

Exercise 4C

1 Find the modulus and Argument of each of the following complex numbers:

a -3

b $5i$

c $i - 1$

d $\sqrt{3} + i$

e $2 - 2\sqrt{3}i$

f $(2 - 2\sqrt{3}i)^2$

2 Find the Argument of each of the following, correct to two decimal places:

a $5 + 12i$

b $-8 + 15i$

c $-4 - 3i$

d $1 - i\sqrt{2}$

e $\sqrt{2} + i\sqrt{3}$

f $-(3 + 7i)$

3 Find the argument of each of the following in the interval stated:

a $1 - i\sqrt{3}$ in $[0, 2\pi]$

b $-7i$ in $[0, 2\pi]$

c $-3 + i\sqrt{3}$ in $[0, 2\pi]$

d $\sqrt{2} + i\sqrt{2}$ in $[0, 2\pi]$

e $\sqrt{3} + i$ in $[-2\pi, 0]$

f $2i$ in $[-2\pi, 0]$

4 Convert each of the following arguments into Arguments:

a $\frac{5\pi}{4}$

b $\frac{17\pi}{6}$

c $\frac{-15\pi}{8}$

d $\frac{-5\pi}{2}$

5 Convert the following complex numbers from the cartesian form $a + bi$ into the form $r \operatorname{cis} \theta$, where $\theta = \operatorname{Arg}(a + bi)$:

a $-1 - i$

b $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

c $\sqrt{3} - i\sqrt{3}$

d $\frac{1}{\sqrt{3}} + \frac{1}{3}i$

e $\sqrt{6} - i\sqrt{2}$

f $-2\sqrt{3} + 2i$

6 Convert each of the following complex numbers to the form $a + bi$:

a $2 \operatorname{cis} \frac{3\pi}{4}$

b $5 \operatorname{cis} \frac{-\pi}{3}$

c $2\sqrt{2} \operatorname{cis} \frac{\pi}{4}$

d $3 \operatorname{cis} \frac{-5\pi}{6}$

e $6 \operatorname{cis} \frac{\pi}{2}$

f $4 \operatorname{cis} \pi$

7 Let $z = \operatorname{cis} \theta$. Show that:

a $|z| = 1$

b $\frac{1}{z} = \operatorname{cis}(-\theta)$

8 Find the conjugate of each of the following:

a $2 \operatorname{cis} \left(\frac{3\pi}{4} \right)$

b $7 \operatorname{cis} \left(\frac{-2\pi}{3} \right)$

c $-3 \operatorname{cis} \left(\frac{2\pi}{3} \right)$

d $5 \operatorname{cis} \left(\frac{-\pi}{4} \right)$

Exercise 4D

1 Simplify each of the following:

$$\begin{array}{lll}
 \text{a} & 4 \operatorname{cis} \frac{2\pi}{3} \times 3 \operatorname{cis} \frac{3\pi}{4} & \text{b} \quad \frac{\sqrt{2} \operatorname{cis} \frac{\pi}{2}}{\sqrt{8} \operatorname{cis} \frac{5\pi}{6}} & \text{c} \quad \frac{1}{2} \operatorname{cis} \frac{-2\pi}{5} \times \frac{7}{3} \operatorname{cis} \frac{\pi}{3} \\
 \text{d} & \frac{4 \operatorname{cis} \frac{-\pi}{4}}{\frac{1}{2} \operatorname{cis} \frac{7\pi}{10}} & \text{e} \quad 2 \operatorname{cis} \frac{5\pi}{6} \times \left(\sqrt{2} \operatorname{cis} \frac{7\pi}{8} \right)^4 & \text{f} \quad \frac{1}{\left(\frac{3}{2} \operatorname{cis} \frac{5\pi}{8} \right)^3} \\
 \text{g} & \frac{4 \operatorname{cis} \frac{2\pi}{3}}{32 \operatorname{cis} \frac{-\pi}{3}} & \text{h} \quad \left(\operatorname{cis} \frac{\pi}{6} \right)^8 \times \left(\sqrt{3} \operatorname{cis} \frac{\pi}{4} \right)^6 & \text{i} \quad \left(\frac{1}{2} \operatorname{cis} \frac{\pi}{2} \right)^{-5} \\
 \text{j} & \left(2 \operatorname{cis} \frac{3\pi}{2} \times 3 \operatorname{cis} \frac{\pi}{6} \right)^3 & \text{k} \quad \left(\frac{1}{2} \operatorname{cis} \frac{\pi}{8} \right)^{-6} \times \left(4 \operatorname{cis} \frac{\pi}{3} \right)^2 & \text{l} \quad \frac{\left(6 \operatorname{cis} \frac{2\pi}{5} \right)^3}{\left(\frac{1}{2} \operatorname{cis} \frac{-\pi}{4} \right)^{-5}}
 \end{array}$$

2 For each of the following find $\operatorname{Arg}(z_1 z_2)$ and $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$ and comment on their relationship:

$$\begin{array}{ll}
 \text{a} & z_1 = \operatorname{cis} \frac{\pi}{4} \text{ and } z_2 = \operatorname{cis} \frac{\pi}{3} \\
 \text{b} & z_1 = \operatorname{cis} \frac{-2\pi}{3} \text{ and } z_2 = \operatorname{cis} \frac{-3\pi}{4} \\
 \text{c} & z_1 = \operatorname{cis} \frac{2\pi}{3} \text{ and } z_2 = \operatorname{cis} \frac{\pi}{2}
 \end{array}$$

3 Show that if $\frac{-\pi}{2} < \operatorname{Arg}(z_1) < \frac{\pi}{2}$ and $\frac{-\pi}{2} < \operatorname{Arg}(z_2) < \frac{\pi}{2}$ then $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$ and $\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2)$.

4 For $z = 1 + i$, find:

$$\begin{array}{lll}
 \text{a} & \operatorname{Arg} z & \text{b} \quad \operatorname{Arg} -z & \text{c} \quad \operatorname{Arg} \frac{1}{z}
 \end{array}$$

9 Simplify each of the following, giving your answer in the modulus–Argument form:

$$\begin{array}{lll}
 \text{a} & (1 + \sqrt{3}i)^6 & \text{b} \quad (1 - i)^{-5} & \text{c} \quad i(\sqrt{3} - i)^7 \\
 \text{d} & (-3 + \sqrt{3}i)^{-3} & \text{e} \quad \frac{(1 + \sqrt{3}i)^3}{i(1 - i)^5} & \text{f} \quad \frac{(-1 + \sqrt{3}i)^4(-\sqrt{2} - \sqrt{2}i)^3}{\sqrt{3} - 3i} \\
 \text{g} & (-1 + i)^5 \left(\frac{1}{2} \operatorname{cis} \frac{\pi}{4} \right)^3 & \text{h} \quad \frac{\left(\operatorname{cis} \frac{2\pi}{5} \right)^3}{(1 - \sqrt{3}i)^2} & \text{i} \quad \left((1 - i) \operatorname{cis} \frac{2\pi}{3} \right)^7
 \end{array}$$

Exercise 4C

- 1 a $3; \pi$ b $5; \frac{\pi}{2}$ c $\sqrt{2}; \frac{3\pi}{4}$
 d $2; \frac{\pi}{6}$ e $4; -\frac{\pi}{3}$ f $16; -\frac{2\pi}{3}$
- 2 a 1.18 b 2.06 c -2.50
 d -0.96 e 0.89 f -1.98
- 3 a $\frac{5\pi}{3}$ b $\frac{3\pi}{2}$ c $\frac{5\pi}{6}$
 d $\frac{\pi}{4}$ e $-\frac{11\pi}{6}$ f $-\frac{3\pi}{2}$
- 4 a $-\frac{3\pi}{4}$ b $\frac{5\pi}{6}$ c $\frac{\pi}{8}$ d $-\frac{\pi}{2}$
- 5 a $\sqrt{2} \operatorname{cis}\left(-\frac{3\pi}{4}\right)$ b $\operatorname{cis}\left(-\frac{\pi}{3}\right)$
 c $\sqrt{6} \operatorname{cis}\left(-\frac{\pi}{4}\right)$ d $\frac{2}{3} \operatorname{cis} \frac{\pi}{6}$
 e $2\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{6}\right)$ f $4 \operatorname{cis} \frac{5\pi}{6}$
- 6 a $-\sqrt{2} + \sqrt{2}i$ b $\frac{5}{2} - \frac{5\sqrt{3}}{2}i$
 c $2 + 2i$ d $\frac{3\sqrt{3}}{2} - \frac{3}{2}i$
 e $6i$ f -4
- 8 a $2 \operatorname{cis}\left(-\frac{3\pi}{4}\right)$ b $7 \operatorname{cis} \frac{2\pi}{3}$
 c $3 \operatorname{cis} \frac{\pi}{3}$ d $5 \operatorname{cis} \frac{\pi}{4}$

Exercise 4D

- 1 a $12 \operatorname{cis}\left(-\frac{7\pi}{12}\right)$ b $\frac{1}{2} \operatorname{cis}\left(-\frac{\pi}{3}\right)$
 c $\frac{7}{6} \operatorname{cis}\left(-\frac{\pi}{15}\right)$ d $8 \operatorname{cis}\left(-\frac{19\pi}{20}\right)$
 e $8 \operatorname{cis} \frac{\pi}{3}$ f $\frac{8}{27} \operatorname{cis} \frac{\pi}{8}$ g $-\frac{1}{8}$
 h $27 \operatorname{cis} \frac{5\pi}{6}$ i $-32i$ j -216
 k $1024 \operatorname{cis} -\frac{\pi}{12}$ l $\frac{27}{4} \operatorname{cis}\left(-\frac{\pi}{20}\right)$
- 2 a $\operatorname{Arg}(z_1 z_2) = \frac{7\pi}{12}$
 $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = \frac{7\pi}{12}$
 $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$
- b $\operatorname{Arg}(z_1 z_2) = \frac{7\pi}{12}$
 $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = \frac{-17\pi}{12}$
 $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) + 2\pi$
- c $\operatorname{Arg}(z_1 z_2) = \frac{-5\pi}{6}$
 $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = \frac{7\pi}{6}$
 $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) - 2\pi$
- 4 a $\frac{\pi}{4}$ b $-\frac{3\pi}{4}$ c $-\frac{\pi}{4}$
- 9 a 64 b $\frac{\sqrt{2}}{8} \operatorname{cis}\left(-\frac{3\pi}{4}\right)$
 c $128 \operatorname{cis}\left(\frac{-2\pi}{3}\right)$ d $\frac{-\sqrt{3}i}{72}$
 e $\sqrt{2} \operatorname{cis}\left(\frac{-\pi}{4}\right)$ f $\frac{64\sqrt{3}}{3} \operatorname{cis}\left(\frac{3\pi}{4}\right)$
 g $\frac{\sqrt{2}}{2}i$ h $\frac{1}{4} \operatorname{cis}\left(-\frac{2\pi}{15}\right)$
 i $8\sqrt{2} \operatorname{cis} \frac{11\pi}{12}$

Example

a Express z in polar form.

Solution

$$\begin{aligned} r &= |z| = \sqrt{1^2 + (-1)^2} = \sqrt{2} \\ \theta &= \text{Arg}z = \text{Tan}^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4} \\ \therefore z &= 1 - i = \sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right) \end{aligned}$$

b Working with z in polar form, find z^8 .

$$\begin{aligned} (1 - i)^8 &= \left(\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)\right)^8 \\ &= 8 \times \sqrt{2}\text{cis}\left(8 \times -\frac{\pi}{4}\right) \\ &= 8\sqrt{2}\text{cis}(-2\pi) \\ &= 8\sqrt{2}\text{cis}(0) \end{aligned}$$

c Express z^8 in Cartesian form.

$$\begin{aligned} &8\sqrt{2}\text{cis}(0) \\ &= 8\sqrt{2}\cos(0) + 8\sqrt{2}i\sin(0) \\ &= 8\sqrt{2} + 0i \\ &= 8\sqrt{2} \end{aligned}$$

Exercise 12I

1 If $z_1 = 4\text{cis}\left(\frac{\pi}{3}\right)$ and $z_2 = 2\text{cis}\left(\frac{\pi}{6}\right)$ express the following in simplest polar form:

a $3z_1$

b $2z_2$

c z_1z_2

d $\frac{z_1}{z_2}$

e $\frac{z_2}{z_1}$

2 Express each of the following in:

i simplest polar form

ii Cartesian form

a $(\cos \pi + i \sin \pi)(\cos 2\pi + i \sin 2\pi)$

b $\left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right)\left(\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right)\right)$

c $\left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right)\left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right)$

d $(\cos 20^\circ + i \sin 20^\circ)(\cos 70^\circ + i \sin 70^\circ)$

e $(\cos 120^\circ + i \sin 120^\circ)(\cos 60^\circ + i \sin 60^\circ)$

f $(\cos 170^\circ + i \sin 170^\circ)(\cos 100^\circ + i \sin 100^\circ)$

g $4\text{cis}\left(\frac{2\pi}{3}\right) \times 3\text{cis}\left(\frac{\pi}{6}\right)$

h $\text{cis}\left(\frac{\pi}{4}\right) \times \text{cis}\left(\frac{3\pi}{4}\right)$

i $4\text{cis}\left(-\frac{\pi}{6}\right) \times \left(-3\text{cis}\left(\frac{\pi}{3}\right)\right)$

j $\left(\text{cis}\left(\frac{5\pi}{4}\right)\right)\left(-\text{cis}\left(-\frac{\pi}{4}\right)\right)$

3 Express each of the following in:

i simplest polar form

ii Cartesian form

a $\frac{(\cos \pi + i \sin \pi)}{(\cos(\frac{\pi}{2}) + i \sin(\frac{\pi}{2}))}$

b $\frac{(\cos(\frac{\pi}{3}) + i \sin(\frac{\pi}{3}))}{(\cos(\frac{\pi}{6}) + i \sin(\frac{\pi}{6}))}$

c $\frac{(\cos 60^\circ + i \sin 60^\circ)}{(\cos 30^\circ + i \sin 30^\circ)}$

d $\frac{(\cos -30^\circ + i \sin -30^\circ)}{(\cos 60^\circ + i \sin 60^\circ)}$

e $\frac{2 \operatorname{cis} 120^\circ}{4 \operatorname{cis} 30^\circ}$

f $\frac{2 \operatorname{cis} 80^\circ}{\sqrt{3} \operatorname{cis} 20^\circ}$

g $\frac{5 \operatorname{cis}(\frac{\pi}{2})}{15 \operatorname{cis} 2\pi}$

h $\frac{3 \operatorname{cis}(\frac{3\pi}{4})}{\operatorname{cis}(\frac{\pi}{4})}$

i $\frac{3 \operatorname{cis}(\frac{\pi}{6})}{\operatorname{cis}(\frac{\pi}{12})}$

j $\frac{4 \operatorname{cis} \frac{\pi}{3}}{2 \operatorname{cis} \frac{\pi}{6}}$

k $\frac{8 \operatorname{cis}(\frac{\pi}{4})}{\sqrt{2} \operatorname{cis}(-\frac{\pi}{12})}$

l $\frac{6 \operatorname{cis}(-\frac{5\pi}{4})}{2 \operatorname{cis}(\frac{\pi}{4})}$

4 Express each of the following in:

i simplest polar form

ii Cartesian form

a $(\cos \pi + i \sin \pi)^6$

b $(\cos 2\pi + i \sin 2\pi)^8$

c $\left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right)^8$

d $\left(\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right)\right)^{12}$

e $\left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right)^9$

f $\left(2 \operatorname{cis} \frac{\pi}{12}\right)^6$

g $\left(3 \operatorname{cis}\left(-\frac{\pi}{12}\right)\right)^8$

h $\left(-2 \operatorname{cis}\left(\frac{\pi}{3}\right)\right)^9$

i $(2 \operatorname{cis} 120^\circ)^6$

j $(4 \operatorname{cis} 60^\circ)^4$

k $(-2 \operatorname{cis} 30^\circ)^4$

l $(-5 \operatorname{cis} 60^\circ)^5$

5 a Express the following complex numbers in polar form:

i $z_1 = 1 + i$

ii $z_2 = -1 + i$

iii $z_3 = -1 - i$

b Use polar algebra to express the following in polar form:

i $z_1 z_2$

ii $z_2 z_3$

iii $z_1 z_3$

iv $\frac{z_1}{z_2}$

v $\frac{z_2}{z_3}$

vi $\frac{z_1}{z_3}$

vii z_1^8

viii z_2^6

ix z_3^{10}

x $(z_1 z_2)^{12}$

c Express each of the answers to part b in Cartesian form.





Multiple choice

- 1 The factorised form of the expression $z^2 + 4z + 20$ is:
A $(z + 2 + 4i)(z + 2 - 4i)$ **B** $(z - 2 + 4i)(z - 2 - 4i)$ **C** $(z + 1 + 2i)(z + 1 - 2i)$
D $(z - 1 + 2i)(z - 1 - 2i)$ **E** $(z - 2 + 4i)(z + 2 - 4i)$
- 2 For $z = \sqrt{2}(1 - \sqrt{2}i)$, the $\text{Re}(z)$ and $\text{Im}(z)$ values respectively are:
A $\sqrt{2}, 2$ **B** $-\sqrt{2}, 2$ **C** $\sqrt{2}, \sqrt{2}$ **D** $\sqrt{2}, -2$ **E** $2, -\sqrt{2}$
- 3 The polynomial whose factored form is $(z + 1 + 2i)(z + 1 - 2i)$ is:
A $z^2 - 4z - 13$ **B** $z^2 - 4z + 13$ **C** $z^2 + 4z - 13$
D $z^2 + 2z + 5$ **E** $-z^2 + 2z - 5$

Questions 4 to 7 use the complex numbers $z_1 = 3 + 4i$ and $z_2 = -1 + 5i$.

- 4 $z_1 + z_2 =$
A $8 + 3i$ **B** $3 + 8i$ **C** $-2 + 9i$ **D** $2 + 9i$ **E** $-2 - 9i$
- 5 $z_1 \times z_2 =$
A $23 + 11i$ **B** $-23 + 11i$ **C** $23 - 11i$ **D** $-17 + 11i$ **E** $17 + 11i$
- 6 $2\bar{z}_1 + z_2 =$
A $5 + 3i$ **B** $5 - 3i$ **C** $2 + i$ **D** $2 + 9i$ **E** $-5 + 3i$
- 7 $\frac{1}{4z_1} =$
A $\frac{3 - 4i}{28}$ **B** $\frac{3 + 4i}{28}$ **C** $\frac{3 - 4i}{100}$ **D** $\frac{3 + 4i}{100}$ **E** $\frac{3 + 4i}{25}$
- 8 The complex number $z = 3 - 4i$ expressed in polar form has $|z|$ and $\text{Arg } z$ respectively:
A $5, \tan^{-1}\left(-\frac{3}{4}\right)$ **B** $5, \tan^{-1}\left(-\frac{4}{5}\right)$ **C** $5, \tan^{-1}\left(\frac{4}{3}\right)$
D $5, \tan^{-1}\left(-\frac{4}{3}\right)$ **E** $5, \tan^{-1}\left(-\frac{3}{5}\right)$
- 9 The complex number $z = 5\text{cis}\left(-\frac{\pi}{4}\right)$ expressed in Cartesian form is:
A $2\frac{1}{2}(1 - i)$ **B** $2\frac{1}{2}(1 + i)$ **C** $-2\frac{1}{2}(1 + i)$ **D** $2\frac{1}{4}(1 + i)$ **E** $2\frac{1}{4}(1 - i)$

- 10 If $z_1 = 2\text{cis}\left(\frac{\pi}{3}\right)$ and $z_2 = 3\text{cis}\left(\frac{5\pi}{6}\right)$ then $z_1 z_2$ expressed using the principal argument is:
- A $6\text{cis}\left(\frac{7\pi}{6}\right)$ B $6\text{cis}\left(-\frac{7\pi}{6}\right)$ C $6\text{cis}\left(\frac{5\pi}{6}\right)$ D $6\text{cis}\left(-\frac{5\pi}{6}\right)$ E $6\text{cis}\left(-\frac{\pi}{6}\right)$
- 11 The complex number $z = 3\sqrt{2}(\cos 135^\circ + i \sin 135^\circ)$ expressed in Cartesian form is:
- A $-3(1 + i)$ B $4(1 + i)$ C $4(-1 + i)$ D $3(1 + i)$ E $3(-1 + i)$
- 12 The complex number $z = 1 - \sqrt{3}i$ expressed in polar form is:
- A $2\text{cis}\left(-\frac{2\pi}{3}\right)$ B $2\text{cis}\left(-\frac{\pi}{3}\right)$ C $2\text{cis}\left(-\frac{\pi}{6}\right)$ D $2\text{cis}\left(\frac{\pi}{3}\right)$ E $2\text{cis}\left(\frac{\pi}{6}\right)$

Short answer

- 1 Expand and simplify the following, expressing each in the form $x + yi$:
- a i^5 b i^{12} c i^{23} d $i^3(2 + 3i)$ e $-4i^4(1 - 2i)$
- 2 If $z_1 = 2 + 3i$, $z_2 = 1 - i$ and $z_3 = 5i$ express the following in the form $x + yi$:
- a $z_1 + z_2$ b $z_2 + z_3$ c $z_3(2 + 4i + z_1)$
- 3 Express the following in Cartesian form:
- a $4\text{cis}\left(-\frac{\pi}{3}\right)$ b $2\text{cis}\left(-\frac{3\pi}{4}\right)$ c $5\text{cis}\left(\frac{2\pi}{3}\right)$ d $10\text{cis}\left(-\frac{4\pi}{3}\right)$ e $3\text{cis}\left(\frac{5\pi}{4}\right)$
- 4 Express the following complex numbers in polar form:
- a $2 + 4i$ b $-1 + 4i$ c $4 - 3i$ d $12 - 5i$
- 5 Express the complex number $z = 2 + 4i$ in polar form and hence express the following in polar form:
- a z^3 b $10z^4$ c $3z^8$ d $10z^{12}$
- 6 Find the linear factors in C for the following:
- a $z^2 - 2z + 2$ b $z^4 - z^2 - 12$ c $z^4 + 3z - 4 = 0$
- 7 Express the following in the form $x + yi$:
- a $\frac{2 + 3i}{1 - i}$ b $\frac{2 - i}{2 + i}$ c $\frac{4 - i}{1 + 2i}$ d $\frac{10i}{3 - i}$
- 8 Solve the following where z is expressed in the form $x + yi$:
- a $(1 - i)z = 1 + 5i$ b $(1 + i)z = 1 - 2i$ c $(2 - i)z = 3 + 4i$
- 9 Solve the following equations in C :
- a $z^2 + 4 = 0$ b $z^2 + 100 = 0$ c $z^2 + 196 = 0$
- d $z^2 + 6z + 10 = 0$ e $z^2 - 2z + 8 = 0$ f $(z - 1)^2 + 16 = 0$

Ch12

Complex numbers Challenging Tasks

- 1 Graphics calculators have the complex number functions inbuilt, which can be used to check answers.

Complex numbers can be raised to powers: i is typed using the keys [2nd] [i].

The mode needs to be put into radian or polar mode in the complex rectangular form of $a + bi$.

```
Normal SCI Eng
Float 0123456789
Radian Degree
Func Par Pol Seq
Connected Dot
Sequential Simul
Real a+bi re^θt
Full Horiz G-I
```

Shown below are the screens used to find:

$$(1 + 2i)^4 \qquad (1 + 2i)(1 - i) \qquad 2 + 3i - (1 - 2i) \qquad \frac{1 + 2i}{1 - 2i}$$

Use a graphics calculator to find the following:

- | | | |
|---|--|-------------------------------|
| a $(1 - 3i)^2$ | b $(2i + 3)^2$ | c $(5 - 3i)^5$ |
| d $\left(\frac{1}{2} + \frac{3}{2}i\right)^4$ | e $(1 - 3i)(4 + i)$ | f $(1 + 2i)(1 + 4i)$ |
| g $(3 - 4i)(1 + i)$ | h $(-1 + 5i)(-2 - 3i)$ | i $-1 + 4 + i(5 - 3i)$ |
| j $(-1 - 2i) - (4 - 3i)$ | k $\frac{1}{2} + 2i - (3 + i)$ | l $-2 - i - (4 + 6i)$ |
| m $\frac{2 + 3i}{3 - 2i}$ | n $\frac{-2 + 3i}{5 + i}$ | o $\frac{4 - i}{3 + 5i}$ |
| p $\frac{2 - 3i}{5 + i}$ | q $(1 - i)(2 + 3i)^3$ | r $(2 + 3i)^2(1 - i)^3$ |
| s $(3(2 + 5i)^5(1 - i))$ | t $(2 - 3i)(2 + 3.5i)^{10}$ | u $\frac{(1 - 4i)^3}{1 - 2i}$ |
| v $\frac{(2 + 3i)^5}{(1 - i)^3}$ | w $\frac{(3 + 2i)^7}{(1 - i)^2}$ | x $\frac{3 + 2i}{(2 - 3i)^2}$ |
| y $\frac{(1 + 2i)^8(1 - i)}{5i}$ | z $\frac{3 + 4i}{(1 - 2i)^3(2 - i)^5}$ | |

2 a Express the following in the form $a + bi$:

$$z_1 = \frac{1}{2+i}, z_2 = \frac{1}{2-i}, z_3 = \frac{1+2i}{1-2i}, z_4 = \frac{1+i}{1-i}$$

b Using the expressions in part a, express the following in the form $a + bi$:

i $z_1 + z_2$

ii $z_1 + z_3$

iii $z_3 + z_4$

iv $2z_1 + 3z_3$

v $z_4 - z_3$

vi $z_1 z_2$

vii $z_3 z_4$

viii $\frac{z_3}{z_4}$

ix $z_1^2 - z_4^2$

x $\frac{z_1 z_4}{z_2 z_3}$

3 If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ express the following in terms of a_1, a_2, b_1 and b_2 :

a $z_1 + z_2$

b $z_1 - z_2$

c $cz_1 + kz_2$

d $z_1 z_2$

e $\frac{z_1}{z_2}$

4 If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ show that:

a $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ **b** $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$ **c** $\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2}$ **d** $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

5 If $z = x + yi$ show that:

a $\frac{z\bar{z}}{|z|^2} = \frac{z\bar{z}}{|\bar{z}|^2} = 1$

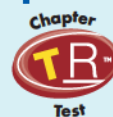
b $\arg z = -\arg \bar{z}$

6 Euler's formula is $e^{i\theta} = \cos\theta + i\sin\theta$. Use the above identity to express the following in the form $a + bi$:

a $e^{i\pi}$

b $2e^{i \times \frac{\pi}{4}}$

c $10e^{i \times \frac{3\pi}{4}}$



Selected answers are given here. For a full set of worked solutions, see *General Maths Dimensions Units 1 & 2 Solutions and Study Guide*.

Exercise 121

- 1 a $12\text{cis}\left(\frac{\pi}{3}\right)$ b $4\text{cis}\left(\frac{\pi}{6}\right)$ c $8\text{cis}\left(\frac{\pi}{2}\right)$
 d $2\text{cis}\left(\frac{\pi}{6}\right)$ e $0.5\text{cis}\left(\frac{-\pi}{6}\right)$
- 2 a $-1+i$ b $0+i$ c $0.97-0.26i$
 d $0+i$ e $-1+0i$ f $0-i$
 g $-6\sqrt{3}+6i$ h $-1+0i$ i $-6\sqrt{3}-6i$
 j $1+0i$
- 3 a $0+i$ b $\frac{\sqrt{3}}{2}+0.5i$ c $\frac{\sqrt{3}}{2}+0.5i$
 d $0-i$ e $0-0.5i$ f $\frac{\sqrt{3}}{3}+i$
 g $\frac{1}{3}i$ h $3i$ i $2.90+0.78i$
 j $\sqrt{3}+i$ k $2\sqrt{2}+2\sqrt{6}i$ l $3i$
- 4 a $1+0i$ b $1+0i$ c $1+0i$
 d $-1+0i$ e $-1+0i$ f $64i$
 g $3^8\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right)$
 h i $-512\text{cis}(3\pi)$ ii 512
 i $64\text{cis}(4\pi)$ ii 64
 j i $256\text{cis}\left(\frac{-2\pi}{3}\right)$ ii $-128-128\sqrt{3}i$
 k i $16\text{cis}\left(\frac{2\pi}{3}\right)$ ii $-8+8\sqrt{3}i$
 l i $-3125\text{cis}\left(\frac{-\pi}{3}\right)$ ii $-3125\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$
- 5 a i $\sqrt{2}\text{cis}\frac{\pi}{4}$ ii $\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$ iii $\sqrt{2}\text{cis}\left(\frac{-3\pi}{4}\right)$
 b i $2\text{cis}\pi$ ii $2\text{cis}0$ iii $2\text{cis}\left(\frac{-\pi}{2}\right)$
 iv $\text{cis}\left(\frac{-\pi}{2}\right)$ v $\text{cis}\left(\frac{-\pi}{2}\right)$ vi $\text{cis}(\pi)$
 vii $16\text{cis}0$ viii $8\text{cis}\left(\frac{\pi}{2}\right)$ ix $32\text{cis}\left(\frac{-\pi}{2}\right)$
 x $4096\text{cis}0$
 c i -2 ii 2 iii $-2i$ iv $-i$
 v $-i$ vi -1 vii 16 viii $8i$
 ix $32i$ x 4096

Revision Questions

Multiple choice

- 1 A 2 D 3 D 4 D
 5 B 6 B 7 C 8 D
 9 [A] $\sqrt{2}10$ D 11 E 12 B

Short answer

- 1 a i b 1 c $-i$
 d $-3-2i$ e $-4+8i$
- 2 a $3+2i$ b $1+4i$ c $-35+20i$
- 3 a $2-2\sqrt{3}i$ b $-\sqrt{2}-\sqrt{2}i$
 c $-2.5+2.5\sqrt{3}i$ d $-5+5\sqrt{3}i$
 e $-\frac{3}{\sqrt{2}}-\frac{3}{\sqrt{2}}i$
- 4 a $2\sqrt{5}\text{cis}1.107$ b $\sqrt{17}\text{cis}1.816$
 c $5\text{cis}-0.6435$ d $13\text{cis}-0.3948$
- 5 a $40\sqrt{5}\text{cis}(-2.96)$
 b $4000\text{cis}(-1.85)$
 c $480000\text{cis}(2.57)$
 d $64 \times 10^8\text{cis}(0.72)$
- 6 a $(z-1+i)(z-1-i)$
 b $(z-2)(z+2)(z-\sqrt{3}i)(z+\sqrt{3}i)$
 c $(z-1)(z+1)(z+2i)(z-2i)$
- 7 a $-0.5+2.5i$ b $0.6-0.8i$
 c $0.4-1.8i$ d $-1+3i$
- 8 a $z=-2+3i$ b $z=-\frac{1}{2}-1\frac{1}{2}i$ c $z=\frac{2}{5}+\frac{11i}{5}$
- 9 a $z=\pm 2i$ b $z=\pm 10i$ c $z=\pm 14i$
 d $z=-3\pm i$ e $z=1\pm 7i$ f $z=1\pm 4i$

Challenging Tasks

- 1 a $-8-6i$ b $5+12i$
 c $-6100-2868i$ d $7/4-6i$
 e $7-11i$ f $-7+6i$
 g $7-i$ h $17+7i$
 i $6+5i$ j $-5+i$
 k $-5/2+i$ l $-6-7i$
 m i $-7/6+17/26i$ n $-7/6+17/26i$
 o $7/34-23/34i$ p $7/26-17/26i$
 q $-37+55i$ r $34-14i$
 s $8421-17271i$ t $-4061222-444148i$
 u $-151/5-42/5i$ v $475/4+719/4i$

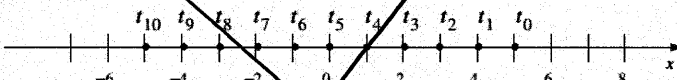
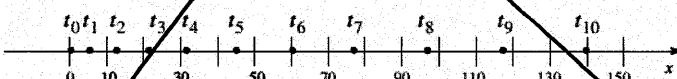
Answers

Selected answers are given here. For a full set of worked solutions, see *General Maths Dimensions Units 1 & 2 Solutions and Study Guide*.

- w $3277 - 2224.5i$ x $-3/13 + 2/13i$ 4 a $\overline{z_1 + z_2} = a_1 + a_2 - (b_1 + b_2)i$
y $-172.6 + 38.2i$ z $24/3125 + 7/3125i$ and $\overline{z_1 + z_2} = a_1 + -b_1i + a_2 - b_2i$
2 a i $0.4 - 0.2i$ ii $0.4 + 0.2i$ They are clearly equal.
iii $-0.6 + 0.8i$ iv $0 + i$
b i 0.8 ii $-0.2 + 0.6i$
iii $-0.6 + 1.8i$ iv $-1 + 2i$
v $0.6 + 0.2i$ vi 0.2
vii $-0.8 - 0.6i$ viii $0.8 + 0.6i$
ix $1.12 - 0.16i$ x $-i$
3 a $a_1 + a_2 + (b_1 + b_2)i$
b $a_1 - a_2 + (b_1 - b_2)i$
c $ca_1 + ka_2 + (cb_1 + kb_2)i$
d $a_1a_2 - b_1b_2 + (a_1b_2 + a_2b_1)i$
e $\frac{(a_1 + b_1i)}{(a_2 + b_2i)}$
b $\overline{z_1 - z_2} = a_1 - a_2 - (b_1 + b_2)i = \overline{z_1} - \overline{z_2}$
c $\overline{z_1 \times z_2} = a_1a_2 + b_1b_2 - (a_1b_2 + a_2b_1)i$ and
 $\overline{z_1} \times \overline{z_2} = a_1a_2 + b_1b_2 - (a_1b_2 + a_2b_1)i$
d $\frac{\overline{z_1}}{\overline{z_2}} = \frac{a_1 - b_1i}{a_2 - b_2i}$ and $\frac{\overline{z_1}}{\overline{z_2}} = \frac{a_1 - b_1i}{a_2 - b_2i}$
5 a 1
b $\text{Arg} z = \tan^{-1} \frac{y}{x}$ and $\text{Arg} \bar{z} = \tan^{-1} \frac{-y}{x} = -\tan^{-1}(\frac{y}{x})$
so $\text{Arg} z = -\text{Arg}(\bar{z})$
6 a -1 b $(\sqrt{2} + \sqrt{2}i)$ c $(-5\sqrt{2} + 5\sqrt{2}i)$

Chapter 13

Exercise 13A

- 1 a 5 cm
b i $t = 1, x = 4$ cm ii $t = 5, x = 0$ cm
iii $t = 10, x = -5$ cm
c 10 m
d 
2 a 0 m
b 5 m, 45 m, 140 m
c 140 m
d i 12 m ii 32 m iii 60 m
iv 96 m v 140 m
e 
3 a i -12 m
ii $t = 25$ $t = 45$ $t = 65$
 $x = 0$ m $x = 28$ m $x = 72$ m
 $t = 85$ $t = 105$
 $x = 132$ m $x = 192$ m
iii 192 m
b

t (s)	0	1	2	3	4	5	6	7	8	9	10
x (cm)	-12	-8	0	12	28	48	72	100	132	168	208

COMPLEX NOS . POLAR FORM . PRODUCT . QUOTIENT . DE MOIURES

1. Find the polar form of the following complex numbers:

- (a) $z = 1 + \sqrt{3}i$ (b) $z = -1 + i$ (c) $z = -1 - i$ (d) $z = 1 - i$
 (e) $z = 3 + 4i$ (f) $z = -3 - 4i$ (g) $z = -3 + 4i$ (h) $z = 4 - 3i$
 (i) $z = 2i$ (j) $z = -2i$

2. Express each of the following complex numbers in cartesian form:

- (a) $z = 5\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$ (b) $z = 4 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$
 (c) $z = 6 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$ (d) $z = 2 \operatorname{cis} \left(-\frac{\pi}{3} \right)$
 (e) $z = 3\sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4} \right)$ (f) $z = 10 \operatorname{cis} \left(-\frac{5\pi}{6} \right)$

3. If $z = 3 + i$, and $w = 1 - 3i$, find:

- (a) $|z|$ (b) $|w|$ (c) zw (d) $|zw|$ (e) $\operatorname{Arg} z$
 (f) $\operatorname{Arg} w$ (g) $\operatorname{Arg} (zw)$ (h) $\frac{|z|}{|w|}$ (i) $|z||w|$

4. If $z = x + yi$, show that:

- (a) $\frac{z\bar{z}}{|z|^2} = 1$ (b) $\operatorname{Arg} z + \operatorname{Arg} \bar{z} = 0$

5. Express the following in polar form:

- (a) $-1 + i$ (b) $1 + \sqrt{3}i$

Hence simplify $\frac{(-1 + i)^4}{(1 + \sqrt{3}i)^5}$.

6. Simplify $\frac{(\sqrt{3} + i)^6}{(1 - i)^8}$.

7. If $z = -1 + i$, express:

- (a) $\bar{z}$ (b) $\frac{i}{z}$ (c) z^7

in the form $x + yi$, where x and y are real numbers.

8. If $z = 1 + \sqrt{3}i$, find:

- (a) $|z|$ (b) $\operatorname{Arg} z$ (c) the polar form of z
 (d) z^4 in polar form (e) z^4 in Cartesian form
 (f) $\frac{1}{z}$ in the form $x + yi$, where x and y are real numbers.

9. If $w = 1 + i$, and $z = 1 - \sqrt{3}i$, find:

- (a) $|w|$ (b) $|z|$ (c) $\operatorname{Arg} w$ (d) $\operatorname{Arg} z$

Hence determine $\left| \frac{w}{z} \right|$, $\operatorname{Arg} \left(\frac{w}{z} \right)$ and $\operatorname{Arg} (wz)$.

$$\begin{aligned} & \frac{z}{w} = \frac{1 - \sqrt{3}i}{1 + i} = \frac{(1 - \sqrt{3}i)(1 - i)}{(1 + i)(1 - i)} = \frac{1 - i - \sqrt{3}i + \sqrt{3}}{2} = \frac{1 + \sqrt{3} - i(1 + \sqrt{3})}{2} \\ & \frac{w}{z} = \frac{1 + i}{1 - \sqrt{3}i} = \frac{(1 + i)(1 + \sqrt{3}i)}{(1 - \sqrt{3}i)(1 + \sqrt{3}i)} = \frac{1 + i + \sqrt{3}i + \sqrt{3}}{4} = \frac{1 + \sqrt{3} + i(1 + \sqrt{3})}{4} \\ & wz = (1 + i)(1 - \sqrt{3}i) = 1 - \sqrt{3}i + i - \sqrt{3} = 1 - \sqrt{3} - i(\sqrt{3} - 1) \\ & \operatorname{Arg} \left(\frac{w}{z} \right) = \operatorname{Arg} \left(\frac{1 + \sqrt{3} - i(1 + \sqrt{3})}{2} \right) = \operatorname{Arg} \left(\frac{1 + \sqrt{3}}{2} - i \frac{1 + \sqrt{3}}{2} \right) \\ & \operatorname{Arg} \left(\frac{w}{z} \right) = \operatorname{Arg} \left(\frac{1 + \sqrt{3}}{2} \right) - \frac{\pi}{2} \\ & \operatorname{Arg} (wz) = \operatorname{Arg} (1 - \sqrt{3} - i(\sqrt{3} - 1)) = \operatorname{Arg} (1 - \sqrt{3} - i(\sqrt{3} - 1)) \end{aligned}$$

Factorization of Cubic and Quartic involving grouping and the Factor Theorem

1. (a) $P(z) = z^3 - 2z^2 + z - 2$
(b) $P(z) = z^3 + 2z^2 + 4z + 8$
2. (a) $P(z) = z^4 - z^3 - 5z^2 - z - 6$
(b) $P(z) = z^4 - 3z^3 + 6z^2 - 12z + 8$
3. (a) $P(z) = z^4 - 3z^2 - 4$
(b) $P(z) = z^4 + 3z^2 - 4$
4. (a) $P(z) = z^4 + 10z^2 + 9$
(b) $P(z) = z^4 + 13z^2 + 36$
5. (a) $P(z) = z^3 + z^2 - 7z - 15$
(b) $P(z) = z^3 + 4z^2 + 9z + 10$

Extension Question: $P(z) = z^3 + iz^2 + i + z$

Extension: $P(z) = (z + i)^2(z - i)$

- 1a. $P(z) = (z - 2)(z + i)(z - i)$
- b. $P(z) = (z + 2)(z + 2i)(z - 2i)$
- 2a. $P(z) = (z + 2)(z - 3)(z + i)(z - i)$
- b. $P(z) = (z - 1)(z - 2)(z + 2i)(z - 2i)$
- 3a. $P(z) = (z - 2)(z + 2)(z + i)(z - i)$
- b. $P(z) = (z - 1)(z + 1)(z + z)(z + 2i)(z - 2i)$
- 4a. $P(z) = (z + i)(z - i)(z + 3i)(z - 3i)$
- b. $P(z) = (z + 2i)(z - 2i)(z + 3i)(z - 3i)$
- 5a. $P(z) = (z - 3)(z + 2 + i)(z + 2 - i)$
- b. $P(z) = (z + 2)(z + 1 + 2i)(z + 1 - 2i)$

Answers:

Find the solutions (the roots of Z^n)

1. $Z^3=1$
2. $Z^3+8=0$
3. $Z^4-1=0$
4. $Z^3=i$
5. $Z^4+4=0$

Challenging question: Find the 6th root of -1

(hint: let $-1 = 1 \operatorname{cis}(\pi)$)

Challenging question $\frac{\sqrt{3}}{2} + \frac{1}{2}i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i, -\frac{\sqrt{3}}{2} - \frac{1}{2}i, \frac{\sqrt{3}}{2} - \frac{1}{2}i, i, -i$

1. $1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
2. $-2, 1 - \sqrt{3}i, 1 + \sqrt{3}i$
3. $-1, 1, i, -i$
4. $-i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i, \frac{\sqrt{3}}{2} + \frac{1}{2}i$
5. $-1-i, -1+i, 1-i, 1+i$

Answers: