

Heron's formula is used to find the area of any triangle given the lengths of the three sides of the triangle.

Heron's formula for the area A of any triangle with side lengths a , b and c is:

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{1}{2}(a+b+c)$$

Derivation

The area of the triangle is given by $A = \frac{1}{2}bc \sin \alpha$.

Using the cosine rule:

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$2bc \cos \alpha = b^2 + c^2 - a^2$$

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

Using the identity:

$$\sin^2 \alpha - \cos^2 \alpha = 1$$

$$\sin^2 \alpha = 1 - \cos^2 \alpha$$

$$\sin \alpha = \sqrt{1 - \cos^2 \alpha}$$

Using the cosine rule result:

$$\sin \alpha = \sqrt{1 - \frac{b^2 + c^2 - a^2}{2bc}}$$

$$= \sqrt{\frac{4b^2c^2 - (b^2 - c^2 + a^2)^2}{4b^2c^2}}$$

$$= \sqrt{\frac{4b^2c^2 - (a^4 + b^4 + c^4 + 2b^2c^2 - 2a^2b^2 - 2a^2c^2)}{4b^2c^2}}$$

$$= \sqrt{\frac{-a^4 - b^4 - c^4 + 2b^2c^2 + 2a^2b^2 + 2a^2c^2}{4b^2c^2}}$$

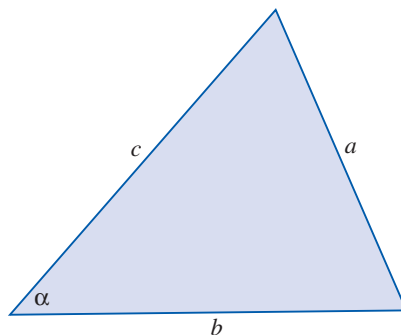
The area of a triangle is given by:

$$A = \frac{1}{2}bc \sin \alpha$$

$$= \frac{1}{2}bc \sqrt{\frac{-a^4 - b^4 - c^4 + 2b^2c^2 + 2a^2b^2 + 2a^2c^2}{4b^2c^2}}$$

$$= \sqrt{\frac{(bc)^2(-a^4 - b^4 - c^4 + 2b^2c^2 + 2a^2b^2 + 2a^2c^2)}{4 \times 4b^2c^2}}$$

$$= \sqrt{\frac{-a^4 - b^4 - c^4 + 2b^2c^2 + 2a^2b^2 + 2a^2c^2}{16}}$$



Learning task 4K

- 1 Show that:

$$(a+b+c)(b+c-a)(a+b-c) \\ = -a^4 - b^4 - c^4 + 2b^2c^2 + 2a^2b^2 + 2a^2c^2$$

- 2 If $s = \frac{1}{2}(a+b+c)$ show that:

a $2s = a+b+c$

b $2(s-a) = b+c-a$

c $2(s-b) = a+c-b$

d $2(s-c) = a+b-c$

- 3 Use the previous results to show that:

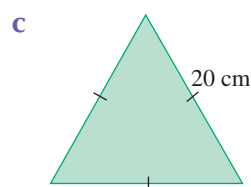
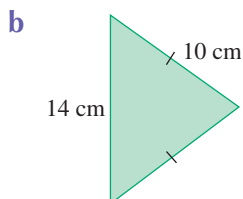
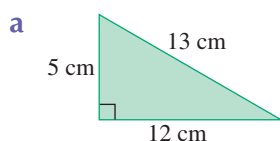
$$A = \sqrt{\frac{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}{16}}$$

$$A = \sqrt{\frac{2s \times 2(s-a) \times 2s(s-b) \times 2s(s-c)}{16}}$$

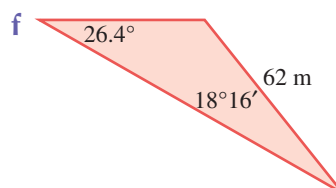
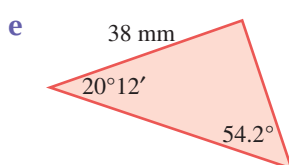
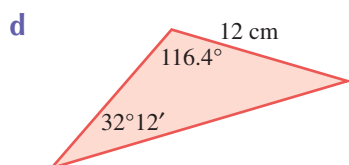
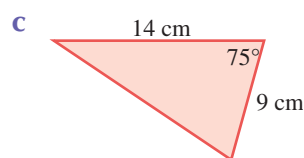
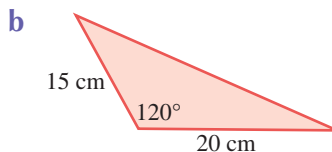
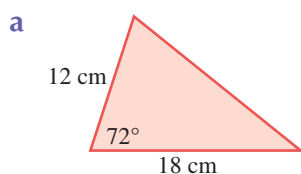
$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

and hence verify Heron's formula.

- 4 Find the area of the following triangles using Heron's formula:



- 5 Find the length of the missing side(s) and so use Heron's formula to find the area of each triangle:



Learning task 4K

4 a 30 cm² **b** 49.99 cm² **c** 173.20 cm²

5 a 102.6 cm² **b** 130.02 cm² **c** 60.94 cm²

d 63.13 cm² **e** 295.81 mm² **f** 935.43 m²