

CHAPTER 3

Cubic and quartic functions

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10 Quick Questions

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Polynomials of degree 3 and 4

This chapter will deal mainly with polynomials of degree 3 (cubics). The general equation of a cubic polynomial is $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, more commonly written as $y = ax^3 + bx^2 + cx + d$.

Degree 4 polynomials (quartics) will also be considered. The general equation of a quartic polynomial is $P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, more commonly written as $y = ax^4 + bx^3 + cx^2 + dx + e$.

3A Expanding

If we expand three linear factors, for example, $(x + 1)(x + 2)(x - 7)$, we get a cubic polynomial (a polynomial of degree 3) as the following worked example shows.

WORKED EXAMPLE 1

Expand:

- a** $x(x + 2)(x - 3)$ **b** $(x - 1)(x + 5)(x + 2)$.

THINK

- a**
- 1 Write the expression.
 - 2 Expand two linear factors and simplify.
 - 3 Multiply by the remaining factor.
- b**
- 1 Write the expression.
 - 2 Expand two linear factors and simplify.
 - 3 Multiply by the remaining factor and simplify.

WRITE

- a** $x(x + 2)(x - 3)$
 $= x(x^2 - 3x + 2x - 6)$
 $= x(x^2 - x - 6)$
 $= x^3 - x^2 - 6x$
- b** $(x - 1)(x + 5)(x + 2)$
 $= (x - 1)(x^2 + 2x + 5x + 10)$
 $= (x - 1)(x^2 + 7x + 10)$
 $= x^3 + 7x^2 + 10x - x^2 - 7x - 10$
 $= x^3 + 6x^2 + 3x - 10$

Note: Just as there is a shortcut for expanding perfect squares, there is also a shortcut for expanding cubes. We can find the shortcut by expanding $(a + b)^3$ as usual.

$$\begin{aligned}(a + b)^3 &= (a + b)(a + b)(a + b) = (a + b)(a^2 + 2ab + b^2) \\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

Similarly, $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

$$\begin{aligned}(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\ (a - b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3\end{aligned}$$

WORKED EXAMPLE 2

Expand the perfect cube $(x - 4)^3$ using the appropriate rule.

THINK

- 1 Use the rule $(a - b)^3 = a^3 - 3a^2b + 3ab^2 + b^3$.
In this case a is x and b is 4.
- 2 Simplify.

WRITE

$$\begin{aligned}(x - 4)^3 &= x^3 - 3 \times x^2 \times 4 + 3 \times x \times 4^2 - 4^3 \\ (x - 4)^3 &= x^3 - 12x^2 + 48x - 64\end{aligned}$$



WORKED EXAMPLE 3

Expand $-2x(x + 5)(x - 12)$.

THINK

- 1 Write the expression.
- 2 Expand the two linear factors and simplify.
- 3 Multiply the remaining factor. No simplification is needed.

WRITE

$$\begin{aligned}-2x(x + 5)(x - 12) &= -2x(x^2 - 12x + 5x - 60) \\ &= -2x(x^2 - 7x - 60) \\ &= -2x^3 + 14x^2 + 120\end{aligned}$$

Exercise 3A Expanding

1 **WE1a** Expand each of the following.

- | | | | |
|------------------------------|-----------------------------|-----------------------------|-----------------------------|
| a $x(x + 6)(x + 1)$ | b $x(x - 9)(x + 2)$ | c $x(x - 3)(x + 11)$ | d $2x(x + 2)(x + 3)$ |
| e $-3x(x - 4)(x + 4)$ | f $5x(x + 8)(x + 2)$ | g $x^2(x + 4)$ | h $-2x^2(7 - x)$ |
| i $(5x)(-6x)(x + 9)$ | j $-7x(x + 4)^2$ | | |

2 **WE1b** Expand each of the following.

- | | | |
|------------------------------------|----------------------------------|----------------------------------|
| a $(x + 7)(x + 2)(x + 3)$ | b $(x - 2)(x + 4)(x - 5)$ | c $(x - 1)(x - 4)(x + 8)$ |
| d $(x - 1)(x - 2)(x - 3)$ | e $(x + 6)(x - 1)(x + 1)$ | f $(x - 7)(x + 7)(x + 5)$ |
| g $(x + 11)(x + 5)(x - 12)$ | h $(x + 5)(x - 1)^2$ | i $(x + 2)(x - 7)^2$ |
| j $(x + 1)(x - 1)(x + 1)$ | | |

3 Expand each of the following.

- | | | |
|------------------------------------|-----------------------------------|------------------------------------|
| a $(x - 2)(x + 7)(x + 8)$ | b $(x + 5)(3x - 1)(x + 4)$ | c $(4x - 1)(x + 3)(x - 3)$ |
| d $(5x + 3)(2x - 3)(x - 4)$ | e $(1 - 6x)(x + 7)(x + 5)$ | f $3x(7x - 4)(x - 4)$ |
| g $-9x(1 - 2x)(3x + 8)$ | h $(6x + 5)(2x - 7)^2$ | i $(3 - 4x)(2 - x)(5x + 9)$ |
| j $2(7 + 2x)(x + 3)(x + 4)$ | | |

4 **WE2** Expand the following using the appropriate rule for expanding cubes.

- | | | |
|----------------------|-----------------------|-----------------------|
| a $(x + 2)^3$ | b $(x + 5)^3$ | c $(x - 1)^3$ |
| d $(x - 3)^3$ | e $(2x - 6)^3$ | f $(3x + 4)^3$ |

5 **WE3** Expand each of the following.

- | | | |
|-----------------------------------|-----------------------------|-------------------------------------|
| a $(x + 5)(x - 11)(x + 2)$ | b $3x(x + 6)(x - 1)$ | c $6(x - 5)(-x + 15)(x + 8)$ |
| d $(-x + 5)(x - 12)^2$ | e $-x(x + 10)^2$ | f $-(x + 13)^3$ |

3B Long division of polynomials

The reverse of expanding is factorising (expressing a polynomial as a product of its linear factors). Before learning how to factorise cubics, you must be familiar with long division of polynomials. You may remember in earlier levels doing long division questions.

Consider $745 \div 3$, or $3 \overline{)745}$

The process used is as follows.

3 into 7 goes 2 times. Write 2 at the top.

$2 \times 3 = 6$. Write down the 6.

Subtract to get 1.

Bring down the 4 to form 14.

$$\begin{array}{r} 2 \\ 3 \overline{)745} \\ - 6 \\ \hline 14 \end{array}$$

3 into 14 goes 4 times. Write 4 at the top.

$4 \times 3 = 12$. Write down the 12.

Subtract to get 2.

Bring down the 5 to form 25.

3 into 25 goes 8 times. Write 8 at the top.

$8 \times 3 = 24$. Write down the 24.

Subtract to get 1.

Answer: $745 \div 3 = 248$ remainder 1

$$\begin{array}{r} \text{Quotient} \leftarrow 248 \\ \text{Divisor} \rightarrow 3 \overline{)745} \leftarrow \text{Dividend} \\ - 6 \\ \hline 14 \\ - 12 \\ \hline 25 \\ - 24 \\ \hline 1 \leftarrow \text{Remainder} \end{array}$$

The same process can be used to divide polynomials by polynomial factors.

Consider $(x^3 + 2x^2 - 13x + 10) \div (x - 3)$ or $x - 3 \overline{)x^3 + 2x^2 - 13x + 10}$

x into x^3 goes x^2 times

(consider only the leading terms).

Write x^2 at the top.

$x^2 \times (x - 3) = x^3 - 3x^2$

Write down the $x^3 - 3x^2$.

$$\begin{array}{r} x^2 \\ x - 3 \overline{)x^3 + 2x^2 - 13x + 10} \\ - (x^3 - 3x^2) \\ \hline 5x^2 - 13x \end{array}$$

Subtract.

$(x^3 - x^3 = 0, 2x^2 - -3x^2 = 5x^2)$

Bring down the $-13x$.

x into $5x^2$ goes $5x$ times. Write $+5x$ at the top.

$5x \times (x - 3) = 5x^2 - 15x$

Write down the $5x^2 - 15x$.

Subtract.

Note: $5x^2 - 5x^2 = 0, -13x - -15x = +2x$

Bring down the 10.

$$\begin{array}{r} \text{Quotient} \leftarrow x^2 + 5x + 2 \\ \text{Divisor} \rightarrow x - 3 \overline{)x^3 + 2x^2 - 13x + 10} \leftarrow \text{Dividend} \\ - (x^3 - 3x^2) \\ \hline 5x^2 - 13x \\ - (5x^2 - 15x) \\ \hline 2x + 10 \\ - (2x - 6) \\ \hline 16 \leftarrow \text{Remainder} \end{array}$$

x into $2x$ goes 2 times. Write $+2$ at the top.

$2 \times (x - 3) = 2x - 6$

Write down the $2x - 6$.

Subtract to get 16.

Answer: $(x^3 + 2x^2 - 13x + 10) \div (x - 3) = x^2 + 5x + 2$ remainder 16

WORKED EXAMPLE 4

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Perform the following long divisions and state the quotient and remainder.

a $(2x^3 + 6x^2 - 3x + 2) \div (x - 6)$

b $(x^3 - 7x + 1) \div (x + 5)$

THINK

WRITE

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Worked example 4

- a** 1 Write the question in long division format.

- 2 Perform the long division process.

- 3 Write down the quotient and remainder.

$$\begin{array}{r} 2x^2 + 18x + 105 \leftarrow Q \\ x-6 \overline{) 2x^3 + 6x^2 - 3x + 2} \\ \underline{-(2x^3 - 12x^2)} \\ 18x^2 - 3x \\ \underline{-(18x^2 - 108x)} \\ 105x + 2 \\ \underline{-(105x - 630)} \\ 632 \leftarrow R \end{array}$$

The quotient is $2x^2 + 18x + 105$; the remainder is 632.

- b** 1 Write the question in long division format. Note that there is no x^2 term in this equation. Include $0x^2$ as a 'place holder'.

- 2 Perform the long division process.

- 3 Write down the quotient and remainder.

$$\begin{array}{r} x^2 - 5x + 18 \leftarrow Q \\ x+5 \overline{) x^3 + 0x^2 - 7x + 1} \\ \underline{-(x^3 + 5x^2)} \\ -5x^2 - 7x \\ \underline{-(-5x^2 - 25x)} \\ 18x + 1 \\ \underline{-(18x + 90)} \\ -89 \leftarrow R \end{array}$$

The quotient is $x^2 - 5x + 18$; the remainder is -89.

WORKED EXAMPLE 5

Find the quotient and remainder when $x^4 - 3x^3 + 2x^2 - 8$ is divided by $x + 2$.

THINK

WRITE

- Write the question in long division format. Include $0x$ as a 'place holder'.
- Divide x into x^4 and write the result above.
- Multiply x^3 by $x + 2$ and write the result underneath.
- Subtract and then bring down the next term.
- Continue to perform the long division process (as you did for cubic polynomials).
- Write down the quotient and remainder.

$$\begin{array}{r} x^3 - 5x^2 + 12x - 24 \\ x+2 \overline{) x^4 - 3x^3 + 2x^2 + 0x - 8} \\ \underline{-(x^4 + 2x^3)} \\ -5x^3 + 2x^2 \\ \underline{-(-5x^3 - 10x^2)} \\ 12x^2 + 0x \\ \underline{-(12x^2 + 24x)} \\ -24x - 8 \\ \underline{-(-24x - 48)} \\ 40 \end{array}$$

The quotient is $x^3 - 5x^2 + 12x - 24$.
The remainder is 40.



WORKED EXAMPLE 6

Calculate the quotient and remainder when $x^3 - 4x^2 - 7x - 5$ is divided by $x - 1$.

THINK

WRITE

- Write the question in long division format.
- Perform the long division process.

$$\begin{array}{r} x^2 - 3x - 10 \\ x-1 \overline{) x^3 - 4x^2 - 7x - 5} \\ \underline{-(x^3 - x^2)} \\ -3x^2 - 7x \\ \underline{-(-3x^2 + 3x)} \\ -10x - 5 \\ \underline{-(-10x + 10)} \\ -15 \end{array}$$

- 3 Write the quotient and the remainder.

The quotient is $x^2 - 3x - 10$.
The remainder is -15 .

Exercise 3B Long division of polynomials

- 1 **WE4a** Perform the following long divisions, and state the quotient and remainder.

- a** $(x^3 + 6x^2 + 3x + 1) \div (x + 3)$ **b** $(x^3 + 4x^2 + 3x + 4) \div (x + 2)$ **c** $(x^3 + x^2 + x + 3) \div (x + 1)$
d $(x^3 + x^2 + 4x + 1) \div (x + 2)$ **e** $(x^3 + 2x^2 - 5x - 9) \div (x - 2)$ **f** $(x^3 + x^2 - 9x - 5) \div (x - 2)$
g $(x^3 - 5x^2 + 3x - 8) \div (x - 3)$ **h** $(x^3 - 9x^2 + 2x - 1) \div (x - 5)$ **i** $3x^3 - x^2 + 6x + 5, x + 2$
j $4x^3 - 4x^2 + 10x - 4, x + 1$ **k** $2x^3 - 7x^2 + 9x + 1, x - 2$ **l** $2x^3 + 8x^2 - 9x - 1, x + 4$

- 2 Divide the first polynomial by the second, and state the quotient and remainder.

- a** $6x^3 - 7x^2 + 4x + 4, 2x - 1$ **b** $6x^3 + 23x^2 + 2x - 31, 3x + 4$ **c** $8x^3 + 6x^2 - 39x - 13, 2x + 5$
d $2x^3 - 15x^2 + 34x - 13, 2x - 7$ **e** $3x^3 + 5x^2 - 16x - 23, 3x + 2$ **f** $9x^3 - 6x^2 - 5x + 9, 3x - 4$

- 3 State the quotient and remainder for each of the following.

- a**
$$\begin{array}{r} -x^3 - 6x^2 - 7x - 16 \\ x + 1 \end{array}$$
 b
$$\begin{array}{r} -3x^3 + 7x^2 + 10x - 15 \\ x - 3 \end{array}$$

c
$$\begin{array}{r} -2x^3 + 9x^2 + 17x + 15 \\ 2x + 1 \end{array}$$
 d
$$\begin{array}{r} 4x^3 - 20x^2 + 23x - 2 \\ -2x + 3 \end{array}$$

- 4 **WE4b** State the quotient and remainder for each of the following.

- a** $(x^3 - 3x + 1) \div (x + 1)$ **b** $(x^3 + 2x^2 - 7) \div (x + 2)$ **c** $(x^3 - 5x^2 + 2x) \div (x - 4)$
d $(-x^3 - 7x + 8) \div (x - 1)$ **e** $(5x^2 + 13x + 1) \div (x + 3)$ **f** $(2x^3 + 8x^2 - 4) \div (x + 5)$
g $(-2x^3 - x + 2) \div (x - 2)$ **h** $(-4x^3 + 6x^2 + 2x) \div (2x + 1)$

- 5 **WE5** Find the quotient and remainder for each of the following.

- a** $(x^4 + x^3 + 3x^2 - 7x) \div (x - 1)$ **b** $(x^4 - 13x^2 + 36) \div (x - 2)$
c $(6x^4 - x^3 + 2x^2 - 4x) \div (x - 3)$

- 6 **WE6** Calculate the quotient and remainder for each of the following.

- a** $(x^3 + 9x^2 + 11x + 25) \div (x + 15)$ **b** $(2x^3 - 18x^2 + 5x - 9) \div (x - 31)$
c $(12x^3 + 32x - 9) \div (3x + 4)$ **d** $(18x^4 + 3x^3 + 45) \div (2x + 7)$

3C Polynomial values

Consider the polynomial $P(x) = x^3 - 5x^2 + x + 1$.

The value of the polynomial when $x = 3$ is denoted by $P(3)$ and is found by substituting $x = 3$ into the equation in place of x . That is,

$$\begin{aligned} P(3) &= (3)^3 - 5(3)^2 + (3) + 1 \\ &= 27 - 5(9) + 3 + 1 \\ &= 27 - 45 + 4 \\ &= -14. \end{aligned}$$

WORKED EXAMPLE 7

If $P(x) = 2x^3 + x^2 - 3x - 4$, determine:

- a** $P(1)$ **b** $P(-2)$ **c** $P(a)$ **d** $P(x + 1)$.

THINK

- a** 1 Write the expression.
 2 Replace x with 1.
 3 Simplify.

WRITE

$$\begin{aligned} \mathbf{a} \quad P(x) &= 2x^3 + x^2 - 3x - 4 \\ P(1) &= 2(1)^3 + (1)^2 - 3(1) - 4 \\ &= 2 + 1 - 3 - 4 \\ &= -4 \end{aligned}$$



b 1 Write the expression.

2 Replace x with -2 .

3 Simplify.

c 1 Write the expression.

2 Replace x with a .

No further simplification is possible.

d 1 Write the expression.

2 Replace x with $(x + 1)$.

3 Expand the right-hand side and collect like terms. Use the rules for expanding cubics and quadratics.

$$\mathbf{b} \quad P(x) = 2x^3 + x^2 - 3x - 4$$

$$\begin{aligned} P(-2) &= 2(-2)^3 + (-2)^2 - 3(-2) - 4 \\ &= 2(-8) + (4) + 6 - 4 \\ &= -16 + 4 + 6 - 4 \\ &= -10 \end{aligned}$$

$$\mathbf{c} \quad P(x) = 2x^3 + x^2 - 3x - 4$$

$$P(a) = 2a^3 + a^2 - 3a - 4$$

$$\mathbf{d} \quad P(x) = 2x^3 + x^2 - 3x - 4$$

$$\begin{aligned} P(x+1) &= 2(x+1)^3 + (x+1)^2 - 3(x+1) - 4 \\ &= 2(x^3 + 3x^2 + 3x + 1) + x^2 + 2x + 1 - 3x - 3 - 4 \\ &= 2x^3 + 6x^2 + 6x + 2 + x^2 - x - 6 \\ &= 2x^3 + 7x^2 + 5x - 4 \end{aligned}$$



WORKED EXAMPLE 8

Determine the following, considering $P(x) = 16x^4 + 3x^3 - 22x + 17$.

a $P(-14)$

b $P(y + 7)$

THINK

a 1 Write the expression.

2 Replace x with -14 .

3 Simplify.

b 1 Write the expression.

2 Replace x with $y + 7$.

3 Expand each term.

4 Collect like terms.

WRITE

$$\mathbf{a} \quad P(x) = 16x^4 + 3x^3 - 22x + 17$$

$$P(-14) = 16(-14)^4 + 3(-14)^3 - 22(-14) + 17$$

$$\begin{aligned} P(-14) &= 614\,656 - 8232 + 308 + 17 \\ &= 606\,749 \end{aligned}$$

$$\mathbf{b} \quad P(x) = 16x^4 + 3x^3 - 22x + 17$$

$$P(y+7) = 16(y+7)^4 + 3(y+7)^3 - 22(y+7) + 17$$

$$\begin{array}{rcll} 16(y+7)^4 & = & 16y^4 + 448y^3 + 4704y^2 + 21\,952y + 38\,416 \\ 3(y+7)^3 & = & 3y^3 + 63y^2 + 441y + 1029 \\ -22(y+7) & = & -22y - 154 \\ 17 & = & 17 \end{array}$$

$$P(y+7) = 16y^4 + 451y^3 + 4767y^2 + 22\,371y + 39\,308$$

Exercise 3C Polynomial values

1 **WE7** If $P(x) = 2x^3 - 3x^2 + 2x + 10$, determine the following.

a $P(0)$

d $P(3)$

g $P(-3)$

j $P(x+2)$

b $P(1)$

e $P(-1)$

h $P(a)$

k $P(x-3)$

c $P(2)$

f $P(-2)$

i $P(2b)$

l $P(-4y)$

2 **WE8** Determine the following considering $P(x) = 16x^4 + 3x^3 - 22x + 17$.

a $P(-11)$

c $P(2x+9)$

b $P(102)$

d $P(x^3+2)$

- 3 Copy the following table.

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8	Column 9
$P(x)$	$P(1)$	$P(2)$	$P(-1)$	$P(-2)$	Rem. when divided by $(x-1)$	Rem. when divided by $(x-2)$	Rem. when divided by $(x+1)$	Rem. when divided by $(x+2)$
a								
b								
c								
d								

Complete columns 2 to 5 of the table for each of the following polynomials.

- a $P(x) = x^3 + x^2 + x + 1$ b $P(x) = x^3 + 2x^2 + 5x + 2$
 c $P(x) = x^3 - x^2 + 4x - 1$ d $P(x) = x^3 - 4x^2 - 7x + 3$

- 4 Find the remainder when each polynomial in question 2 is divided by $(x-1)$ and complete column 6 of the table.
 5 Find the remainder when each polynomial in question 2 is divided by $(x-2)$ and complete column 7 of the table.
 6 Find the remainder when each polynomial in question 2 is divided by $(x+1)$ and complete column 8 of the table.
 7 Find the remainder when each polynomial in question 2 is divided by $(x+2)$ and complete column 9 of the table.
 8 Copy and complete the following sentences, using your answers to questions 3 to 7 to find the pattern.
 a A quick way of finding the remainder when $P(x)$ is divided by $(x+8)$ is to calculate _____.
 b A quick way of finding the remainder when $P(x)$ is divided by $(x-7)$ is to calculate _____.
 c A quick way of finding the remainder when $P(x)$ is divided by $(x-a)$ is to calculate _____.

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3D The remainder and factor theorems

The remainder theorem

In the previous exercise, you may have noticed that:

The remainder when $P(x)$ is divided by $(x-a)$ is equal to $P(a)$.

That is, $R = P(a)$.

This is called the remainder theorem.

We could have derived this result as follows.

If 13 is divided by 4, the quotient is 3, and the remainder is 1. That is,

$$13 \div 4 = 3 + \frac{1}{4} \quad \text{and} \\ 13 = 4 \times 3 + 1.$$

Similarly, if $P(x) = x^3 + x^2 + x + 1$ is divided by $(x-2)$, the quotient is $x^2 + 3x + 7$ and the remainder is 15. That is,

$$(x^3 + x^2 + x + 1) \div (x-2) = x^2 + 3x + 7 + \frac{15}{x-2} \quad \text{and} \\ (x^3 + x^2 + x + 1) = (x^2 + 3x + 7)(x-2) + 15.$$

In general, if $P(x)$ is divided by $(x-a)$, the quotient is $Q(x)$, and the remainder is R , we can write

$$P(x) \div (x-a) = Q(x) + \frac{R}{(x-a)} \quad \text{and} \\ P(x) = (x-a)Q(x) + R.$$

Substituting $x = a$ into this last expression yields

$$\begin{aligned} P(a) &= (a - a)Q(x) + R \\ &= 0 \times Q(x) + R \\ &= R \quad \text{as before.} \end{aligned}$$

WORKED EXAMPLE 9

Without actually dividing, find the remainder when $x^3 - 7x^2 - 2x + 4$ is divided by:

- a** $x - 3$ **b** $x + 6$.

THINK

- a** 1 Name the polynomial.
2 The remainder when $P(x)$ is divided by $(x - 3)$ is equal to $P(3)$.

- b** The remainder when $P(x)$ is divided by $(x + 6)$ is equal to $P(-6)$.

WRITE

a Let $P(x) = x^3 - 7x^2 - 2x + 4$

$$\begin{aligned} R &= P(3) \\ &= 3^3 - 7(3)^2 - 2(3) + 4 \\ &= 27 - 7(9) - 6 + 4 \\ &= 27 - 63 - 6 + 4 \\ &= -38 \end{aligned}$$

b $R = P(-6)$

$$\begin{aligned} &= (-6)^3 - 7(-6)^2 - 2(-6) + 4 \\ &= -216 - 7(36) + 12 + 4 \\ &= -216 - 252 + 12 + 4 \\ &= -452 \end{aligned}$$

WORKED EXAMPLE 10

The remainder when $x^3 + kx^2 + x - 2$ is divided by $(x - 2)$ is equal to 20. Find the value of k .

THINK

- 1 Name the polynomial.
2 The remainder when $P(x)$ is divided by $(x - 2)$ is equal to $P(2)$.
3 We are given $R = 20$.
Put $8 + 4k = 20$.
4 Solve for k .

WRITE

Let $P(x) = x^3 + kx^2 + x - 2$.

$$\begin{aligned} R &= P(2) \\ &= 2^3 + k(2)^2 + 2 - 2 \\ &= 8 + 4k \end{aligned}$$

Since $R = 20$.

$$\begin{aligned} 8 + 4k &= 20 \\ 4k &= 12 \\ k &= 3 \end{aligned}$$

The factor theorem

The remainder when 12 is divided by 4 is zero, since 4 is a factor of 12.

Similarly, if the remainder (R) when $P(x)$ is divided by $(x - a)$ is zero, then $(x - a)$ must be a factor of $P(x)$.

Since $R = P(a)$, all we need to do is to find a value of a that makes $P(a) = 0$, and we can say that $(x - a)$ is a factor.

If $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

This is called the factor theorem.

Imagine $P(x)$ could be factorised as follows:

$P(x) = (x - a)Q(x)$, where $Q(x)$ is 'the other' factor of $P(x)$.

Then we have

$$\begin{aligned} P(a) &= (a - a)Q(a) \\ &= 0 \times Q(a) \\ &= 0. \end{aligned}$$

So if $P(a) = 0$, $(x - a)$ is a factor.

WORKED EXAMPLE 11

Apply the factor theorem to determine which of the following is a factor of $x^4 - 4x^3 - 43x^2 + 58x + 240$.

- a** $(x + 2)$ **b** $(x - 1)$

THINK

- a** 1 Name the polynomial.
2 To find the remainder when $P(x)$ is divided by $(x - a)$, find $P(a)$.
3 State the answer.
- b** 1 To find the remainder when $P(x)$ is divided by $(x - a)$, find $P(a)$.
2 State the answer.

WRITE

- a** Let $P(x) = x^4 - 4x^3 - 43x^2 + 58x + 240$

$$P(-2) = (-2)^4 - 4(-2)^3 - 43(-2)^2 + 58(-2) + 240$$

$$= 16 - 4(-8) - 43(4) - 116 + 240$$

$$= 16 + 32 - 172 - 116 + 240$$

$$= 0$$

As $P(-2) = 0$, the remainder when $P(x)$ is divided by $(x + 2)$ is zero; therefore, $(x + 2)$ is a factor.

b $P(1) = (1)^4 - 4(1)^3 - 43(1)^2 + 58(1) + 240$

$$= 1 - 4 - 43 + 58 + 240$$

$$= 252$$

As $P(1) = 252$, the remainder when $P(x)$ is divided by $(x - 1)$ is 252; therefore, $(x - 1)$ is not a factor.

Exercise 3D The remainder and factor theorems

- 1 **WE9** Without actually dividing, find the remainder when $x^3 + 3x^2 - 10x - 24$ is divided by:

- a** $x - 1$ **b** $x + 2$ **c** $x - 3$ **d** $x + 5$
e $x - 0$ **f** $x - k$ **g** $x + n$ **h** $x + 3c$.

- 2 Find the remainder when the first polynomial is divided by the second without performing long division.

- a** $x^3 + 2x^2 + 3x + 4$, $x - 3$ **b** $x^3 - 4x^2 + 2x - 1$, $x + 1$
c $x^3 + 3x^2 - 3x + 1$, $x + 2$ **d** $x^3 - x^2 - 4x - 5$, $x - 1$
e $2x^3 + 3x^2 + 6x + 3$, $x + 5$ **f** $-3x^3 - 2x^2 + x + 6$, $x + 1$
g $x^3 + x^2 + 8$, $x - 5$ **h** $x^3 - 3x^2 - 2$, $x - 2$
i $-x^3 + 8$, $x + 3$ **j** $x^3 + 2x^2$, $x - 7$

- 3 **a** **WE10** The remainder when $x^3 + kx + 1$ is divided by $(x + 2)$ is -19 . Find the value of k .
b The remainder when $x^3 + 2x^2 + mx + 5$ is divided by $(x - 2)$ is 27. Find the value of m .
c The remainder when $x^3 - 3x^2 + 2x + n$ is divided by $(x - 1)$ is 1. Find the value of n .
d The remainder when $ax^3 + 4x^2 - 2x + 1$ is divided by $(x - 3)$ is -23 . Find the value of a .
e The remainder when $x^3 - bx^2 - 2x + 1$ is divided by $(x + 1)$ is 0. Find the value of b .
f The remainder when $-4x^2 + 2x + 7$ is divided by $(x - c)$ is -5 . Find a possible whole number value of c .
g The remainder when $x^2 - 3x + 1$ is divided by $(x + d)$ is 11. Find the possible values of d .
h The remainder when $x^3 + ax^2 + bx + 1$ is divided by $(x - 5)$ is -14 . When the cubic polynomial is divided by $(x + 1)$, the remainder is -2 . Find a and b .
- 4 **WE11** Apply the factor theorem to determine which of the following are factors of $x^3 + 2x^2 - 11x - 12$.
a $(x - 1)$ **b** $(x - 3)$ **c** $(x + 1)$ **d** $(x + 2)$
- 5 **WE12** Prove that each of the following are linear factors of $x^3 + 4x^2 - 11x - 30$ by substituting values into the cubic function: $(x + 2)$, $(x - 3)$, $(x + 5)$.
- 6 Use the factor theorem to show that the first polynomial is exactly divisible by the second (that is, the second polynomial is a factor of the first).
a $x^3 + 5x^2 + 2x - 8$, $x - 1$ **b** $x^3 - 7x^2 - x + 7$, $x - 7$
c $x^3 - 7x^2 + 4x + 12$, $x - 2$ **d** $x^3 + 2x^2 - 9x - 18$, $x + 2$
e $x^3 + 3x^2 - 9x - 27$, $x + 3$ **f** $-x^3 + x^2 + 9x - 9$, $x - 1$
g $-2x^3 + 9x^2 - x - 12$, $x - 4$ **h** $3x^3 + 22x^2 + 37x + 10$, $x + 5$

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- 7 **MC** a When $x^3 + 2x^2 - 5x - 5$ is divided by $(x + 2)$, the remainder is:
 A -5 B -2 C 0 D 2 E 5
- b Which of the following is a factor of $2x^3 + 15x^2 + 22x - 15$?
 A $(x - 1)$ B $(x - 2)$ C $(x + 3)$ D $(x - 5)$ E $(x + 4)$
- c When $x^3 - 13x^2 + 48x - 36$ is divided by $(x - 1)$, the remainder is:
 A -3 B -2 C -1 D 0 E 1
- d Which of the following is a factor of $x^3 - 5x^2 - 22x + 56$?
 A $(x - 2)$ B $(x + 2)$ C $(x - 5)$ D $(x + 5)$ E $(x + 7)$
- 8 Find one factor of each of the following cubic polynomials.
 a $x^3 - 3x^2 + 3x - 1$ b $x^3 - 7x^2 + 16x - 12$
 c $x^3 + x^2 - 8x - 12$ d $x^3 + 3x^2 - 34x - 120$
- 9 Consider the polynomial $P(x) = 6x^3 + 7x^2 - x - 2$.
 a Determine: i $P(-1)$ ii $P\left(\frac{1}{2}\right)$ iii $P\left(\frac{-2}{3}\right)$
 b i Factorise $P(x)$ as the product of $(x + 1)$ and a quadratic factor.
 ii Further factorise so $P(x)$ is written as the product of three linear factors.
 c Explain how the other two linear factors relate to what you found in parts b and c.
 d Copy and complete the following: In general if $(ax + b)$ is a factor, then $P(\dots) = 0$.

3E Factorising polynomials

Using long division

Once one factor of a polynomial has been found (using the factor theorem as in the previous section), long division may be used to find other factors.

WORKED EXAMPLE 12

Use long division to factorise $x^3 - 19x + 30$.

THINK

- Name the polynomial.
 Note: There is no x^2 term, so include $0x^2$.
- Look at the last term in $P(x)$, which is 30. This suggests it is worth trying $P(5)$ or $P(-5)$. Try $P(-5)$. $P(-5) = 0$, so $(x + 5)$ is a factor.
- Divide $(x + 5)$ into $P(x)$ using long division to find a quadratic factor.
- Write $P(x)$ as a product of the two factors found so far.
- Factorise the second bracket if possible.

WRITE

$$\begin{aligned}
 P(x) &= x^3 - 19x + 30 \\
 P(x) &= x^3 + 0x^2 - 19x + 30 \\
 P(-5) &= (-5)^3 - 19 \times (-5) + 30 \\
 &= -125 + 95 + 30 \\
 &= 0 \\
 \text{Therefore } (x + 5) &\text{ is a factor.}
 \end{aligned}$$

$$\begin{array}{r}
 x^2 - 5x + 6 \\
 x + 5 \overline{) x^3 + 0x^2 - 19x + 30} \\
 \underline{-(x^3 + 5x^2)} \\
 -5x^2 - 19x \\
 \underline{-(-5x^2 - 25x)} \\
 6x + 30 \\
 \underline{-(6x + 30)} \\
 0
 \end{array}$$

$$\begin{aligned}
 P(x) &= (x + 5)(x^2 - 5x + 6) \\
 P(x) &= (x + 5)(x - 2)(x - 3)
 \end{aligned}$$

Note: In this example, $P(x)$ may have been factorised without long division by finding all three values of x that make $P(x) = 0$, and hence three factors, then checking that the three factors multiply to give $P(x)$.

Using short division

The process of long division can take a lot of time (and space). One short division method is shown here; it may take a little longer to understand, but it is quicker than long division once mastered.

Consider $P(x) = x^3 + 2x^2 - 13x + 10$. Using the factor theorem, we can find that $(x - 1)$ is a factor of $P(x)$. So, $P(x) = (x - 1)(ax^2 + bx + c)$.

Actually, we know more than this: as $P(x)$ begins with x^3 and ends with $+10$, we could write:

$$P(x) = (x - 1)(x^2 + bx - 10)$$

Imagine expanding this version of $P(x)$. Our x^2 terms give $-1x^2 + bx^2$.

Since $P(x) = x^3 + 2x^2 - 13x + 10$, we need $+2x^2$. That is, we need $-1x^2 + 3x^2$. To get this, the bx must be $3x$, as when x in the first bracket is multiplied by $3x$ in the second bracket, $+3x^2$ results. That is, we have deduced

$$P(x) = (x - 1)(x^2 + 3x - 10).$$

Factorising the second bracket gives

$$P(x) = (x - 1)(x + 5)(x - 2)$$

WORKED EXAMPLE 13

Factorise the following, using short division where possible.

a $x^3 - 5x^2 - 2x + 24$

b $x^4 + x^3 - 13x^2 - 25x - 12$

THINK

- a** 1 Name the polynomial.
- 2 Look for a value of x such that $P(x) = 0$. Try $P(-2)$.

 $P(-2)$ does equal 0, so $(x + 2)$ is a factor.
- 3 Write the original polynomial as the found factor multiplied by $ax^2 + bx + c$.

The first term in the brackets must be x^2 , and the last term must be 12.
- 4 Imagine the expansion of the expression in step 3. We have $2x^2$, and require $-5x^2$. We need an extra $-7x^2$. So $b = -7$.
- 5 Factorise the second bracket if possible.

- b** 1 Name the polynomial.
- 2 Look for a value of x such that $P(x) = 0$. Try $P(-1)$.
- 3 It is difficult to factorise a quartic using short division, so we will use long division here.
- 4 Name the cubic factor and try to find another factor using the factor theorem.
- 5 Factorise the cubic using short division.

WRITE

a Let $P(x) = x^3 - 5x^2 - 2x + 24$

$$\begin{aligned} P(-2) &= (-2)^3 - 5 \times (-2)^2 - 2 \times (-2) + 24 \\ &= -8 - 20 + 4 + 24 \\ &= -28 + 28 \\ &= 0 \end{aligned}$$

So $(x + 2)$ is a factor.

$$\begin{aligned} P(x) &= x^3 - 5x^2 - 2x + 24 \\ P(x) &= (x + 2)(ax^2 + bx + c) \\ &= (x + 2)(x^2 + bx + 12) \end{aligned}$$

$$\begin{aligned} 2x^2 + bx^2 &= -5x^2 \\ b &= -7 \\ P(x) &= (x + 2)(x^2 - 7x + 12) \\ P(x) &= (x + 2)(x - 3)(x - 4) \end{aligned}$$

b Let $P(x) = x^4 + x^3 - 13x^2 - 25x - 12$

$$\begin{aligned} P(-1) &= 0 \\ \text{So } (x + 1) &\text{ is a factor.} \end{aligned}$$

$$\begin{array}{r} x^3 + 0x^2 - 13x - 12 \\ x + 1 \overline{) x^4 + x^3 - 13x^2 - 25x - 12} \\ \underline{-(x^4 + x^3)} \\ 0 - 13x^2 - 25x \\ \underline{-(-13x^2 - 13x)} \\ -12x - 12 \\ \underline{-(-12x - 12)} \\ 0 \end{array}$$

$$\begin{aligned} \text{Let } Q(x) &= x^3 - 13x - 12. \\ Q(-3) &= 0 \\ \text{So } (x + 3) &\text{ is a factor.} \end{aligned}$$

$$\begin{aligned} Q(x) &= x^3 - 13x - 12 \\ &= (x + 3)(x^2 - 3x - 4) \end{aligned}$$





6 Factorise the quadratic if possible.

$$Q(x) = (x + 3)(x - 4)(x + 1)$$

7 Write the original polynomial in factorised form.

$$\begin{aligned} P(x) &= x^4 + x^3 - 13x^2 - 25x - 12 \\ &= (x + 1)(x + 3)(x - 4)(x + 1) \\ &= (x + 1)^2(x + 3)(x - 4) \end{aligned}$$



WORKED EXAMPLE 14

Factorise the expression $2x^3 - 9x^2 - 2x + 24$.

THINK

- 1 Name the polynomial.
- 2 Look for values of x such that $P(x) = 0$.
- 3 Write $P(x)$ as the product of $x - 2$ and an unknown quadratic.
- 4 Consider the x^3 term ($2x^3$). This must equal ax^3 .
- 5 Consider the constant term (24). This must equal $-2c$.
- 6 Rewrite $P(x)$.
- 7 Consider the x^2 term from step 6. This must equal $-9x^2$ from the original cubic.
- 8 Consider the x term from step 6. This must equal $-2x$ from the original cubic. This confirms step 7.
- 9 Write $P(x)$.
- 10 Factorise the quadratic term.

WRITE

$$\begin{aligned} P(x) &= 2x^3 - 9x^2 - 2x + 24 \\ x = 1: P(1) &= 2 - 9 - 2 + 24 \neq 0 \\ x = 2: P(2) &= 16 - 36 - 8 + 24 = 0 \\ \text{So } x - 2 &\text{ is a factor.} \\ P(x) &= (x - 2)(ax^2 + bx + c) \\ (x)ax^2 &= ax^3 \\ &= 2x^3 \\ \text{Therefore } a &= 2. \\ (-2)c &= -2c \\ &= 2c \\ \text{Therefore } c &= -12. \\ P(x) &= (x - 2)(2x^2 + bx - 12) \\ -4x^2 + bx^2 &= -9x^2 \\ bx^2 &= -5x^2 \\ \text{Therefore } b &= -5. \\ -2bx - 12x &= -22 \\ -2bx &= 10x \\ b &= -5 \\ P(x) &= (x - 2)(2x^2 - 5x - 12) \\ P(x) &= (x - 2)(2x + 3)(x - 4) \end{aligned}$$

Exercise 3E Factorising polynomials

1 **WE12** Use long division to factorise each dividend.

a $x + 1 \overline{)x^3 + 10x^2 + 27x + 18}$

b $x + 2 \overline{)x^3 + 8x^2 + 17x + 10}$

c $x + 9 \overline{)x^3 + 12x^2 + 29x + 18}$

d $x + 1 \overline{)x^3 + 8x^2 + 19x + 12}$

e $x + 3 \overline{)x^3 + 14x^2 + 61x + 84}$

f $x + 7 \overline{)x^3 + 12x^2 + 41x + 42}$

g $x + 2 \overline{)x^3 + 4x^2 + 5x + 2}$

h $x + 3 \overline{)x^3 + 7x^2 + 16x + 12}$

i $x + 5 \overline{)x^3 + 14x^2 + 65x + 100}$

j $x \overline{)x^3 + 13x^2 + 40x}$

k $x \overline{)x^3 + 7x^2 + 12x}$

l $x + 5 \overline{)x^3 + 10x^2 + 25x}$

m $x + 1 \overline{)x^3 + 6x^2 + 5x}$

n $x + 6 \overline{)x^3 + 6x^2}$

2 **WE12, 13** Factorise the following as fully as possible.

- a** $x^3 + x^2 - x - 1$ **b** $x^3 - 2x^2 - x + 2$ **c** $x^3 + 7x^2 + 11x + 5$
d $x^3 + x^2 - 8x - 12$ **e** $x^3 + 9x^2 + 24x + 16$ **f** $x^3 - 5x^2 - 4x + 20$
g $x^3 + 2x^2 - x - 2$ **h** $x^3 - 7x - 6$ **i** $x^3 + 3x^2 - 4$
j $x^3 + x^2 + x + 6$ **k** $x^3 + 8x^2 + 17x + 10$ **l** $x^3 + x^2 - 9x - 9$
m $x^3 - x^2 - 8x + 12$ **n** $x^3 + 9x^2 - 12x - 160$ **o** $x^4 + 4x^3 + 3x^2 - 4x - 4$
p $x^4 + 3x^3 - 6x^2 - 28x - 24$ **q** $x^4 + 6x^3 + 8x^2 - 6x - 9$ **r** $x^4 - 5x^3 - 17x^2 + 21x$

3 **WE14** Use a CAS calculator to factorise the following as fully as possible.

- a** $3x^3 - x^2 - 10x$ **b** $4x^3 + 2x^2 - 2x$ **c** $3x^3 - 6x^2 - 24x$
d $-2x^3 - 12x^2 - 18x$ **e** $6x^3 - 6x^2$ **f** $-x^3 - 7x^2 - 12x$
g $-x^3 - 3x^2 + x + 3$ **h** $-2x^3 + 10x^2 - 12x$ **i** $-6x^3 - 5x^2 + 12x - 4$
j $-5x^3 + 24x^2 - 36x + 16$ **k** $-x^5 - x^4 + 21x^3 + 49x^2 - 8x - 60$ **l** $24x^4 - 53x^3 - 71x^2 + 152x + 20$

4 Factorise the following as fully as possible.

- a** $2x^3 + 5x^2 - x - 6$ **b** $3x^3 + 14x^2 + 7x - 4$ **c** $3x^3 + 2x^2 - 12x - 8$
d $4x^3 + 35x^2 + 84x + 45$ **e** $5x^3 + 9x^2 + 3x - 1$ **f** $x^3 + x^2 + 4$
g $4x^3 + 16x^2 + 21x + 9$ **h** $6x^3 - 23x^2 + 26x - 8$ **i** $10x^3 + 19x^2 - 94x - 40$
j $7x^3 + 12x^2 - 60x + 16$ **k** $2x^4 - x^3 - 11x^2 - 11x - 3$ **l** $6x^4 + 11x^3 - 22x^2 - x + 6$

3F Sum and difference of two cubes

Two special cases of cubic polynomials, called ‘sum of cubes’ and ‘difference of cubes’, are discussed in this section. There are shortcuts for factorising such cubic expressions. Examples of each are shown in the table below.

Sum of cubes	Difference of cubes
$x^3 + 2^3$	$x^3 - 27$
$125 + 64b^3$	$\frac{x^3}{1000} - 81y^3$
$x^3y^3 + 1$	$w^6 - 1$
$(2x + 1)^3 + 8$	$216 - (uv)^3$

Consider the following expansions.

$$\begin{aligned}
 (a+b)(a^2-ab+b^2) &= a^3 - a^2b + ab^2 + ba^2 - ab^2 + b^3 \\
 &= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3 \\
 &= a^3 + b^3
 \end{aligned}
 \quad \text{and} \quad
 \begin{aligned}
 (a-b)(a^2+ab+b^2) &= a^3 + a^2b + ab^2 - ba^2 - ab^2 - b^3 \\
 &= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3 \\
 &= a^3 - b^3
 \end{aligned}$$

These expansions show that:

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2) \quad \text{and} \quad a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

That is, we have two formulas that may be used to factorise sums and differences of cubes.

WORKED EXAMPLE 15

Factorise the following using the sum or difference of cubes formula.

a $x^3 - 1000y^3$ **b** $2(x+6)^3 + 16$

THINK

- 1 Write the expression.
- 2 Recognise a difference of cubes.
- 3 Identify a and b for use with the formula
 $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.
- 4 Use the formula to factorise.
- 5 Simplify.

WRITE

$$\begin{aligned}
 \text{a } x^3 - 1000y^3 &= x^3 - (10y)^3 \\
 &= (x - 10y)[x^2 + x(10y) + (10y)^2] \\
 &= (x - 10y)(x^2 + 10xy + 100y^2)
 \end{aligned}$$

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Worked example 15



b 1 Write the expression.

2 Take out a common factor of 2 to produce a sum of cubes.

3 Identify a and b for use with the formula
 $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

4 Apply the sum of cubes formula.

5 Simplify.

$$\mathbf{b} \ 2(x + 6)^3 + 16$$

$$= 2[(x + 6)^3 + 8]$$

$$= 2[(x + 6)^3 + 2^3]$$

$$a = (x + 6), b = 2$$

$$= 2[(x + 6) + 2][(x + 6)^2 - (x + 6)(2) + 2^2]$$

$$= 2(x + 8)(x^2 + 12x + 36 - 2x - 12 + 4)$$

$$= 2(x + 8)(x^2 + 10x + 28)$$

Exercise 3F Sum and difference of two cubes

1 Identify a and b (as used in the above sum and difference of cubes expressions) in each of the following (do not factorise).

a $x^3 + 6^3$

b $8y^3 + z^3$

c $(x + 5)^3 + 27$

d $1 - 64h^3$

e $\frac{1}{8}c^3 - e^3g^3$

f $t^3 - \frac{u^3}{216}$

2 **WE15a** Factorise the following using the sum or difference of cubes formula.

a $x^3 - 125$

b $j^3 + k^3$

c $y^3 - 8$

d $27x^3 + y^3$

e $64t^3 - 216u^3$

f $x^3 - 1$

g $\frac{x^3}{729} + 8p^3$

h $27r^3 - 1$

i $(3k)^3 - \frac{1}{8}$

j $s^3t^3 + g^6$

3 **WE15b** Factorise the following.

a $(a - 1)^3 + a^3$

b $(x + 2)^3 - 8$

c $(2x + 3)^3 + 1$

d $(w - 5)^3 - w^3$

e $(2m + p)^3 + (3m - p)^3$

f $27x^3 - (x + 3)^3$

g $(2y + 7)^3 + (y - 2)^3$

h $(3x + y)^3 + (x - 4y)^3$

i $(2 - 4p)^3 - (p + 1)^3$

j $(5x - 9)^3 - (7 - x)^3$

k $x^6 + y^9$

l $2x^3 - 54$

m $3a^3 + 3$

n $6(x^2 + 1)^3 + 162$

4 When $mx^3 - ny^3$ is fully factorised it gives $(3x - y)(9x^2 + 3xy + y^2)$. What are the values of m and n ?

5 **a** Write $3x^3 + my^3$ in the form $a^3 + b^3$.

b Identify the values of a and b .

c Factorise using the rule for sum of cubes.

3G Solving polynomial equations

Making x the subject of the equation

Cubic equations of the form $a(x - b)^3 + c = 0$ may be solved by isolating x as follows.

$$a(x - b)^3 = -c$$

$$(x - b)^3 = \frac{-c}{a}$$

$$x - b = \sqrt[3]{\frac{-c}{a}}$$

$$x = b + \sqrt[3]{\frac{-c}{a}}$$

Unlike a square root, a cube root can be only positive or negative, not both; for example,
 $\sqrt[3]{-8} = -2, \sqrt[3]{8} = 2$.

WORKED EXAMPLE 16

Solve $3(x+2)^3 + 192 = 0$ by making x the subject.

THINK

- 1 Write the equation.
- 2 Subtract 192 from both sides.
- 3 Divide both sides by 3.
- 4 Take the cube root of both sides.
- 5 Subtract 2 from both sides and simplify.

WRITE

$$\begin{aligned}
 3(x+2)^3 + 192 &= 0 \\
 3(x+2)^3 &= -192 \\
 (x+2)^3 &= -64 \\
 x+2 &= -4 \\
 x &= -4 - 2 \\
 &= -6
 \end{aligned}$$

Factorising to solve polynomial equations

The Null Factor Law applies to cubic and quartic equations just as it does for quadratics.

If $P(x) = (x-a)(x-b)(x-c) = 0$, then the solutions are $x = a$, $x = b$ and $x = c$.

If $P(x) = k(lx-a)(mx-b)(nx-c) = 0$, solutions are found by solving the following equations:

$$lx - a = 0, mx - b = 0 \text{ and } nx - c = 0$$

WORKED EXAMPLE 17

Solve each of the following equations.

a $x^4 = 16x^2$ **b** $2x^3 - 11x^2 + 18x - 9 = 0$

THINK

- a**
- 1 Write the equation.
 - 2 Rearrange so that all terms are on the left.
 - 3 Take out a common factor of x .
 - 4 Factorise the brackets using a difference of squares.
 - 5 Use the Null Factor Law to solve.
- b**
- 1 Name the polynomial.
 - 2 Use the factor theorem to find a factor (search for a value a such that $P(a) = 0$). Consider factors of the constant term (that is, factors of 9 such as 1, 3). The simplest value to try is 1.
 - 3 Use long or short division to find the other factors of $P(x)$.
 - 4 Factorise the brackets.
 - 5 Consider the factorised equation to solve.
 - 6 Use the Null Factor Law to solve.

WRITE

a $x^4 = 16x^2$

$$\begin{aligned}
 x^4 - 16x^2 &= 0 \\
 x^2(x^2 - 16) &= 0 \\
 x^2(x+4)(x-4) &= 0 \\
 x = 0, x+4 = 0 \text{ or } x-4 = 0 \\
 \text{so } x = 0, x = -4 \text{ or } x = 4
 \end{aligned}$$

b Let $P(x) = 2x^3 - 11x^2 + 18x - 9$.

$$\begin{aligned}
 P(1) &= 2 - 11 + 18 - 9 \\
 &= 0
 \end{aligned}$$

So $(x-1)$ is a factor.

$$\begin{array}{r}
 2x^2 - 9x + 9 \\
 x-1 \overline{) 2x^3 - 11x^2 - 18x - 9} \\
 \underline{2x^3 - 2x^2} \\
 -9x^2 + 18x \\
 \underline{-9x^2 + 9x} \\
 9x - 9 \\
 \underline{9x - 9} \\
 0
 \end{array}$$

$$\begin{aligned}
 P(x) &= (x-1)(2x^2 - 9x + 9) \\
 P(x) &= (x-1)(2x-3)(x-3) \\
 \text{For } (x-1)(2x-3)(x-3) &= 0 \\
 x-1 = 0, 2x-3 = 0 \text{ or } x-3 = 0 \\
 \text{so } x = 1, x = \frac{3}{2} \text{ or } x = 3
 \end{aligned}$$

Polynomial equations can also be solved using a CAS calculator, which is useful when the solutions are not rational. The solutions can be found by using the solve function on a calculator page, or by constructing a graph.



WORKED EXAMPLE 18

Use a CAS calculator to solve $x^3 + 3x^2 - 8x - 5 = 0$.

THINK

- 1 Use the CAS calculator to solve the equation.
- 2 Read the results supplied by the calculator.
- 3 Write the answer.

WRITE

Enter
solve ($x^3 + 3x^2 - 8x - 5 = 0$, x)
into the CAS calculator.

$x = -4.524\ 04$ or
 $x = -0.536\ 396$ or
 $x = 2.060\ 43$

The solutions of $x^3 + 3x^2 - 8x - 5 = 0$ are
 $x = -4.52$, -0.54 and 2.06 , rounded to 2 decimal places.

Exercise 3G Solving polynomial equations

- 1 **WE16** Solve the following equations by making x the subject.

- | | |
|--|---|
| a $2(x-1)^3 - 250 = 0$ | b $3(x+2)^3 + 81 = 0$ |
| c $(x-4)^3 - 1000 = 0$ | d $(x+7)^3 - 8 = 0$ |
| e $-2(x-5)^3 - 2 = 0$ | f $-(x+3)^3 + 1 = 0$ |
| g $(2x+3)^3 - 27 = 0$ | h $4(3x-1)^3 + 500 = 0$ |
| i $\frac{2}{3}(x+1)^3 + 18 = 0$ | j $\frac{1}{2}(5-x)^3 - 32 = 0$ |
| k $(x-5)^3 = 343$ | l $4 - \frac{4}{5}(x+8)^3 = 104$ |

- 2 Find all solutions of the following equations.

- | | |
|-------------------------------------|--------------------------------------|
| a $(x-1)(x-2)(x-5) = 0$ | b $(x+3)(x+4)(x+7) = 0$ |
| c $(x-5)(x+2)(x-9)(x-1) = 0$ | d $(2x-4)(x+1)(x-3)(x+2) = 0$ |
| e $(3x+12)(x-4)(x+4) = 0$ | f $(2-x)(x+2)(1-x)(1+x) = 0$ |
| g $(x+5)(x-8)^2 = 0$ | h $(x-1)^3 = 0$ |
| i $x^2(x+1)(x-1) = 0$ | j $-3x(x-9)^3 = 0$ |
| k $(6-x)^2(2+x) = 0$ | l $x^2(2x+7) = 0$ |
| m $x(5x-6)(2x+3)(6-7x) = 0$ | n $(3-4x)^2(5x-1) = 0$ |

- 3 **WE17a** Solve the following equations.

- | | | |
|----------------------------------|---------------------------------|-----------------------------|
| a $x^3 - 4x = 0$ | b $x^3 - 16x = 0$ | c $2x^4 - 50x^2 = 0$ |
| d $-3x^4 + 81 = 0$ | e $x^3 + 5x^2 = 0$ | f $x^3 - 2x^2 = 0$ |
| g $-4x^3 + 8x = 0$ | h $12x^3 + 3x^2 = 0$ | i $4x^3 - 20x^4 = 0$ |
| j $x^4 - 5x^3 + 6x^2 = 0$ | k $x^3 - 8x^2 + 16x = 0$ | l $x^3 + 6x^2 = 7x$ |
| m $9x^2 = 20x + x^3$ | n $x^3 + 6x = 4x^2$ | |

- 4 **WE17b** Use the Null Factor Law to solve the following equations.

- | | |
|--|---------------------------------------|
| a $x^3 - x^2 - 16x + 16 = 0$ | b $x^3 - 6x^2 - x + 30 = 0$ |
| c $x^3 - x^2 - 25x + 25 = 0$ | d $x^3 + 4x^2 - 4x - 16 = 0$ |
| e $x^3 - 4x^2 + x + 6 = 0$ | f $x^3 - 4x^2 - 7x + 10 = 0$ |
| g $x^4 - 3x^3 - 7x^2 + 15x = -18$ | h $x^4 + 2x^3 - 13x^2 + 10x$ |
| i $2x^3 + 15x^2 + 19x + 6 = 0$ | j $-4x^3 + 16x^2 - 9x - 9 = 0$ |
| k $-2x^3 - 9x^2 - 7x + 6 = 0$ | l $2x^3 + 4x^2 - 2x - 4 = 0$ |

- 5 **WE18** Use a CAS calculator to find all solutions to the following equations.

- | | |
|--|--|
| a $x^3 - 17x^2 - 56x + 1153 = 0$ | b $x^3 + 12x^2 - 49x - 588 = 0$ |
| c $-x^3 + 17x^2 + 65x - 1521 = 0$ | d $x^3 - 48x^2 + 768x - 4096 = 0$ |

e $x^3 + 6x^2 - 6x + 2 = 0$

g $3x^2 + 2x + 1 = x^3$

i $x^4 + 4x^3 - 5x^2 + 15 = 0$

f $x^3 - 14x^2 - 4x + 13 = 0$

h $-2x^3 - 3x^2 + 2x + 0.5 = 0$

j $2x^4 + 25x^3 + 17x = 9$

6 MC Which of the following is a solution to $x^3 - 7x^2 + 2x + 40 = 0$?

A -5

B -4

C -2

D 1

E 2

7 MC A solution of $x^3 - 9x^2 + 15x + 25 = 0$ is $x = 5$. How many other (distinct) solutions are there?

A 0

B 1

C 2

D 3

E 4

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DIGITAL DOC

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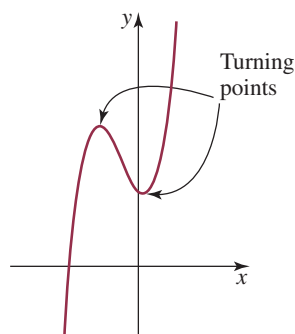
WorkSHEET 3.2

3H Cubic graphs — intercepts method

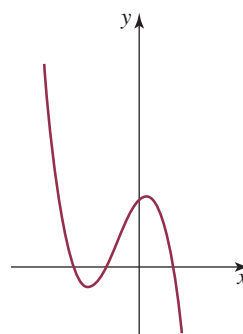
A good sketch graph of a function shows:

1. x - and y -intercepts
2. the behaviour of the function at extreme values of x , that is, as x approaches infinity ($x \rightarrow +\infty$) and as x approaches negative infinity ($x \rightarrow -\infty$)
3. the general location of turning points.

The graphs below show the two main types of cubic graph.



A positive cubic



A negative cubic

For positive cubic graphs, as positive values of x become larger and larger, y -values also become larger.

For negative cubic graphs, as positive values of x become larger and larger, y -values become smaller.

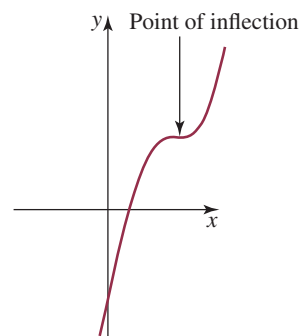
Sometimes, instead of two turning points, there is a **point of inflection**, where the graph changes from a decreasing gradient to zero to an increasing gradient (or vice versa). When this occurs, there is only one x -intercept.

Consider the general factorised cubic $f(x) = (x - a)(x - b)(x - c)$.

The Null Factor Law tells us that $f(x) = 0$ when $x = a$ or $x = b$ or $x = c$.

The y -intercept occurs when $x = 0$; that is, the y -intercept is

$$\begin{aligned} f(0) &= (0 - a)(0 - b)(0 - c) \\ &= -abc \end{aligned}$$



Cubic with a point of inflection

