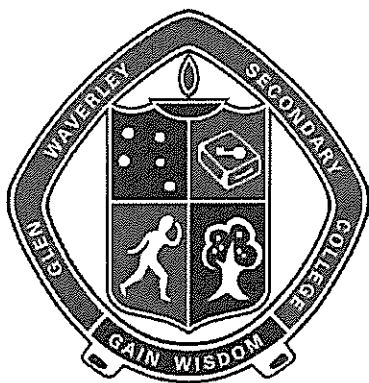




Glen Waverley Secondary College 2017

VCE Specialist Mathematics – Tech Free

Unit 2 Examination

Student Name: _____

SOLUTIONS

Student ID Code: _____

Form: _____

Teacher: CAL PLG GMN
(circle your teacher's code)

QUESTION AND ANSWER BOOK

Reading Time: ¹⁵~~10~~ minutes
Writing Time: 60 minutes

Structure of Book

Section	Number of questions	Number of questions to be answered	Number of marks
Short Answer	11	11	50

- Students are permitted to bring into the room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the exam room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and Answer book.
- Formula sheet.

Instructions

- Write your **student name and student ID code** in the space provided above on this page.
- CAS calculators, a bound reference and/or summary notes are **NOT** allowed.
- Marks are allocated for neat and clear working as well as answers.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Question 1

Given $x^2 - 8x - 29 \equiv (x + a)^2 + b$, where a and b are constants,

a) find the value of a and the value of b .

$$(x+a)^2 + b = x^2 + 2ax + a^2 + b = x^2 - 8x - 29$$

$$2a = -8$$

$$a = -4$$

(1)

$$a^2 + b = -29$$

$$b = -29 - a^2$$

$$= -29 - 16$$

$$= -45$$

(1)

$$\therefore \begin{array}{l} a = -4 \\ b = -45 \end{array}$$

b) Hence, or otherwise, show that the roots of $x^2 - 8x - 29 = 0$ are of the form $c \pm d\sqrt{5}$, where c and d are integers.

$$x^2 - 8x - 29 = (x - 4)^2 - 45 = 0$$

(1)

$$(x - 4)^2 - (3\sqrt{5})^2 = 0$$

$$(x - 4 - 3\sqrt{5})(x - 4 + 3\sqrt{5}) = 0$$

(2)

$$x = 4 \pm 3\sqrt{5} \quad \left(\frac{1}{2}\right) \text{ as required.}$$

$$c = 4$$

$$d = 3$$

(2 + 2 = 4 marks)

Question 2

Find the first term and common difference of an arithmetic sequence whose 16th term is 22, where the sum of the first 6 terms is 57.

$$t_{16} = a + 15d = 22 \Rightarrow 15d = 22 - a$$

(1)

$$S_6 = \frac{6}{2}(2a + 5d) = 57$$

$$3(2a + 5d) = 57$$

$$6a + 15d = 57$$

$$6a + 22 - a = 57$$

$$5a = 35$$

$$a = 7$$

(1)

$$15d = 22 - a$$

$$= 22 - 7$$

$$d = \frac{15}{15}$$

$$= 1$$

(1)

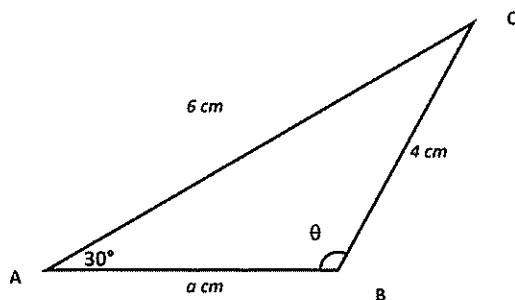
$$\begin{array}{l} \text{First term} = 7 \\ \text{Difference} = 1 \end{array}$$

(3 marks)

Question 3

In triangle ABC side AC is of length 6 cm, side BC is of length 4 cm and $\angle BAC = 30^\circ$. Let $\angle ABC = \theta^\circ$ and let side AB be of length a cm.

"Diagram **not** to scale"



- a) Use the sine rule in triangle ABC to evaluate $\sin \theta$.

$$\frac{\sin \theta}{6} = \frac{\sin 30^\circ}{4} \quad (1)$$

$$\begin{aligned} \sin \theta &= \frac{6}{4} \times \frac{1}{2} \\ &= \frac{3}{4} \end{aligned} \quad (1)$$

- b) Hence, obtain two possible values for $\cos \theta$ in triangle ABC.

$$\begin{aligned} \cos^2 \theta + \sin^2 \theta &= 1 \\ \cos^2 \theta &= 1 - \sin^2 \theta \\ &= 1 - \frac{9}{16} \\ &= \frac{7}{16} \\ \cos \theta &= \pm \frac{\sqrt{7}}{4} \end{aligned} \quad \left. \begin{array}{l} (1) \\ (1) \end{array} \right\}$$

(2 + 2 = 4 marks)

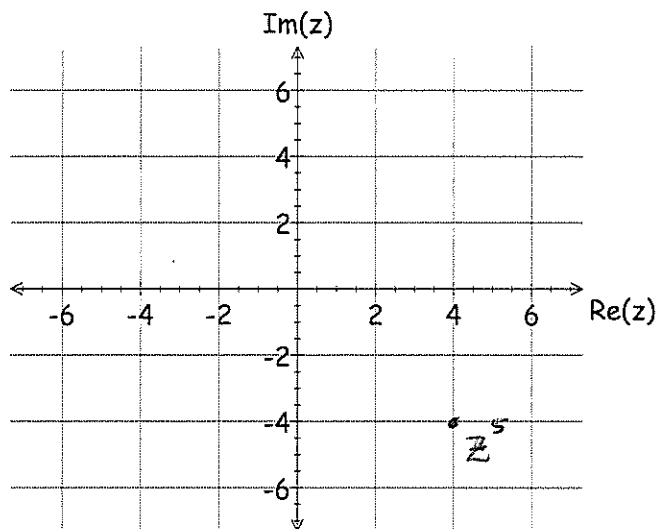
Question 4

Let $z = \sqrt{2} \operatorname{cis} \left(\frac{3\pi}{4} \right)$

- (a) Use De Moivre's theorem to evaluate z^5 . Express your answer in Cartesian form.

$$\begin{aligned} z^5 &= (\sqrt{2})^5 \operatorname{cis} \left(5 \times \frac{3\pi}{4} \right) \quad (1) \\ &= 4\sqrt{2} \operatorname{cis} \left(\frac{15\pi}{4} \right) \quad \left(\frac{1}{2} \right) \\ &= 4\sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4} \right) \\ &= \left(4\sqrt{2} \times \frac{1}{\sqrt{2}} \right) + \left(4\sqrt{2} \times -\frac{1}{\sqrt{2}} \right) i \quad \left(\frac{1}{2} \right) \\ &= 4 - 4i \quad (1) \end{aligned}$$

(b) Plot z^5 on the Argand diagram below.



(3 + 1 = 4 marks)

Question 5

(a) Show that the matrix $\begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}$ is its own inverse.

$$\text{Let } M = \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix} \quad \det(M) = -4 + 3 = -1 \quad (1)$$

$$M^{-1} = - \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix} \quad (1)$$

$$= M$$

as required.

OR $M \cdot M^{-1} = I$

$$\begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore M = M^{-1}$$

as required.

(b) Determine an equation that shows how the values of a , b and c are related if a matrix of the form

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \text{ is such that } \begin{bmatrix} a & b \\ c & -a \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}^2 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + bc & ab - ab \\ ac - ac & bc + a^2 \end{bmatrix} \quad (1)$$

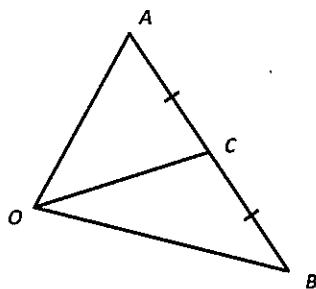
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore a^2 + bc = 1 \quad (1)$$

(2 + 2 = 4 marks)

Question 6

In the triangle AOB point C is the midpoint of side AB as shown in the diagram below:



Let $\underline{a}$ be the vector representing the side OA and $\underline{c}$ be the vector representing the median OC .

- (a) Show that $\vec{OB}$ is represented by the vector $2\underline{c} - \underline{a}$.

$$\begin{aligned}
 \vec{OB} &= \vec{OA} + \vec{AB} & (1) \\
 &= \underline{a} + 2\vec{AC} \\
 &= \underline{a} + 2(\underline{c} - \underline{a}) & (1) \\
 &= 2\underline{c} - \underline{a} \quad \text{as required.}
 \end{aligned}$$

- (b) Express $\vec{OB} \cdot \vec{OB}$ in terms of $\underline{a} \cdot \underline{a}$, $\underline{c} \cdot \underline{c}$ and $\underline{a} \cdot \underline{c}$

$$\begin{aligned}
 \vec{OB} \cdot \vec{OB} &= (2\underline{c} - \underline{a}) \cdot (2\underline{c} - \underline{a}) & (1) \\
 &= 4\underline{c} \cdot \underline{c} - 2\underline{c} \cdot \underline{a} - 2\underline{c} \cdot \underline{a} + \underline{a} \cdot \underline{a} & (1) \\
 &= 4\underline{c} \cdot \underline{c} - 4\underline{a} \cdot \underline{c} + \underline{a} \cdot \underline{a}
 \end{aligned}$$

- (c) Hence prove that if triangle AOB is isosceles with side OA of equal length to side OB , then the median OC will be perpendicular to side AB .

$$\begin{aligned}
 |\vec{OB}| &= |\vec{OA}| & (1) \\
 |\vec{OB}|^2 &= |\vec{OA}|^2 \\
 4\underline{c} \cdot \underline{c} - 4\underline{a} \cdot \underline{c} + \underline{a} \cdot \underline{a} &= \underline{a} \cdot \underline{a} & (1) \\
 4\underline{c} \cdot \underline{c} - 4\underline{a} \cdot \underline{c} &= 0 & (1) \\
 \underline{c} \cdot \underline{c} - \underline{a} \cdot \underline{c} &= 0 & (1) \\
 \underline{c} \cdot (\underline{c} - \underline{a}) &= 0 & (1) \\
 \underline{c} \perp (\underline{c} - \underline{a}) & \text{ Hence Median } \vec{OC} \perp \vec{AB} & (1) \\
 \therefore \triangle AOB \text{ isosceles} & & (1)
 \end{aligned}$$

(2+2+3=7 marks)

Question 7

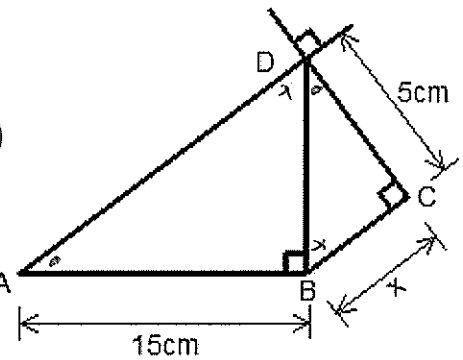
(a) Show that $\triangle ABD \sim \triangle BCD$.

$$\begin{aligned} \angle ABD &= \angle BCD = 90^\circ \text{ (given)} \\ \angle ADB &= \angle DBC \text{ (alternate angles)} \\ \angle BAD &= \angle BDC \text{ (sum of } \Delta \text{ add to } 180^\circ) \end{aligned}$$

$$\therefore \triangle ABD \sim \triangle BCD \text{ (AAA)}$$

as required

must state reason



(b) Calculate the value of x . Express the answer with a rationalized denominator.

$$\begin{aligned} BD &= \sqrt{x^2 + 25} \\ \frac{x}{\sqrt{x^2 + 25}} &= \frac{5}{15} = \frac{1}{3} \\ \frac{x^2}{x^2 + 25} &= \frac{1}{9} \\ 9x^2 - x^2 &= 25 \\ 8x^2 &= 25 \\ x &= \pm \frac{5}{2\sqrt{2}} \end{aligned}$$

As $x > 0$
 $\therefore x = \frac{5\sqrt{2}}{4}$

OR $\frac{5\sqrt{2}}{4}$ (2+4=6 marks)

Question 8

Suppose a fair "six sided" die is rolled three times and the number of "sixes" observed.

(a) What is the probability p that a "six" is observed on a single roll of the dice?

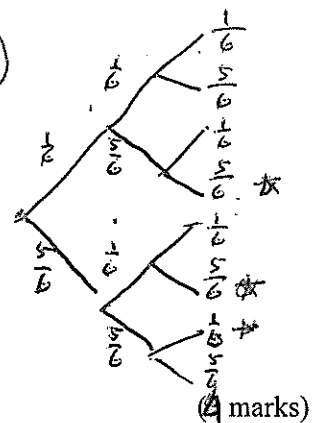
$$p(X=1) = \frac{1}{6}$$

(b) What are the possible values of the sample proportion $\hat{p}$ of "sixes" in the sample?

$$\hat{p} = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$$

(c) What is the probability that the sample proportion is equal to $\frac{1}{3}$.

$$\begin{aligned} Pr(\hat{p} = \frac{1}{3}) &= \left(\frac{25}{216}\right) + \left(\frac{25}{216}\right) + \left(\frac{25}{216}\right) \\ &= \frac{75}{216} \\ &= \frac{25}{72} \end{aligned}$$



Question 9

- (a) A sample of n people were asked whether education was important, and 80% said yes. What is the value of the sample proportion $\hat{p}$ for this sample?

$$\hat{p} = 0.8$$

- (b) A second sample of m people were asked whether education was important, and 95% said yes. What is the value of the sample proportion $\hat{p}$ for this sample?

$$\hat{p} = 0.95$$

- (c) Determine the sample proportion based on the combined results of both samples in terms of n and m .

$$\hat{p} = \frac{0.8n + 0.95m}{n+m} \quad \text{or} \quad \frac{16n + 19m}{20n + 20m}$$

(1+1+1=3 marks)

Question 10

A body initially at a point 4 metres to the right of the origin, undergoes constant acceleration resulting in the following velocity-time diagram.

- (a) By first deriving an equation for the velocity of the body v m/s at any time t seconds, $t \geq 0$, show that the position of the body x metre at any time t is given by the equation

$$x = -t^2 + 10t + 4$$

$$a(t) = -2$$

$$v(t) = \int -2 dt = -2t + C$$

$$\text{When } t=0, v=0 \Rightarrow C=10$$

$$\therefore v(t) = -2t + 10$$

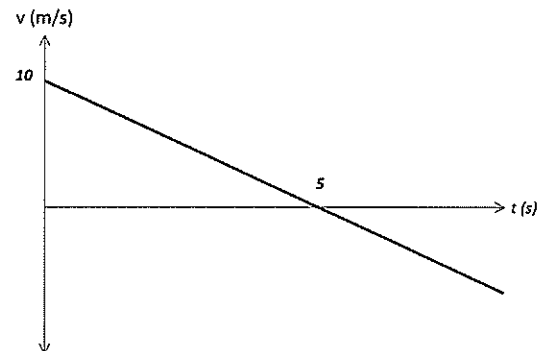
$$x(t) = \int -2t + 10 dt$$

$$= -t^2 + 10t + C$$

$$\text{At } t=0, x=4$$

$$4 = 0 + 0 + C$$

$$\therefore C=4$$



$$\therefore x(t) = -t^2 + 10t + 4 \text{ as required.}$$

(b) Calculate the average velocity over the first 8 seconds of motion.

$$\begin{aligned} \text{Ave. Velocity} &= \frac{x(8) - x(0)}{8} \quad (1) \\ &= \frac{(-64 + 80) - 0}{8} \\ &= \frac{16}{8} \\ &= 2 \text{ m/s.} \quad (1) \end{aligned}$$

(3 + 2 = 5 marks)

Question 11

The velocity, v m/s of a body moving in a straight line is given by $v(t) = 24 + 6t - 3t^2$, $t \geq 0$ where t is measured in seconds. If the body is initially 5m to the right of the origin, find:

(a) the acceleration at any time, t

$$a(t) = 6 - 6t, \quad t \geq 0 \quad (1)$$

(b) the position at any time, t

$$x(t) = 24t + 3t^2 - t^3 + C \quad (1)$$

$$\text{At } t=0, x=5 \Rightarrow C=5 \quad \left(\frac{1}{2}\right)$$

$$\therefore x(t) = 24t + 3t^2 - t^3 + 5, \quad t \geq 0 \quad \left(\frac{1}{2}\right)$$

(c) the time when the body is stationary

when $v=0$

$$\left. \begin{aligned} 24 + 6t - 3t^2 &= 0 \\ (t-4)(t+2) &= 0 \end{aligned} \right\} \quad \left(\frac{1}{2}\right)$$

$$t = 4 \text{ or } t = -2 \quad \left(\frac{1}{2}\right)$$

$$\text{As } t \geq 0 \quad \therefore t = 4 \text{ seconds.} \quad (1)$$

(d) the position when it is stationary

$$\begin{aligned} x(4) &= (24 \times 4) + (3 \times 4^2) - 4^3 + 5 \\ &= 96 + 48 - 64 + 5 \\ &= 144 - 59 \\ &= 85 \text{ m.} \end{aligned} \quad \left. \right\} \left(\frac{1}{2}\right) \quad \left(\frac{1}{2}\right)$$

(1+2+2+1=6 marks)

End of Exam