

Application Task Part 2 – Complex Numbers & Trigonometry

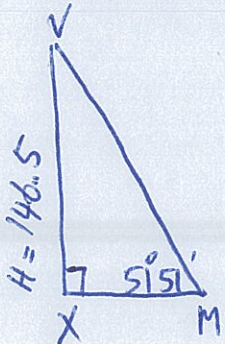
Name:

Solutions

Question 1 (11 Marks)

The pyramid of Cheops had a square base and a height of 146.5 metres when constructed and the angle of the sloping face to the horizontal was $51^\circ 51'$.

a) Calculate the side length of the square base. (Answer to two decimal places)



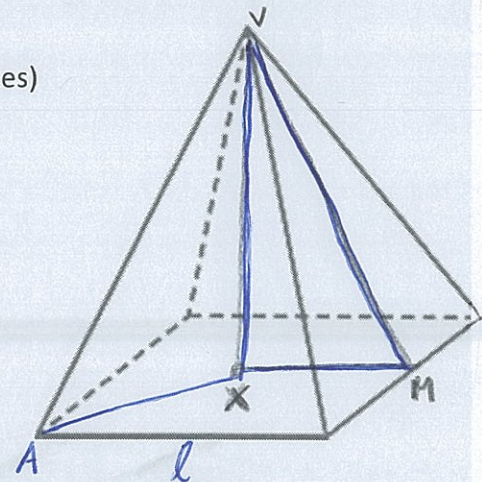
$$\overline{XM} = \frac{146.5}{\tan 51^\circ 51'}$$

(1/2)^M

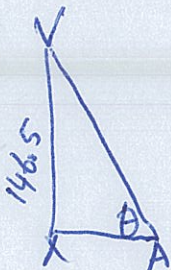
$$\begin{aligned} \text{sidelength } (l) &= 2 \times \overline{XM} \\ &= 2 \times \frac{146.5}{\tan 51^\circ 51'} \\ &\approx 230.15 \text{ m} \end{aligned}$$

(1/2)^M(1)^A

-1/2 m if not 2.d.p.



b) Calculate the angle of the sloping edge to the horizontal (Answer to the nearest minute).



$$\overline{XA} = \frac{1}{2} \times \text{diagonal of base.}$$

$$= \frac{1}{2} \times \sqrt{2} l = \frac{1}{2} \times \sqrt{2} \times 230.15$$

(1)^M

$$\theta = \tan^{-1} \left(\frac{146.5}{\overline{XA}} \right)$$

(1)^M

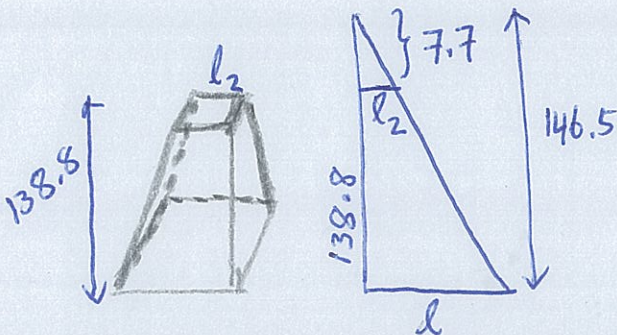
$$= 41.99315996$$

$$\approx 42^\circ 0'$$

(1)^A

deduct 1/2 m if not nearest to minute

c) Since construction the top of the pyramid has worn away so that the top is now a horizontal square and the new height of the truncated pyramid is 138.8 metres. Find the side length of the top. (Answer to two decimal places). Hint: use the concept of similarity.



Using similarity.

$$\frac{l}{146.5} = \frac{l_2}{7.7}$$

(1)^M

$$l_2 = \frac{l \times 7.7}{146.5}$$

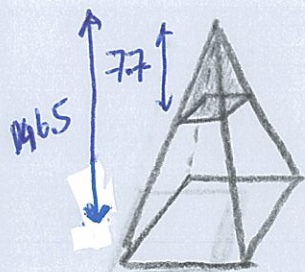
(1)^M

$$= 12.09686$$

$$\approx 12.10 \text{ m}$$

(1)^A

- d) (i) Find the ratio of the volume of the smaller pyramid (top part) to the volume of the original pyramid, given that the ratio of volume between two similar shapes equals to the cube of the ratio of side lengths. (Answer to five decimal places).



$$\begin{aligned}\frac{\text{Top Pyramid}}{\text{Original}} &= \frac{V_T}{V_O} = \left(\frac{H_T}{H_O}\right)^3 \\ &= \left(\frac{7.7}{146.5}\right)^3 \\ &= 1.45198 \times 10^{-4} \\ &(\approx 0.00015)\end{aligned}$$

(1)^M

(1)^A

- (ii) Hence, show that the volume ratio of the truncated pyramid of today to the original volume of the pyramid is 0.99985.

$$\begin{aligned}\frac{\text{Vol. Truncated Pyramid}}{\text{Vol. Original}} &= \frac{\text{Vol. Original} - \text{Vol. Top Pyramid}}{\text{Vol. Original}} \\ &= 1 - \frac{V_T}{V_O} \\ &= 1 - 1.45198 \times 10^{-4} \\ &= 0.99985\end{aligned}$$

(1)^M

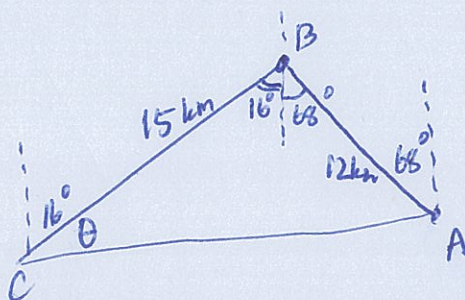
(1)^A

(2 + 3 + 3 + 3 = 11 marks)

Question 2 (7 Marks)

An army division on exercises starts from Point A and walks 12 km on a true bearing of 292° to Point B and then 15 km on a true bearing of 196° to Point C.

- a) Draw a diagram showing all three points and label all relevant lengths and angles.



$$\angle ABC = 84^\circ$$

1m labelled points
1m relatively accurate
1m labelled angles

b) What distance will they walk from Point C to Point A? (Answer to two decimal places)

$$AC^2 = 15^2 + 12^2 - 2 \times 15 \times 12 \times \cos(16^\circ + 68^\circ)$$

$$= 331.3697532$$

$$AC = 18.20 \text{ km}$$

c) The army division then walks directly from Point C to Point A. On what true bearing should they travel? (Answer to two decimal places)

$$\frac{12}{\sin \theta} = \frac{AC}{\sin 84^\circ}$$

$$\theta = \sin^{-1} \left(\frac{12 \times \sin 84^\circ}{AC} \right)$$

$$= \sin^{-1} \left(\frac{12 \times \sin 84^\circ}{18.20} \right)$$

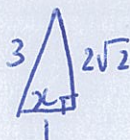
$$\approx 40.97^\circ$$

(3 + 2 + 2 = 7 marks)

Question 3 (9 marks)

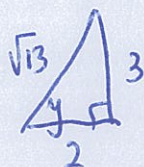
If $\sec(x) = 3$, $0 < x < \frac{\pi}{2}$ and $\cot(y) = -\frac{2}{3}$, $\frac{\pi}{2} < y < \pi$
 Find: (quadrant I) (quadrant 4)

a) $\cos(x)$



$$\therefore \cos(x) = \frac{1}{3}$$

b) $\sin(y)$



$$\therefore \sin(y) = \frac{3}{\sqrt{13}} \quad \text{or} \quad \frac{3\sqrt{13}}{13}$$

c) $\cos(x+y)$

$$\begin{aligned}\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\ &= \left(\frac{1}{3} \times \frac{-2}{\sqrt{13}}\right) - \left(\frac{2\sqrt{2}}{3} \times \frac{3}{\sqrt{13}}\right) \\ &= \frac{-2 - 6\sqrt{2}}{3\sqrt{13}}\end{aligned}$$

(1)^M

(1)^A

d) $\operatorname{cosec}(2x)$

$$\begin{aligned}\operatorname{cosec}(2x) &= \frac{1}{\sin(2x)} = \frac{1}{2\sin x \cos x} \\ &= \frac{1}{2 \times \frac{2\sqrt{2}}{3} \times \frac{1}{3}} \\ &= \frac{9}{4\sqrt{2}}\end{aligned}$$

(1)^M

(1)^A

e) $\tan(x+2y)$

$$\begin{aligned}\tan(x+2y) &= \frac{\tan(x) + \tan(2y)}{1 - \tan(x)\tan(2y)} \\ &= \frac{2\sqrt{2} + \frac{12}{5}}{1 - \left(2\sqrt{2} \times \frac{12}{5}\right)} \\ &= \frac{\frac{10\sqrt{2} + 12}{5}}{1 - \left(\frac{24\sqrt{2}}{5}\right)} \\ &= \frac{10\sqrt{2} + 12}{5 - 24\sqrt{2}}\end{aligned}$$

($\frac{1}{2}$)^M

($\frac{1}{2}$)^M

(1)^A

Note:

$$\begin{aligned}\tan(2y) &= \frac{2\tan(y)}{1 - \tan^2(y)} \\ &= \frac{-3}{1 - \left(\frac{3}{2}\right)^2} \\ &= \frac{12}{5}\end{aligned}$$

(1)^A

(1 + 1 + 2 + 2 + 3 = 9 marks)

Question 4 (13 marks)

If $z_1 = 1 - \sqrt{3}i$ and $z_2 = 4 + 4i$

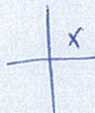
a) Express z_1 and z_2 in polar form

$$z_1 = 1 - \sqrt{3}i$$


$$r = 2$$

$$\theta = \tan^{-1}(\sqrt{3})$$

$$\therefore z_1 = 2 \operatorname{cis}\left(-\frac{\pi}{3}\right) \quad (1)^A$$

$$z_2 = 4 + 4i$$


$$r = 4\sqrt{2}$$

$$\theta = \tan^{-1}(1)$$

$$\therefore z_2 = 4\sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right) \quad (1)^A$$

b) Find $\frac{z_1}{z_2}$ in polar form

$$\frac{z_1}{z_2} = \frac{2 \operatorname{cis}\left(-\frac{\pi}{3}\right)}{4\sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)}$$

$$= \frac{1}{2\sqrt{2}} \operatorname{cis}\left(-\frac{\pi}{3} - \frac{\pi}{4}\right) \quad (1)^M$$

$$= \frac{1}{2\sqrt{2}} \operatorname{cis}\left(-\frac{7\pi}{12}\right) \quad (1)^A$$

c) If $w_1 = 3 + i$ and $w_2 = 4 + 5i$, find $\frac{w_1}{w_2}$ in Cartesian form.

$$\frac{w_1}{w_2} = \frac{3+i}{4+5i} \times \frac{4-5i}{4-5i} \quad (1)^M$$

$$= \frac{12 - 15i + 4i + 5}{16 + 25}$$

$$= \frac{17 - 11i}{41} \quad (1)^A$$

d) Find w_1^2 using Cartesian form.

$$w_1^2 = (3+i)^2$$

$$= 9 + 6i - 1 \quad (1)^M$$

$$= 8 + 6i \quad (1)^A$$

e) By expressing w_1 in polar form show that $w_1^2 = 10 \operatorname{cis}(2 \times \tan^{-1}(\frac{1}{3}))$.

$$w_1 = 3+i$$

$$= \sqrt{10} \operatorname{cis}(\tan^{-1}(\frac{1}{3}))$$

$$r = \sqrt{9+1} = \sqrt{10} \quad (1)$$

$$\theta = \tan^{-1}(\frac{1}{3}) \quad (1)$$

$$w_1^2 = 10 \operatorname{cis}(2 \times \tan^{-1}(\frac{1}{3})) \quad \text{as required.}$$

f) By substituting $\theta = \tan^{-1}(\frac{1}{3})$ and therefore $\tan(\theta) = \frac{1}{3}$ use a double angle formula to find w_1^2 in polar form and confirm that it is the same as your answer to (d) in Cartesian form.

$$\left. \begin{array}{l} \theta = \tan^{-1}(\frac{1}{3}) \\ \tan \theta = \frac{1}{3} \end{array} \right\} w_1^2 = 10 \operatorname{cis}(2\theta)$$

$$\tan(2\theta) = \frac{2 \times \tan \theta}{1 - \tan^2 \theta}$$

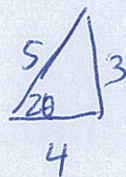
$$= \frac{2 \times \frac{1}{3}}{1 - (\frac{1}{3})^2}$$

(1)^M

$$= \frac{\frac{2}{3}}{1 - \frac{1}{9}}$$

$$= \frac{\frac{2}{3} \times \frac{9}{8}}{\frac{8}{8}}$$

$$= \frac{3}{4} \quad (1)^A$$



(2 + 2 + 2 + 2 + 2 + 3 = 13 marks)

$$w_1^2 = 10 \operatorname{cis}(2\theta)$$

$$= 10 (\cos(2\theta) + i \sin(2\theta))$$

(1/2)^M

$$= 10 \left(\frac{4}{5} + \frac{3}{5}i \right)$$

(1/2)^A

$$= 8 + 6i \quad \text{as required.}$$