

7.2 Distance between two points

study on

Units 1 & 2

AOS 5

Topic 1

Concept 1

Distance between two points

Concept summary
Practice questions

Coordinate geometry is a branch of mathematics with many practical applications. The distance between two points can be calculated easily using Pythagoras' theorem. It is particularly useful when trying to find a distance that is difficult to measure directly; for example, finding the distance from a point on one side of a lake to a point on the other side.



Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on the Cartesian plane as shown below right.

Triangle ABC is a right-angled triangle.

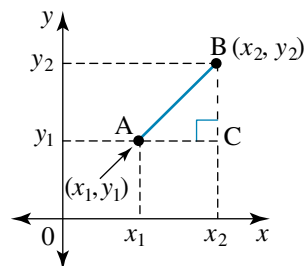
$$AC = x_2 - x_1$$

$$BC = y_2 - y_1$$

By Pythagoras' theorem:

$$\begin{aligned} AB^2 &= AC^2 + BC^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \end{aligned}$$

$$\text{Hence, } AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



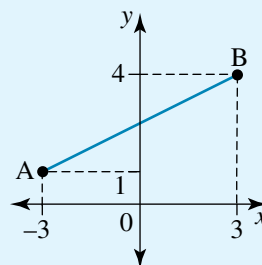
The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

WORKED EXAMPLE

1

- Find the distance between the points A and B in the figure at right, correct to 2 decimal places.
- Find the distance between the points P $(-1, 5)$ and Q $(3, -2)$, correct to 2 decimal places.



THINK

- From the graph find points A and B.
- Let A have coordinates (x_1, y_1) .
- Let B have coordinates (x_2, y_2) .
- Find the length AB by applying the formula for the distance between two points.

WRITE

- A $(-3, 1)$ and B $(3, 4)$

$$\text{Let } (x_1, y_1) = (-3, 1).$$

$$\text{Let } (x_2, y_2) = (3, 4).$$

$$\begin{aligned} AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - (-3))^2 + (4 - 1)^2} \\ &= \sqrt{(6)^2 + (3)^2} \\ &= \sqrt{36 + 9} \\ &= \sqrt{45} \\ &= 3\sqrt{5} \text{ units} \\ &= 6.71 \text{ units (correct to 2 decimal places)} \end{aligned}$$

- b 1** Let P have coordinates (x_1, y_1) .
- 2** Let Q have coordinates (x_2, y_2) .
- 3** Find the length PQ by applying the formula for the distance between two points.

b Let $(x_1, y_1) = (-1, 5)$.

Let $(x_2, y_2) = (3, -2)$.

$$\begin{aligned}
 PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
 &= \sqrt{(3 - (-1))^2 + (-2 - 5)^2} \\
 &= \sqrt{(4)^2 + (-7)^2} \\
 &= \sqrt{16 + 49} \\
 &= \sqrt{65} \text{ units} \\
 &= 8.06 \text{ units (correct to 2 decimal places)}
 \end{aligned}$$

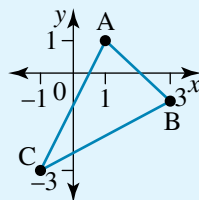
WORKED EXAMPLE 2 Prove that the points A (1, 1), B (3, -1) and C (-1, -3) are the vertices of an isosceles triangle.

THINK

- 1** Plot the points.

Note: For triangle ABC to be isosceles, two sides must have the same magnitude.

WRITE/DRAW



From the diagram, AC appears to have the same length as BC.

- 2** Find the length AC.

$$\begin{aligned}
 AC &= \sqrt{[1 - (-1)]^2 + [1 - (-3)]^2} \\
 &= \sqrt{(2)^2 + (4)^2} \\
 &= \sqrt{20} \\
 &= 2\sqrt{5}
 \end{aligned}$$

- 3** Find the length BC.

$$\begin{aligned}
 BC &= \sqrt{[3 - (-1)]^2 + [-1 - (-3)]^2} \\
 &= \sqrt{(4)^2 + (2)^2} \\
 &= \sqrt{20} \\
 &= 2\sqrt{5}
 \end{aligned}$$

- 4** Find the length AB.

$$\begin{aligned}
 AB &= \sqrt{[3 - (1)]^2 + [-1 - (1)]^2} \\
 &= \sqrt{(2)^2 + (-2)^2} \\
 &= \sqrt{4 + 4} \\
 &= \sqrt{8} \\
 &= 2\sqrt{2}
 \end{aligned}$$

- 5** State your proof.

Since $AC = BC \neq AB$, triangle ABC is an isosceles triangle.

EXERCISE 7.2 Distance between two points

PRACTISE

- 1 **WE1** a Find the distance between the points A and B shown at right.

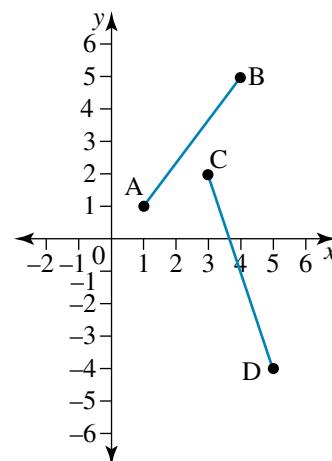
b Find the distance between the points (2, 5), (6, 8).

- 2 a Find the distance between the points C and D shown at right.

b Find the distance between the points (-1, 2) and (4, 14).

- 3 **WE2** Prove that the points A (0, -3), B (-2, -1) and C (4, 3) are the vertices of an isosceles triangle.

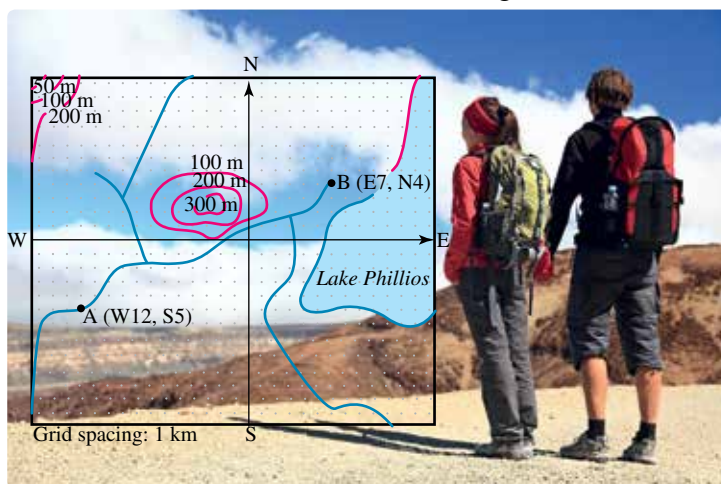
- 4 Prove that the points A (3, 1), B (-3, 7) and C (-1, 3) are the vertices of an isosceles triangle.



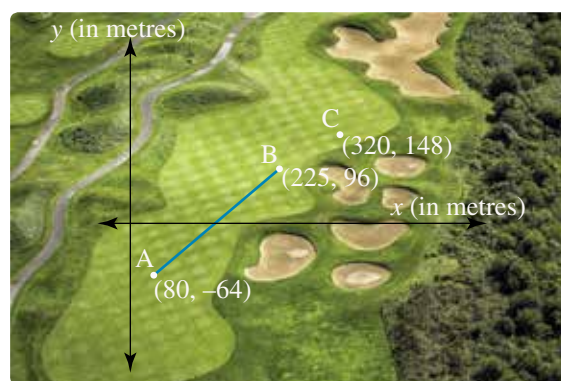
CONSOLIDATE

- 5 The points P (2, -1), Q (-4, -1) and R (-1, $3\sqrt{3} - 1$) are joined to form a triangle. Prove that triangle PQR is equilateral.
- 6 Prove that the quadrilateral with vertices A (-1, 3), B (5, 3), C (1, 0) and D (-5, 0) is a parallelogram.
- 7 Prove that the triangle with vertices D (5, 6), E (9, 3) and F (5, 3) is a right-angled triangle.
- 8 The vertices of a quadrilateral are A (1, 4), B (-1, 8), C (1, 9) and D (3, 5).
 a Find the lengths of the sides.
 b Find the lengths of the diagonals.
 c What type of quadrilateral is it?
- 9 Calculate the distance between each of the pairs of points below, correct to 3 decimal places.
 a (-14, 10) and (-8, 14) b (6, -7) and (13, 6) c (-11, 1) and (2, 2)
- 10 Find the distance between each of the following pairs of points in terms of the given variables.
 a (a, 1), (2, 3) b (5, 6), (0, b)
 c (c, 2), (4, c) d (d, 2d), (1, 5)
- 11 If the distance between the points (3, b) and (-5, 2) is 10 units, then a possible value of b is:
 A -8 B -4 C 4 D 0 E 2
- 12 A rhombus has vertices A (1, 6), B (6, 6), C (-2, 2) and D (x, y). The coordinates of D are:
 A (2, -3) B (2, 3) C (-2, 3)
 D (3, 2) E (3, -2)
- 13 A rectangle has vertices A (1, 5), B (10.6, z), C (7.6, -6.2) and D (-2, 1). Find:
 a the length of CD b the length of AD
 c the length of the diagonal AC d the value of z.
- 14 Show that the triangle ABC with coordinates A (a, a), B (m, -a) and C (-a, m) is isosceles.

- 15 Two hikers are about to hike from A to B (shown on the map below). How far is it from A to B ‘as the crow flies’, that is, in a straight line?



- 16 Using the coordinates shown on the aerial photo of the golf course, calculate (to the nearest metre):
- the horizontal distance travelled by the golf ball for the shot down the fairway, from A to B
 - the horizontal distance that needs to be covered in the next shot to reach the point labelled C in the bunker.



7.3 Midpoint of a line segment

We can determine the coordinates of the midpoint of a line segment by applying the midpoint formula shown below.

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Topic 1

Concept 2

Midpoint of a line segment

Concept summary
Practice questions

Midpoint formula

Consider the line segment connecting the points

A (x_1, y_1) and B (x_2, y_2) .

Let P (x, y) be the midpoint of AB.

AC is parallel to PD. PC is parallel to BD.

AP is parallel to PB (collinear, that is, the points lie on the same straight line).

Hence, triangle APC is similar to triangle PBD.

But AP = PB (since P is the midpoint of AB).

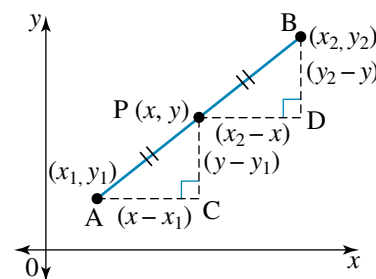
Hence, triangle APC is congruent, to triangle PBD.

Therefore $x - x_1 = x_2 - x$

$$2x = x_1 + x_2$$

$$x = \frac{x_1 + x_2}{2}$$

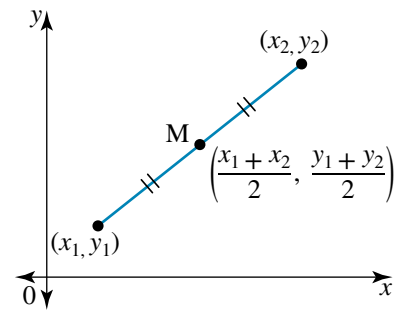
Similarly it can be shown that $y = \frac{y_1 + y_2}{2}$.



In general, the coordinates of the midpoint of a line segment joining the points (x_1, y_1) and (x_2, y_2) can be found by averaging the x - and y -coordinates of the end points, respectively.

The coordinates of the midpoint of the line segment joining

(x_1, y_1) and (x_2, y_2) are: $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$.



WORKED EXAMPLE 3 Find the coordinates of the midpoint of the line segment joining $(-2, 5)$ and $(7, 1)$.

THINK

1 Label the given points (x_1, y_1) and (x_2, y_2) .

2 Find the x -coordinate of the midpoint.

3 Find the y -coordinate of the midpoint.

4 Give the coordinates of the midpoint.

WRITE

Let $(x_1, y_1) = (-2, 5)$ and $(x_2, y_2) = (7, 1)$.

$$\begin{aligned} x &= \frac{x_1 + x_2}{2} \\ &= \frac{-2 + 7}{2} \\ &= 2\frac{1}{2} \end{aligned}$$

$$\begin{aligned} y &= \frac{y_1 + y_2}{2} \\ &= \frac{5 + 1}{2} \\ &= 3 \end{aligned}$$

Hence, the coordinates of the midpoint are $\left(2\frac{1}{2}, 3 \right)$.

WORKED EXAMPLE 4 The coordinates of the midpoint, M , of the line segment AB are $(7, 2)$. If the coordinates of A are $(1, -4)$, find the coordinates of B .

THINK

1 Label the start of the line segment (x_1, y_1) and the midpoint (x, y) .

2 Find the x -coordinate of the end point.

WRITE/DRAW

Let $(x_1, y_1) = (1, -4)$ and $(x, y) = (7, 2)$.

$$\begin{aligned} x &= \frac{x_1 + x_2}{2} \\ 7 &= \frac{1 + x_2}{2} \\ 14 &= 1 + x_2 \\ x_2 &= 13 \end{aligned}$$

3 Find the y-coordinate of the end point.

$$y = \frac{y_1 + y_2}{2}$$

$$2 = \frac{-4 + y_2}{2}$$

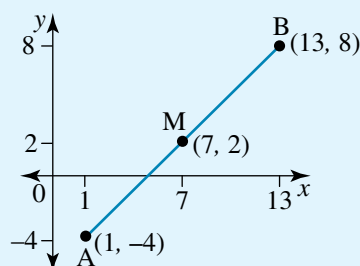
$$4 = -4 + y_2$$

$$y_2 = 8$$

4 Give the coordinates of the end point.

Hence, the coordinates of point B are (13, 8).

5 Check that the coordinates are feasible.



EXERCISE 7.3 Midpoint of a line segment

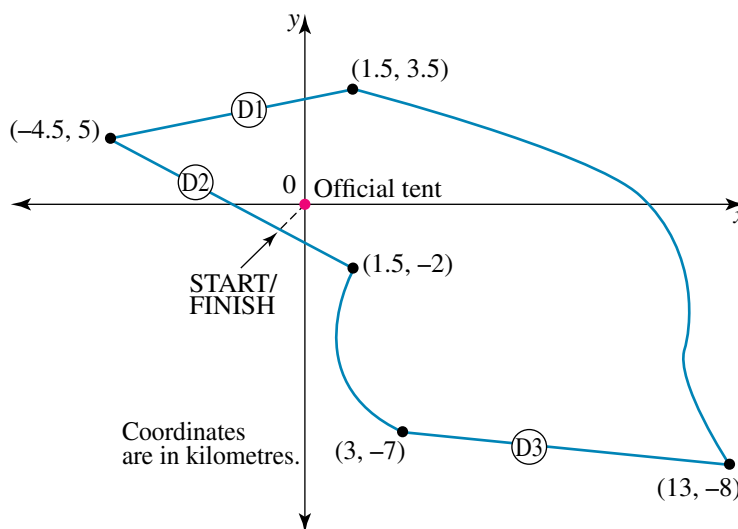
PRACTISE

- WE3** Find the coordinates of the midpoint of the line segment joining $(-5, 1)$ and $(-1, -8)$.
- Find the coordinates of the midpoint of the line segment joining $(4, 2)$ and $(11, -2)$.
- WE4** The coordinates of the midpoint, M, of the line segment AB are $(2, -3)$. If the coordinates of A are $(7, 4)$, find the coordinates of B.
- The coordinates of the midpoint, M, of the line segment AB are $(-2, 4)$. If the coordinates of A are $(1, 8)$, find the coordinates of B.

CONSOLIDATE

- The vertices of a square are A $(0, 0)$, B $(2, 4)$, C $(6, 2)$ and D $(4, -2)$. Find:
 - the coordinates of the centre of the square
 - the side length
 - the length of the diagonals.
- The midpoint of the line segment joining the points $(-2, 1)$ and $(8, -3)$ is:
A $(6, -2)$ **B** $(5, 2)$ **C** $(6, 2)$ **D** $(3, -1)$ **E** $(5, -2)$
- If the midpoint of AB is $(-1, 5)$ and the coordinates of B are $(3, 8)$, then A has coordinates:
A $(1, 6.5)$ **B** $(2, 13)$ **C** $(-5, 2)$ **D** $(4, 3)$ **E** $(7, 11)$
- Find the coordinates of the midpoint of each of the following pairs of points, in terms of a variable or variables where appropriate.
 - $(2a, a)$, $(6a, 5a)$
 - $(5, 3c)$, $(11, 3c)$
 - $(3f, 5)$, $(g, -1)$
- Find the value of a so M is the midpoint of the segment joining points A and B.
 - A $(-2, a)$, B $(-6, 5)$, M $(-4, 5)$
 - A $(a, 0)$, B $(7, 3)$, M $\left(8, \frac{3}{2}\right)$
- a** The vertices of a triangle are A $(2, 5)$, B $(1, -3)$ and C $(-4, 3)$. Find:
 - the coordinates of P, the midpoint of AC
 - the coordinates of Q, the midpoint of AB
 - the length of PQ
 - the length of BC.**b** Hence show that $BC = 2PQ$.

- 11 a** A quadrilateral has vertices A (6, 2), B (4, -3), C (-4, -3) and D (-2, 2). Find:
- the midpoint of the diagonal AC
 - the midpoint of the diagonal BD.
- b** Comment on your finding.
- 12 a** The points A (-5, 3.5), B (1, 0.5) and C (-6, -6) are the vertices of a triangle. Find:
- the midpoint, P, of AB
 - the length of PC
 - the length of AC
 - the length of BC.
- b** Describe the triangle ABC. What could PC represent?
- 13** Find the equation of the straight line that passes through the midpoint of A (-2, 5) and B (-2, 3) and has a gradient of -3.
- 14** Find the equation of the straight line that passes through the midpoint of A (-1, -3) and B (3, -5) and has a gradient of $\frac{2}{3}$.
- 15** A fun-run course is drawn (not to scale) at right. If drink stations D1, D2 and D3 are to be placed at the middle of each straight section, give the map coordinates of each drink station.
- 16** Find the equation of a line that has a gradient of 5 and passes through the midpoint of the segment joining (-1, -7) and (3, 3).



MASTER

7.4 Parallel lines and perpendicular lines

Parallel lines

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Topic 1

Concept 3

Parallel lines

Concept summary
Practice questions

The equation of a straight line may be expressed in the form:

$$y = mx + c$$

where m is the gradient of the line and c is the y -intercept.

The gradient can be calculated if two points, (x_1, y_1) and (x_2, y_2) , are given.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

An alternative form for the equation of a straight line is:

$$ax + by + c = 0$$

where a , b and c are constants.

Another alternative form is:

$$y - y_1 = m(x - x_1)$$

where m is the gradient and (x_1, y_1) is a point on the line.

**WORKED
EXAMPLE 5**

Show that AB is parallel to CD given that A has coordinates $(-1, -5)$, B has coordinates $(5, 7)$, C has coordinates $(-3, 1)$, and D has coordinates $(4, 15)$.

THINK

1 Find the gradient of AB.

2 Find the gradient of CD.

3 Compare the gradients to determine if they are parallel. (Note: $\parallel$ means 'is parallel to'.)

WRITE

Let $A(-1, -5) = (x_1, y_1)$ and $B(5, 7) = (x_2, y_2)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\begin{aligned} m_{AB} &= \frac{7 - (-5)}{5 - (-1)} \\ &= \frac{12}{6} \\ &= 2 \end{aligned}$$

Let $C(-3, 1) = (x_1, y_1)$ and $D(4, 15) = (x_2, y_2)$.

$$\begin{aligned} m_{CD} &= \frac{15 - 1}{4 - (-3)} \\ &= \frac{14}{7} \\ &= 2 \end{aligned}$$

Parallel lines have the same gradient. $m_{AB} = m_{CD} = 2$, hence $AB \parallel CD$.

Collinear points lie on the same straight line.

**WORKED
EXAMPLE 6**

Show that the points A $(2, 0)$, B $(4, 1)$ and C $(10, 4)$ are collinear.

THINK

1 Find the gradient of AB.

2 Find the gradient of BC.

WRITE

Let $A(2, 0) = (x_1, y_1)$
and $B(4, 1) = (x_2, y_2)$.

$$\begin{aligned} \text{Since } m &= \frac{y_2 - y_1}{x_2 - x_1} \\ m_{AB} &= \frac{1 - 0}{4 - 2} \\ &= \frac{1}{2} \end{aligned}$$

Let $B(4, 1) = (x_1, y_1)$
and $C(10, 4) = (x_2, y_2)$.

$$\begin{aligned} m_{BC} &= \frac{4 - 1}{10 - 4} \\ &= \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$



3 Show that A, B and C are collinear.

$$\text{Since } m_{AB} = m_{BC} = \frac{1}{2}$$

then $AB \parallel BC$.

Since B is common to both line segments, A, B and C must lie on the same straight line. That is, A, B and C are collinear.

WORKED
EXAMPLE

7

Find the equation of the straight line that passes through the point (2, 5) and is parallel to the line $y = 3x + 1$.

THINK

- 1 In order to find the equation of a straight line, we need to know the gradient and a point on the line. One point is given, and because the line is parallel to $y = 3x + 1$, the gradients will be the same.
- 2 Use the formula $y - y_1 = m(x - x_1)$ and substitute the coordinates of the point and the gradient to find the equation of the line.

WRITE

Point on the line: (2, 5)

Gradient: $m = 3$.

$$y - y_1 = m(x - x_1)$$

$$y - 5 = 3(x - 2)$$

$$y - 5 = 3x - 6$$

$$y = 3x - 1$$

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Topic 1

Concept 4

Perpendicular lines

Concept summary
Practice questions

Perpendicular lines

In this section, we examine some of the properties of perpendicular lines. Observing the graphs can be very useful in investigating these properties.

Consider the diagram below right, where the line segment AB is perpendicular to the line segment BC. Line AC is parallel to the x-axis. Line BD is the height of the resulting triangle ABC.

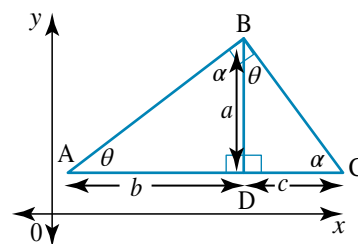
$$\begin{aligned} \text{Let } m_{AB} &= m_1 \\ &= \frac{a}{b} \\ &= \tan(\theta) \end{aligned}$$

$$\begin{aligned} \text{Let } m_{BC} &= m_2 \\ &= -\frac{a}{c} \\ &= -\tan(\theta) \\ &= -\frac{b}{a} \\ &= -\frac{1}{m_1} \end{aligned}$$

$$\text{Hence } m_2 = -\frac{1}{m_1}$$

$$\text{or } m_1 m_2 = -1$$

Hence, if two lines are perpendicular to each other, then the product of their gradients is -1 .



Two lines are perpendicular if and only if:

$$m_1 m_2 = -1$$

or

$$m_2 = -\frac{1}{m_1}$$

WORKED EXAMPLE 8 Show that the lines $y = -5x + 2$ and $5y - x + 15 = 0$ are perpendicular to one another.

THINK

- 1 Find the gradient of equation 1.
- 2 Find the gradient of equation 2.
- 3 Test for perpendicularity. (The two lines are perpendicular if the product of their gradients is -1 .)

WRITE

$$y = -5x + 2$$

Hence $m_1 = -5$

$$5y - x + 15 = 0$$

Rewrite in the form $y = mx + c$:

$$5y = x - 15$$

$$y = \frac{x}{5} - 3$$

Hence $m_2 = \frac{1}{5}$

$$\begin{aligned} m_1 m_2 &= -5 \times \frac{1}{5} \\ &= -1 \end{aligned}$$

Hence, the two lines are perpendicular to each other.

EXERCISE 7.4 Parallel lines and perpendicular lines

PRACTISE

- 1 **WE5** Show that AB is parallel to CD given that A has coordinates (2, 4), B has coordinates (8, 1), C has coordinates (−6, −2) and D has coordinates (2, −6).
- 2 Show that AB is parallel to CD given that A has coordinates (1, 0), B has coordinates (2, 5), C has coordinates (3, 15) and D has coordinates (7, 35).
- 3 **WE6** Show that the points A (0, −2), B (5, 1) and C (−5, −5) are collinear.
- 4 Show that the points A (3, 1), B (5, 2) and C (11, 5) are collinear.
- 5 **WE7** Find the equation of a straight line given the following conditions. The line passes through the point (−1, 3) and is parallel to $y = -2x + 5$.
- 6 Find the equation of a straight line given that the line passes through the point (4, −3) and is parallel to $3y + 2x = -3$.
- 7 **WE8** Show that the lines $y = 6x - 3$ and $x + 6y - 6 = 0$ are perpendicular to one another.
- 8 Show that the lines $y = 2x - 4$ and $x + 2y - 10 = 0$ are perpendicular to one another.

CONSOLIDATE

- # MASTER

7 Answers

EXERCISE 7.2

- 1 a $AB = 5$
b 5
- 2 a $CD = 2\sqrt{10}$
b 13
- 3–7 Answers will vary.
- 8 a $AB = 2\sqrt{5}$, $BC = \sqrt{5}$, $CD = 2\sqrt{5}$, $DA = \sqrt{5}$
b $AC = 5$, $BD = 5$
c Rectangle
- 9 a 7.211
b 14.765
c 13.038
- 10 a $\sqrt{a^2 - 4a + 8}$
b $\sqrt{b^2 - 12b + 61}$
c $\sqrt{2(c^2 - 6c + 10)}$
d $\sqrt{5d^2 - 22d + 26}$
- 11 B
- 12 D
- 13 a 12
b 5
c 13
d -2.2
- 14 Answers will vary.
- 15 $\sqrt{442}$ km
- 16 a $5\sqrt{1865}$ m
b $\sqrt{11729}$ m

EXERCISE 7.3

- 1 $(-3, -3\frac{1}{2})$
2 $(7\frac{1}{2}, 0)$
3 $(-3, -10)$
4 $(-5, 0)$
- 5 a $(3, 1)$
b $2\sqrt{5}$ units
c $3\sqrt{10}$ units
- 6 D
- 7 C
- 8 a $(4a, 3a)$
b $(8, 3c)$
c $(\frac{3f+g}{2}, 2)$
- 9 a 5
b 9

- 10 a i $(-1, 4)$
ii $(1\frac{1}{2}, 1)$
iii $\frac{\sqrt{61}}{2}$ units
iv $\sqrt{61}$
b Answers will vary.
- 11 a i $(1, -0.5)$
ii $(1, -0.5)$
b Answers will vary.
- 12 a i $(-2, 2)$
ii $4\sqrt{5}$ units
iii $\frac{\sqrt{365}}{2}$ units
iv $\frac{\sqrt{365}}{2}$ units
b Isosceles triangle, perpendicular height
- 13 $y = -3x - 2$
- 14 $3y - 2x + 14 = 0$
- 15 D1 $(-1.5, 4.25)$, D2 $(-1.5, 1.5)$, D3 $(8, -7.5)$
- 16 $y = 5x - 7$

EXERCISE 7.4

- 1–4 Answers will vary.
- 5 $y = -2x + 1$
- 6 $3y + 2x + 1 = 0$
- 7 and 8 Answers will vary.
- 9 b, f; c, e
- 10–12 Answers will vary.
- 13 B
- 14 E
- 15 a, e; b, f; c, h; d, g
- 16, 17 Answers will vary.
- 18 $y = -\frac{1}{2}x + \frac{3}{2}$
- 19 a i $m = 10$ ii $m = -\frac{8}{5}$
b i $m = \frac{-5}{2}$ ii $m = \frac{18}{5}$
- 20 Answers will vary.

EXERCISE 7.5

- 1 $y = 2x - 9$
- 2 $y = -x - 1$
- 3 $3x + 2y - 8 = 0$
- 4 $x - 2y - 2 = 0$
- 5 a $x = 1$
b $y = -7$