

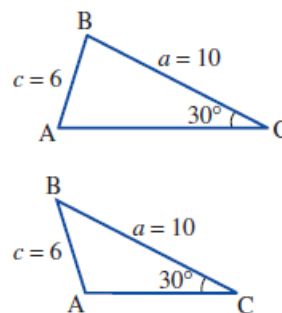
The ambiguous case

When using the sine rule there is one important issue to consider. If we are given two side lengths and an angle opposite one of these side lengths, then two different triangles may be drawn. For example, if $a = 10$, $c = 6$ and $C = 30^\circ$, two possible triangles could be created.

In the first case, angle A is an acute angle, while in the second case, angle A is an obtuse angle. The two values for A will add to 180° .

The ambiguous case does not work for each example. It would be useful to know, before commencing a question, whether or not the ambiguous case exists and, if so, to then find both sets of solutions.

The ambiguous case exists if C is an acute angle and $a > c > a \times \sin(C)$, or any equivalent statement; for example, if B is an acute angle and $a > b > a \times \sin(B)$, and so on.



WORKED EXAMPLE 8

In the triangle ABC, $a = 10$ m, $c = 6$ m and $C = 30^\circ$.

a Show that the ambiguous case exists.

b Find two possible values of A , and hence two possible values of B and b .

THINK

- a** 1 Check that the conditions for an ambiguous case exist, i.e. that C is an acute angle and that $a > c > a \times \sin(C)$.

- 2 State the answer.

WRITE/DRAW

- a** $C = 30^\circ$ so C is an acute angle.

$$\sin(C) = \sin(30^\circ) = 0.5$$

$$a > c > a \times \sin(C)$$

$$10 > 6 > 10 \sin(30^\circ)$$

$$10 > 6 > 5$$

This is correct.

This is an ambiguous case of the sine rule.

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Worked example 8

Case 1

- b** 1 Draw a labelled diagram of the triangle ABC and fill in the given information.

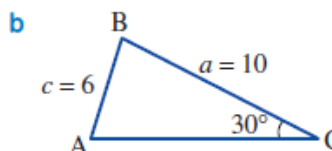
- 2 Write the sine rule to find A .

- 3 Substitute the known values into the rule.

- 4 Transpose the equation to make $\sin(A)$ the subject.

- 5 Evaluate angle A , in degrees and minutes.

- 6 Determine the value of angle B , using the fact that the angle sum of any triangle is 180° .



To find angle A :

$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$$

$$\frac{10}{\sin(A)} = \frac{6}{\sin(30^\circ)}$$

$$10 \times \sin(30^\circ) = 6 \times \sin(A)$$

$$\frac{10 \times \sin(30^\circ)}{6} = \sin(A)$$

$$\sin(A) = \frac{10 \times \sin(30^\circ)}{6}$$

$$A = \sin^{-1}\left(\frac{10 \times \sin(30^\circ)}{6}\right)$$

$$A = 56^\circ 27'$$

$$B = 180^\circ - (30^\circ + 56^\circ 27') \\ = 93^\circ 33'$$

- 7 Write the sine rule to find b .

To find side length b :

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

- 8 Substitute the known values into the rule.

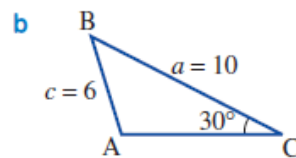
$$\frac{b}{\sin(93^\circ 33')} = \frac{6}{\sin(30^\circ)}$$

- 9 Transpose the equation to make b the subject and evaluate.

$$\begin{aligned} b &= \frac{6 \times \sin(93^\circ 33')}{\sin(30^\circ)} \\ &= 11.98 \text{ m} \end{aligned}$$

Case 2

- b 1 Draw a labelled diagram of the triangle ABC and fill in the given information.



- 2 Write the alternative value for angle A. Subtract the value obtained for A in Case 1 from 180° .

To find the alternative angle A:

If $\sin(A) = 0.8333$, then A could also be:

$$\begin{aligned} A &= 180^\circ - 56^\circ 27' \\ &= 123^\circ 33' \end{aligned}$$

- 3 Determine the alternative value of angle B, using the fact that the angle sum of any triangle is 180° .

$$\begin{aligned} B &= 180^\circ - (30^\circ + 123^\circ 33') \\ &= 26^\circ 27' \end{aligned}$$

- 4 Write the sine rule to find the alternative b .

To find side length b :

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

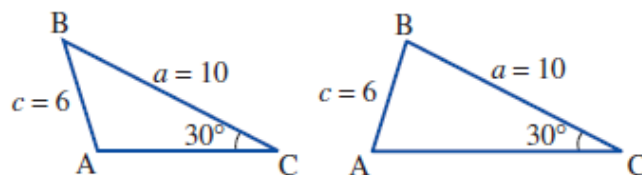
- 5 Substitute the known values into the rule.

$$\frac{b}{\sin(26^\circ 27')} = \frac{6}{\sin(30^\circ)}$$

- 6 Transpose the equation to make b the subject and evaluate.

$$\begin{aligned} b &= \frac{6 \times \sin(26^\circ 27')}{\sin(30^\circ)} \\ &= 5.35 \text{ m} \end{aligned}$$

Hence, for this example there were two possible solutions as shown by the diagram below.



Exercise 5C The sine rule

- 1 **WE7** In the triangle ABC, $a = 10$, $b = 12$ and $B = 58^\circ$. Find A , C and c .
- 2 In the triangle ABC, $c = 17.35$, $a = 26.82$ and $A = 101^\circ 47'$. Find C , B and b .
- 3 In the triangle ABC, $a = 5$, $A = 30^\circ$ and $B = 80^\circ$. Find C , b and c .
- 4 In the triangle ABC, $c = 27$, $C = 42^\circ$ and $A = 105^\circ$. Find B , a and b .
- 5 In the triangle ABC, $a = 7$, $c = 5$ and $A = 68^\circ$. Find the perimeter of the triangle.
- 6 Find all unknown sides and angles for the triangle ABC, given $A = 57^\circ$, $B = 72^\circ$ and $a = 48.2$.
- 7 Find all unknown sides and angles for the triangle ABC, given $a = 105$, $B = 105^\circ$ and $C = 15^\circ$.
- 8 Find all unknown sides and angles for the triangle ABC, given $a = 32$, $b = 51$ and $A = 28^\circ$.
- 9 Find the perimeter of the triangle ABC if $a = 7.8$, $b = 6.2$ and $A = 50^\circ$.
- 10 **MC** In a triangle ABC, $A = 40^\circ$, $C = 80^\circ$ and $c = 3$. The value of b is:

A 2.64	B 2.86	C 14
D 4.38	E 4.60	
- 11 **WE8** In the triangle ABC, $a = 10$, $c = 8$ and $C = 50^\circ$. Find two possible values of A and hence two possible values of b .
- 12 In the triangle ABC, $a = 20$, $b = 12$ and $B = 35^\circ$. Find two possible values for the perimeter of the triangle.
- 13 Find all unknown sides and angles for the triangle ABC, given $A = 27^\circ$, $B = 43^\circ$ and $c = 6.4$.
- 14 Find all unknown sides and angles for the triangle ABC, given $A = 100^\circ$, $b = 2.1$ and $C = 42^\circ$.
- 15 Find all unknown sides and angles for the triangle ABC, given $A = 25^\circ$, $b = 17$ and $a = 13$.

ANSWERS

Exercise 5C The sine rule

- 1 $44^\circ 58'$, $77^\circ 2'$, 13.79
- 2 $39^\circ 18'$, $38^\circ 55'$, 17.21
- 3 70° , 9.85, 9.4
- 4 33° , 38.98, 21.98
- 5 19.12
- 6 $C = 51^\circ$, $b = 54.66$, $c = 44.66$
- 7 $A = 60^\circ$, $b = 117.11$, $c = 31.38$
- 8 $B = 48^\circ 26'$, $C = 103^\circ 34'$, $c = 66.26$; or
 $B = 131^\circ 34'$, $C = 20^\circ 26'$, $c = 23.8$
- 9 24.17
- 10 A
- 11 $A = 73^\circ 15'$, $b = 8.73$; or $A = 106^\circ 45'$,
 $b = 4.12$
- 12 51.9 or 44.86
- 13 $C = 110^\circ$, $a = 3.09$, $b = 4.64$
- 14 $B = 38^\circ$, $a = 3.36$, $c = 2.28$
- 15 $B = 33^\circ 33'$, $C = 121^\circ 27'$, $c = 26.24$; or
 $B = 146^\circ 27'$, $C = 8^\circ 33'$, $c = 4.57$

Area of triangles

The area of any triangle is given by the rule $\text{Area} = \frac{1}{2}bh$ where b is the base length and h is the perpendicular height of the triangle.

However, often the perpendicular height is not given directly and needs to be calculated first. In the triangle ABC, b is the base length and h is the perpendicular height of the triangle.

Using the trigonometric ratio for sine:

$$\sin(A) = \frac{h}{c}$$

Transposing the equation to make h the subject, we obtain:

$$h = c \times \sin(A)$$

Therefore, the area of triangle ABC becomes:

$$\text{Area} = \frac{1}{2}bc \sin(A)$$

Depending on how the triangle is labelled, the formula could read:

$$\text{Area} = \frac{1}{2}ab \sin(C)$$

$$\text{Area} = \frac{1}{2}ac \sin(B)$$

$$\text{Area} = \frac{1}{2}bc \sin(A)$$

The area formula may be used on any triangle provided that two sides of the triangle and the included angle (that is, the angle between the two given sides) are known.

Heron's formula

If we know the lengths of all the sides of the triangle but none of the angles, we could use the cosine rule to find an angle, then use $\frac{1}{2}bc \sin(A)$ to find the area. Alternatively, we could use *Heron's formula* to find the area.

Heron's formula states that the area of a triangle is:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where s is the *semi-perimeter* of the triangle; that is,

$$s = \frac{1}{2}(a + b + c)$$

Note: The proof of this formula is beyond the scope of this course.

REMEMBER

1. If two sides of any triangle and the included angle (that is, the angle between the two given sides) are known, the following rules may be used to determine the area of that triangle.

$$\text{Area} = \frac{1}{2}ab \sin(C)$$

$$\text{Area} = \frac{1}{2}ac \sin(B)$$

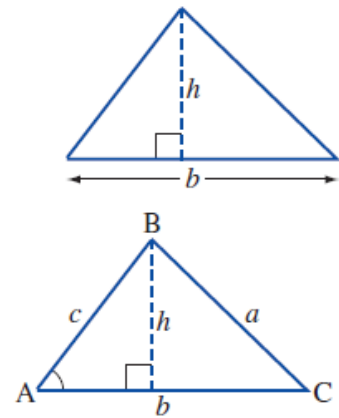
$$\text{Area} = \frac{1}{2}bc \sin(A)$$

2. Alternatively, if the lengths of three sides of a triangle are known, Heron's formula may be used to find the area of the triangle:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

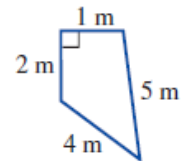
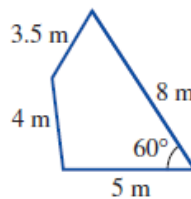
where s is the semi-perimeter of the triangle; that is,

$$s = \frac{1}{2}(a + b + c)$$

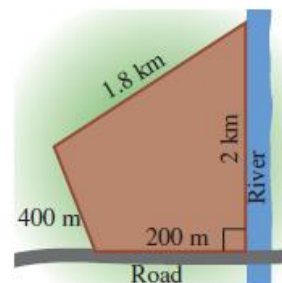


Exercise 5E Area of triangles

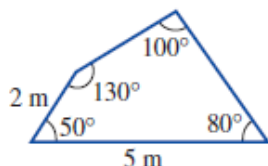
- 1 **WE12** Find the area of the triangle ABC with $a = 7$ cm, $b = 4$ cm and $C = 68^\circ$.
- 2 Find the area of the triangle ABC with $a = 7.3$ cm, $c = 10.8$ cm and $B = 104^\circ 40'$.
- 3 Find the area of the triangle ABC with $b = 23.1$ m, $c = 18.6$ m and $A = 82^\circ 17'$.
- 4 Find the exact area of the triangle DEF with $d = 6$, $e = 9$ and $F = 60^\circ$.
- 5 Find the exact area of the triangle QPR with $p = 12$, $r = 10$ and $Q = 45^\circ$.
- 6 **WE13 MC** In a triangle, $a = 15$ m, $b = 20$ m and $B = 50^\circ$. The area of the triangle is:
A 86.2 m^2 **B** 114.9 m^2 **C** 149.4 m^2 **D** 172.4 m^2 **E** 181.7 m^2
- 7 **WE14** Find the area of the triangle with sides of 5 cm, 6 cm and 8 cm.
- 8 Find the area of the triangle with sides of 40 mm, 30 mm and 5.7 cm.
- 9 Find the area of the triangle with sides of 16 mm, 3 cm and 2.7 cm.
- 10 Find the area of the equilateral triangle with sides 4 cm. Leave your answer in simplified surd form.
- 11 **MC** A triangle has sides of length 10 cm, 14 cm and 20 cm. The area of the triangle is:
A 41 cm^2 **B** 65 cm^2 **C** 106 cm^2 **D** 137 cm^2 **E** 1038 cm^2
- 12 A triangle has $a = 10$ cm, $c = 14$ cm and $C = 48^\circ$. Find A and B and hence the area.
- 13 A triangle has $a = 17$ m, $c = 22$ m and $C = 56^\circ$. Find A and B and hence the area.
- 14 A triangle has $b = 32$ mm, $c = 15$ mm and $B = 38^\circ$. Find A and C and hence the area.
- 15 A piece of metal is in the shape of a triangle with sides of length 114 mm, 72 mm and 87 mm. Find its area using Heron's formula.
- 16 In a triangle, the largest angle is 115° , the longest side is 62 cm and another side is 35 cm. Find the area of the triangle.
- 17 A triangle has two sides of 25 cm and 30 cm. The angle between the two sides is 30° . Find:
a its area **b** the length of its third side **c** its area using Heron's formula.
- 18 The surface of a fish pond has the shape shown in the diagram at right. How many goldfish can the pond support if each fish requires 0.3 m^2 surface area of water?
- 19 Find the area of this quadrilateral.



- 20 A parallelogram has diagonals of length 10 cm and 17 cm. An angle between them is 125° . Find:
a the area of the parallelogram **b** the dimensions of the parallelogram.
- 21 A lawn is to be made in the shape of a triangle with sides of length 11 m, 15 m and 17.2 m. How much grass seed, to the nearest kilogram, is needed if it is sown at the rate of 1 kg per 5 m^2 ?
- 22 A bushfire burns out an area of level grassland shown in the diagram. What is the area, in hectares, of the land that is burned?



- 23 An earth embankment is 27 m long and has a cross-section shown in the diagram. Find the volume of earth needed to build the embankment.



- 24 **MC** A parallelogram has sides of 14 cm and 18 cm, and an angle between them of 72° . The area of the parallelogram is:
 A 86.2 cm^2 B 118.4 cm^2 C 172.4 cm^2 D 239.7 cm^2 E 252 cm^2
- 25 **MC** An advertising hoarding is in the shape of an isosceles triangle with sides of length 15 m, 15 m and 18 m. It is to be painted with two coats of purple paint. If the paint covers 12 m^2 per litre, the amount of paint needed, to the nearest litre, would be:
 A 9 L B 18 L C 24 L D 36 L E 42 L

ANSWERS

Exercise 5E Area of triangles

- 1 12.98 cm^2
- 2 38.14 cm^2
- 3 212.88 m^2
- 4 $\frac{27\sqrt{3}}{2}$
- 5 $30\sqrt{2}$
- 6 C
- 7 14.98 cm^2
- 8 570.03 mm^2
- 9 2.15 cm^2
- 10 $4\sqrt{3}$
- 11 B
- 12 $A = 32^\circ 4'$, $B = 99^\circ 56'$, area = 68.95 cm^2
- 13 $A = 39^\circ 50'$, $B = 84^\circ 10'$, area = 186.03 m^2
- 14 $A = 125^\circ 14'$, $C = 16^\circ 46'$,
area = 196.03 mm^2
- 15 3131.41 mm^2
- 16 610.38 cm^2
- 17 a 187.5 cm^2
b 15.03 cm
c 187.47 cm^2
- 18 17 goldfish
- 19 22.02 m^2
- 20 a Area = 69.63 cm^2
b The dimensions are 12.08 cm and 6.96 cm.
- 21 17 kg
- 22 52.2 hectares
- 23 174.5 m^3
- 24 D
- 25 B