frac{\cos^2\left(\frac{\pi x}{12}\right)}{x(6-x)} dx + \int_1^5 \frac{\sin^2\left(\frac{\pi x}{12}\right)}{x(6-x)} dx$$

$$2I = \int_1^5 \frac{\sin^2\left(\frac{\pi x}{12}\right) + \cos^2\left(\frac{\pi x}{12}\right)}{x(6-x)} dx$$

$$2I = \int_1^5 \frac{1}{x(6-x)} dx \text{ by partial fractions}$$

$$\frac{1}{x(6-x)} = \frac{A}{x} + \frac{B}{6-x} = \frac{A(6-x) + Bx}{x(6-x)} = \frac{x(B-A) + 6A}{x(6-x)}$$
 M1

$$B - A = 0 \text{ and } 6A = 1 \Rightarrow A = B = \frac{1}{6}$$

$$2I = \frac{1}{6} \int_1^5 \left(\frac{1}{x} + \frac{1}{6-x} \right) dx = \frac{1}{6} [\log_e |x| - \log_e |6-x|]_1^5$$
 M1

$$2I = \frac{1}{6} \left[\log_e \left| \frac{x}{6-x} \right| \right]_1^5 = \frac{1}{6} \left[\log_e (5) - \log_e \left(\frac{1}{5} \right) \right] = \frac{1}{6} \log_e (25)$$

$$I = \frac{1}{12} \log_e (25) = \frac{1}{12} \log_e (5^2) = \frac{1}{6} \log_e (5)$$
 A1

END OF SUGGESTED SOLUTIONS