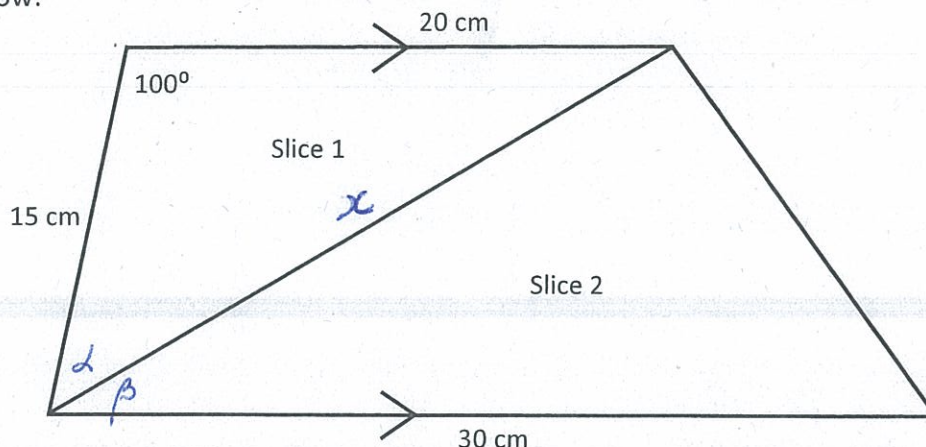


Application Task Part 1 – Complex Numbers & Trigonometry

Name: Solutions

Question 1 (7 Marks) – Give all answers in this question to two decimal places

A pizza shop is experimenting with a new trapezium shaped pizza cut in two pieces as shown in the diagram below:



a) Find the area of Slice 1 in cm^2 .

$$A = \frac{1}{2} \times 15 \times 20 \times \sin 100^\circ$$

$$= 147.72 \text{ cm}^2$$

(1)^M
(1)^A

b) Find the length of the side common to both slices.

$$x^2 = 20^2 + 15^2 - 2 \times 20 \times 15 \times \cos 100^\circ$$

$$= 729.1889066$$

$$x = 27.00 \text{ cm}$$

(1)^M
(1)^A

c) Find the area of Slice 2 in cm^2 .

$$\frac{20}{\sin \alpha} = \frac{x}{\sin 100^\circ}$$

$$\sin \alpha = \left(\frac{20 \sin 100^\circ}{x} \right)$$

(1/2)^M

$$\alpha = 46.83550831^\circ$$

$$\beta = 80 - \alpha = 33.16449169^\circ$$

(1/2)^M

(1/2)^A

$$A = \frac{1}{2} \times 30 \times x \times \sin \beta$$

$$= 221.58 \text{ cm}^2$$

(1/2)^M

(1)^A

(2 + 2 + 3 = 7 marks)

Question 2 (6 marks)

a) Use a compound angle formula to expand $\sin(x + \alpha)$

$$\sin(x + \alpha) = \sin(x) \cos(\alpha) + \cos(x) \sin(\alpha)$$

b) If $y = a \sin(x) + b \cos(x)$ can be expressed as $y = r \sin(x + \alpha)$, use your previous answer to find expressions for a and b .

$$a = r \cos(\alpha) \quad \left(\frac{1}{2}\right)^A$$

$$b = r \sin(\alpha) \quad \left(\frac{1}{2}\right)^A$$

c) Hence show that $r^2 = a^2 + b^2$ and $\tan(\alpha) = b/a$

$$a^2 + b^2 = (r \cos \alpha)^2 + (r \sin \alpha)^2 \quad (1)^M$$

$$= r^2 [\cos^2 \alpha + \sin^2 \alpha] \quad (1)^A$$

$$\therefore a^2 + b^2 = r^2 \quad \text{QED}$$

$$\begin{aligned} \tan \alpha &= \frac{\sin \alpha}{\cos \alpha} \\ &= \frac{b/r}{a/r} \quad (1)^M \\ &= \frac{b}{a} \quad \text{QED} \end{aligned}$$

$$\begin{aligned} r \sin \alpha &= b \\ \sin \alpha &= \frac{b}{r} \quad \left(\frac{1}{2}\right)^A \end{aligned}$$

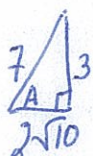
$$\begin{aligned} r \cos \alpha &= a \\ \cos \alpha &= \frac{a}{r} \quad \left(\frac{1}{2}\right)^A \end{aligned}$$

(1 + 1 + 4 = 6 marks)

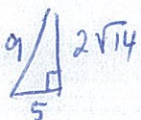
Question 3 (11 Marks) Give answer in exact values.

Given that: $\sin(A) = \frac{3}{7}$, $0 < A < \frac{\pi}{2}$ and $\cos(B) = \frac{5}{9}$, $\frac{3\pi}{2} < B < 2\pi$

a) Calculate $\cos(A)$ and $\sin(B)$.



$$\cos(A) = \frac{2\sqrt{10}}{7} \quad (1)^A$$



$$\sin(B) = -\frac{2\sqrt{14}}{9} \quad (1)^A$$

b) Calculate $\sin(2A)$ and $\cos(2A)$.

$$\begin{aligned}\sin(2A) &= 2 \sin(A) \cos(B) \\ &= 2 \times \frac{3}{7} \times \frac{2\sqrt{10}}{7} \\ &= \frac{12\sqrt{10}}{49}\end{aligned}$$

$$\begin{aligned}\cos(2A) &= 1 - 2\sin^2(A) \\ &= 1 - 2 \times \frac{9}{49} \\ &= 1 - \frac{18}{49} \\ &= \frac{31}{49}\end{aligned}$$

c) Hence calculate $\sin(2A+B)$.

$$\begin{aligned}\sin(2A+B) &= \sin(2A) \cos(B) + \cos(2A) \sin(B) \\ &= \left(\frac{12\sqrt{10}}{49} \times \frac{5}{9} \right) + \left(\frac{31}{49} \times \frac{-2\sqrt{14}}{9} \right) \\ &= \frac{20\sqrt{10}}{147} - \frac{62\sqrt{14}}{441} \\ &= \frac{60\sqrt{10} - 62\sqrt{14}}{441}\end{aligned}$$

d) Using $z_1 = \cos(A) + i\sin(A)$ and $z_2 = \cos(B) + i\sin(B)$ show that $z_1^2 z_2 = \cos(2A+B) + i\sin(2A+B)$

$$\begin{aligned}z_1^2 &= \cos(2A) + i\sin(2A) \\ z_2 &= \cos(B) + i\sin(B)\end{aligned}$$

$$z_1^2 z_2 = \cos(2A) \cos(B) - \sin(2A) \sin(B) + i[\sin(2A) \cos(B) + \cos(2A) \sin(B)]$$

$$= \cos(2A+B) + i\sin(2A+B)$$

e) By using z_1 and z_2 in cartesian form evaluate $z_1^2 z_2$.

$$\begin{aligned}z_1 &= \cos A + i\sin A \\ &= \frac{2\sqrt{10}}{7} + \frac{3}{7}i\end{aligned}$$

$$\begin{aligned}z_2 &= \cos B + i\sin B \\ &= \frac{5}{9} - \frac{2\sqrt{14}}{9}i\end{aligned}$$

$$\begin{aligned}z_1^2 z_2 &= \left(\frac{2\sqrt{10}}{7} + \frac{3}{7}i \right)^2 \left(\frac{5}{9} - \frac{2\sqrt{14}}{9}i \right) \\ &= \frac{16\sqrt{35}}{147} + \frac{155}{441} - \frac{62\sqrt{14}}{441}i + \frac{20\sqrt{10}}{149}i \\ &= \frac{48\sqrt{35} + 155}{441} - \frac{62\sqrt{14} - 60\sqrt{10}}{441}i\end{aligned}$$

f) Compare your answers to (c) and (e). Comment.

$$\text{Im}(z_1^2 z_2) = \sin(2A+B)$$

(2 + 2 + 2 + 1 + 3 + 1 = 11 marks)

Question 4 (6 marks)

Let $z_1 = 2 - 2i$ and $z_2 = 3 + 3\sqrt{3}i$

a) Express z_1 and z_2 in polar form

$$z_1 = 2 - 2i$$

$$r = 2\sqrt{2}$$

$$\theta = \tan^{-1}(-1) = -\frac{\pi}{4}$$

$$\therefore z_1 = 2\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right) \text{ (1)}$$

$$z_2 = 3 + 3\sqrt{3}i$$

$$r = \sqrt{9 + 27} = 6$$

$$\theta = \tan^{-1}\left(\frac{3\sqrt{3}}{3}\right) = \frac{\pi}{3}$$

$$\therefore z_2 = 6 \operatorname{cis}\left(\frac{\pi}{3}\right) \text{ (1)}$$

b) (i) Express z_1^n in polar form

$$z_1^n = \left(2\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)\right)^n = (2\sqrt{2})^n \operatorname{cis}\left(-\frac{n\pi}{4}\right) \text{ (1)}$$

(ii) Express kz_2^n in polar form

$$kz_2^n = k \left(6 \operatorname{cis}\frac{\pi}{3}\right)^n = k \left(6^n \operatorname{cis}\left(\frac{n\pi}{3}\right)\right) \text{ (1)}$$

(iii) Hence, if n is a positive whole number find all possible values of n such that: $z_1^n = kz_2^n$, $k \in \mathbb{R}$

$$(2\sqrt{2})^n \operatorname{cis}\left(-\frac{n\pi}{4}\right) = k 6^n \operatorname{cis}\left(\frac{n\pi}{3}\right) \text{ (1/2)}^m$$

$$-\frac{n\pi}{4} = 2m\pi + \frac{n\pi}{3}$$

$$-n\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = 2m\pi$$

$$-\frac{7n\pi}{12} = 2m\pi$$

$$n = \frac{24m}{7}, m \in \mathbb{Z} \text{ (1)}^A$$

OR

$$2m\pi - \frac{n\pi}{4} = \frac{n\pi}{3}$$

$$2m\pi = n\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$$

$$2m\pi = \frac{7n\pi}{12}$$

$$n = \frac{24m}{7}, m \in \mathbb{Z}$$

$$\text{OR } (2\sqrt{2})^n \operatorname{cis}\left(-\frac{n\pi}{4}\right) = k 6^n \operatorname{cis}\left(\frac{n\pi}{3}\right)$$

$$\frac{(2\sqrt{2})^n}{k 6^n} = \frac{\operatorname{cis}\left(\frac{n\pi}{3}\right)}{\operatorname{cis}\left(-\frac{n\pi}{4}\right)}$$

$$\frac{(2\sqrt{2})^n}{k 6^n} = \operatorname{cis}\left(\frac{n7\pi}{12}\right)$$

Imaginary part equals zero

$$\sin\left(\frac{n7\pi}{12}\right) = 0$$

$$\frac{n7\pi}{12} = 2m\pi$$

$$n = \frac{24m}{7}$$

(2 + 4 = 6 marks)

Question 5 (10 marks)

a) Show that the solutions for $z^5 = -i$ are:

$$z_1 = \text{cis}\left(-\frac{\pi}{10}\right) \quad z_2 = \text{cis}\left(\frac{3\pi}{10}\right) \quad z_3 = \text{cis}\left(\frac{7\pi}{10}\right) \quad z_4 = \text{cis}\left(-\frac{\pi}{2}\right) \quad z_5 = \text{cis}\left(-\frac{9\pi}{10}\right)$$

$\frac{1}{2}$ $z^5 = r^5 \text{cis}(5\theta)$

$\frac{1}{2}$ $-i = \text{cis}\left(-\frac{\pi}{2}\right)$

Hence,

$$r^5 \text{cis}(5\theta) = \text{cis}\left(-\frac{\pi}{2}\right)$$

$$r^5 = 1$$

$$r = 1$$

$$5\theta = -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{-5\pi}{2}, \frac{-9\pi}{2}$$

$$\theta = -\frac{\pi}{10}, \frac{3\pi}{10}, \frac{7\pi}{10}, \frac{-5\pi}{10}, \frac{-9\pi}{10}$$

$\frac{1}{2}$

$\frac{1}{2}$

$$\therefore z_1 = \text{cis}\left(-\frac{\pi}{10}\right)$$

$$z_2 = \text{cis}\left(\frac{3\pi}{10}\right)$$

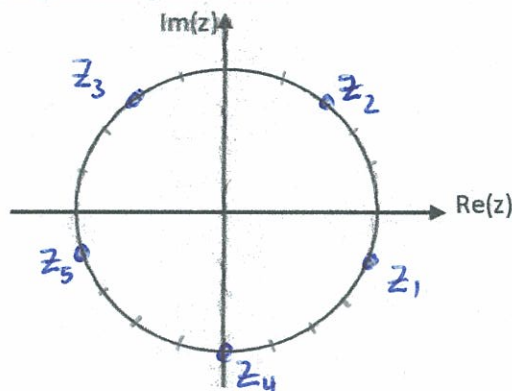
$$z_4 = \text{cis}\left(\frac{7\pi}{10}\right)$$

$$z_4 = \text{cis}\left(-\frac{5\pi}{10}\right)$$

$$z_5 = \text{cis}\left(-\frac{9\pi}{10}\right)$$

as required.

b) Plot these solutions on the argand diagram below.



1 mark - labels
1 mark - accuracy (location)
1 mark - all 5 points shown

c) Let the solutions of the equation be z_1, z_2, z_3, z_4, z_5 .

Find the product of the solutions: $z_1 \times z_2 \times z_3 \times z_4 \times z_5$ in polar form and cartesian form

$$z_1 \cdot z_2 \cdot z_3 \cdot z_4 \cdot z_5 = \text{cis}\left(-\frac{\pi}{10} + \frac{3\pi}{10} + \frac{7\pi}{10} - \frac{5\pi}{10} - \frac{9\pi}{10}\right) \quad \textcircled{1}$$

$$= \text{cis}\left(-\frac{5\pi}{10}\right)$$

$$= \text{cis}\left(-\frac{\pi}{2}\right)$$

$$= -i$$

$\textcircled{1}$

d) Find the five solutions to the equation: $z^5 = \text{cis}(\alpha)$ in terms of α .

$$r^5 \text{cis}(5\theta) = \text{cis}(\alpha)$$

$$r^5 = 1$$

$$r = 1$$

$$5\theta = \alpha, \alpha + 2\pi, \alpha + 4\pi, \alpha - 2\pi, \alpha - 4\pi$$

$$= \frac{\alpha}{5}, \frac{\alpha + 2\pi}{5}, \frac{\alpha + 4\pi}{5}, \frac{\alpha - 2\pi}{5}, \frac{\alpha - 4\pi}{5}$$

$$\therefore z_1 = \text{cis}\left(\frac{\alpha}{5}\right)$$

$$z_3 = \text{cis}\left(\frac{\alpha + 4\pi}{5}\right)$$

$$z_5 = \text{cis}\left(\frac{\alpha - 4\pi}{5}\right)$$

$$z_2 = \text{cis}\left(\frac{\alpha + 2\pi}{5}\right)$$

$$z_4 = \text{cis}\left(\frac{\alpha - 2\pi}{5}\right)$$

e) Find the product of these five solutions: $z_1 \times z_2 \times z_3 \times z_4 \times z_5$

$$z_1 \cdot z_2 \cdot z_3 \cdot z_4 \cdot z_5 = \text{cis}\left(\frac{\alpha + \alpha + 2\pi + \alpha + 4\pi + \alpha - 2\pi + \alpha - 4\pi}{5}\right)$$

$$= \text{cis}\left(\frac{5\alpha}{5}\right)$$

$$= \text{cis}(\alpha)$$

f) Compare your results in (c) and (e) and comment.

Product of the solutions equals to z^5

OR

Product of the solutions gives the right-hand side of the equation.

(2 + 3 + 2 + 2 + 2 + 1 = 10 marks)