

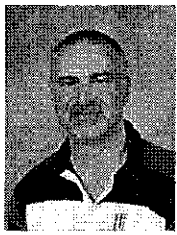


Glen Waverley Secondary College
2016 Semester 2: Technology Free exam

UNIT 2 Specialist Maths

Teachers:

Please circle

		
CAL	GMN	JLS

Student Name: *Solutions*

Student Code: **Form Group:**

Instructions:

Reading Time:	15 minutes
Writing Time:	60 minutes

Total Marks	50 marks
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This examination consists of 12 short answer questions.

For each question correct working must be shown to gain full marks

Neither calculators nor summary notes of any type are permitted.

A formula sheet is provided

Question 1

Express $\frac{2x+3}{(x-3)^2}$ as partial fractions

$$\frac{2x+3}{(x-3)^2} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2} \quad \left(\frac{1}{2}\right)$$

$$2x+3 = A(x-3) + B \quad \left(\frac{1}{2}\right)$$

$$\text{let } x=3 \quad 9=B \quad \left(\frac{1}{2}\right)$$

$$\text{let } x=0 \quad 3=-3A+B$$

$$3=-3A+9$$

$$3A=6$$

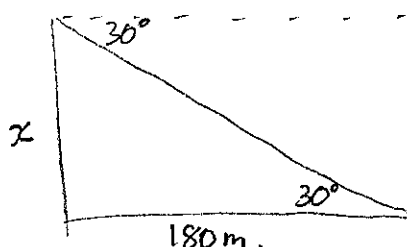
$$A=2 \quad \left(\frac{1}{2}\right)$$

$$\therefore \frac{2x+3}{(x-3)^2} = \frac{2}{(x-3)} + \frac{9}{(x-3)^2} \quad (1)$$

(3 marks)

Question 2

From a point on a cliff x m above sea level, the angle of depression to a boat is 30° . The distance from the foot of the cliff to the boat is 180 metres. Find the height of the cliff.



$$\tan 30^\circ = \frac{x}{180}$$

$$x = 180 \times \tan 30^\circ \quad (1)$$

$$= 180 \times \frac{\sqrt{3}}{3}$$

$$= 60\sqrt{3} \text{ m} \quad (1)$$

(2 marks)

Question 3

If $\cos \theta = -\frac{1}{3}$ and $\pi < \theta < \frac{3\pi}{2}$ calculate the exact value of $\operatorname{cosec} 2\theta$

$$\frac{x}{x}$$

$$\sin \theta = -\frac{2\sqrt{2}}{3} \quad (1)$$

$$\operatorname{cosec} 2\theta = \frac{1}{\sin 2\theta}$$

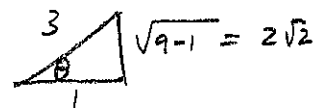
$$= \frac{1}{2 \sin \theta \cos \theta} \quad \left(\frac{1}{2}\right)$$

$$= \frac{1}{2 \left(-\frac{2\sqrt{2}}{3}\right) \left(-\frac{1}{3}\right)}$$

$$= \frac{9}{4\sqrt{2}} \quad \left(\frac{1}{2}\right)$$

$$= \frac{9\sqrt{2}}{8}$$

(3 marks)



$$\therefore \operatorname{cosec} 2\theta = \frac{9\sqrt{2}}{8} \quad (1)$$

Question 4

The lengths of the sides of a triangle form a geometric sequence and the largest side has a length of 36 cm.

Given that the perimeter is 76 cm, find the length of the shortest side.

$$ar^2 = 36$$

$$a = \frac{36}{r^2}$$

(1)

OR

sequence $\{x, y, z\}$ with $z = 36$

$$\frac{y}{x} = \frac{z}{y}$$

$$(1) \quad y^2 = xz = 36x \dots (1)$$

$$x + y + 36 = 76$$

$$y = 40 - x$$

$$(1) \quad y^2 = (40 - x)^2 \dots (2)$$

$$(1) = (2)$$

$$36x = (40 - x)^2$$

$$36x = 1600 - 80x + x^2$$

$$(1) \quad x^2 - 116x + 1600 = 0$$

$$(x - 16)(x - 100) = 0$$

$$(1) \quad x = 16 \text{ or } x = 100$$

(reject)

$\therefore$ Shortest side is 16

$$a + ar + ar^2 = 76$$

(1)

$$\frac{36}{r^2} + \frac{36r}{r^2} + 36 = 76$$

$$36 + 36r + 36r^2 = 76r^2$$

$$40r^2 - 36r - 36 = 0$$

(1)

$$10r^2 - 9r - 9 = 0$$

$$(5r + 3)(2r - 3) = 0$$

(1)

$$r = -\frac{3}{5} \text{ (reject)}$$

$$\text{or}$$

$$r = \frac{3}{2}$$

$$a = \frac{36}{r^2} = \frac{36}{(\frac{3}{2})^2} = \frac{36 \times 4}{9} \quad (1)$$

Shortest side is 16

(4 marks)

Question 5

Find the value of x if $AC = 6$ cm, $A'C' = 10$ cm, $A'A = 2$ cm and $AB = (x + 1)$ cm.

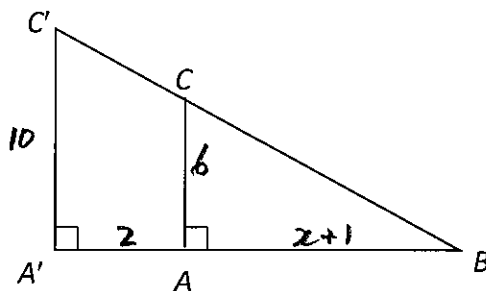
$$\frac{x+1}{6} = \frac{(x+1)+2}{10} \quad (1)$$

$$10x + 10 = 6x + 18$$

$$4x = 8$$

$$x = 2$$

(1)



(2 marks)

Question 6

Find the Cartesian equation of the curve whose parametric equations are $x = 3\cos(\theta)$ and $y = 2\sin(\theta)$

$$\textcircled{1/2} \quad \begin{aligned} \cos \theta &= \frac{x}{3} \\ \sin \theta &= \frac{y}{2} \end{aligned}$$

$$\begin{aligned} \cos^2 \theta + \sin^2 \theta &= 1 \\ \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 &= 1 \end{aligned}$$

$$\textcircled{1/2}$$

$$\therefore \frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$\textcircled{1}$$

(2 marks)

Question 7

Express $\frac{2\sqrt{3} + 2i}{1 - \sqrt{3}i}$ in polar form.

$$\frac{2\sqrt{3} + 2i}{1 - \sqrt{3}i} = \frac{2\sqrt{3} + 2i}{1 - \sqrt{3}i} \times \frac{1 + \sqrt{3}i}{1 + \sqrt{3}i} \quad \textcircled{1}$$

$$= \frac{2\sqrt{3} + 6i + 2i - 2\sqrt{3}}{1 + 3} \quad \textcircled{1}$$

$$= \frac{8i}{4}$$

$$= 2i$$

$$\therefore z = 2 \operatorname{cis} \frac{\pi}{2} \quad \textcircled{1}$$



$$\textcircled{1}$$

(4 marks)

Question 8

Prove that $\sin \theta + \sin\left(\theta + \frac{2\pi}{3}\right) + \sin\left(\theta + \frac{4\pi}{3}\right) = 0$

$$\text{LHS} = \sin \theta + \sin \theta \cos \frac{2\pi}{3} + \cos \theta \sin \frac{2\pi}{3} + \sin \theta \cos \frac{4\pi}{3} + \cos \theta \sin \frac{4\pi}{3} \quad \textcircled{1}$$

$$= \sin \theta - \frac{1}{2} \sin \theta + \frac{\sqrt{3}}{2} \cos \theta - \frac{1}{2} \sin \theta - \frac{\sqrt{3}}{2} \cos \theta \quad \textcircled{1}$$

$$= \sin \theta - \sin \theta + \frac{\sqrt{3}}{2} \cos \theta - \frac{\sqrt{3}}{2} \cos \theta$$

$$= 0 \quad \textcircled{1}$$

= RHS as required

(3 marks)

Question 9

$\underline{u}$ and $\underline{v}$ are vectors defined by $\underline{u} = \cos(\theta)\underline{i} + \sin(\theta)\underline{j}$ and $\underline{v} = \sin(\theta)\underline{i} + \cos(\theta)\underline{j}$ where $0 < \theta < \frac{\pi}{2}$

a) Show that $\underline{u}$ and $\underline{v}$ are unit vectors

$$\begin{aligned} |\underline{u}| &= \sqrt{\cos^2 \theta + \sin^2 \theta} \\ &= \sqrt{1} \\ &= 1 \end{aligned} \quad \textcircled{\frac{1}{2}}$$

$$\begin{aligned} |\underline{v}| &= \sqrt{\sin^2 \theta + \cos^2 \theta} \\ &= \sqrt{1} \\ &= 1 \end{aligned} \quad \textcircled{\frac{1}{2}}$$

$\therefore \underline{u}$ and $\underline{v}$ are unit vectors $\textcircled{1}$

b) Let α be the angle between the vectors $\underline{u}$ and $\underline{v}$. Show, using an appropriate double angle formula that

$$\alpha = \cos^{-1}(\sin(2\theta))$$

$$\cos \alpha = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| |\underline{v}|}$$

$$= (\cos \theta \underline{i} + \sin \theta \underline{j}) \cdot (\sin \theta \underline{i} + \cos \theta \underline{j}) \quad \textcircled{1}$$

$$= \sin \theta \cos \theta + \sin \theta \cos \theta \quad \textcircled{1}$$

$$= 2 \sin \theta \cos \theta \quad \textcircled{\frac{1}{2}}$$

$$= \sin(2\theta) \quad \textcircled{\frac{1}{2}}$$

$$\therefore \alpha = \cos^{-1}(\sin(2\theta)) \quad \text{as required}$$

c) Find α when $\theta = \frac{\pi}{6}$

$$\alpha = \cos^{-1}(\sin(2 \times \frac{\pi}{6})) \quad \textcircled{\frac{1}{2}}$$

$$= \cos^{-1}(\sin \frac{\pi}{3}) \quad \textcircled{\frac{1}{2}}$$

$$= \cos^{-1}(\frac{\sqrt{3}}{2}) \quad \textcircled{1}$$

$$= \frac{\pi}{6}$$

$$\therefore \alpha = \frac{\pi}{6}$$

d) If $\theta = \frac{\pi}{3}$, find the vector resolute of $\underline{v}$ in the direction of $\underline{u}$.

$$\begin{aligned} \hat{\underline{u}} &= \cos\left(\frac{\pi}{3}\right)\underline{i} + \sin\left(\frac{\pi}{3}\right)\underline{j} \\ &= \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j} \end{aligned} \quad \textcircled{1}$$

$$(\underline{u} \cdot \hat{\underline{u}}) = \left(\frac{\sqrt{3}}{2}\underline{i} + \frac{1}{2}\underline{j}\right) \cdot \left(\frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}\right) = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \quad \textcircled{1}$$

$$(\underline{v} \cdot \hat{\underline{u}})\hat{\underline{u}} = \frac{\sqrt{3}}{2} \left(\frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}\right)$$

$$= \frac{\sqrt{3}}{4}\underline{i} + \frac{3}{4}\underline{j} \quad \textcircled{1}$$

(2+3+2+3=10 marks)

Question 10

The straight line with equation $2x + 4y = 3$ is transformed by the rule $(x, y) \rightarrow (x - y, x)$. Find the equation of its image.

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & \textcircled{1} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} & \textcircled{1} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \\ &= \begin{bmatrix} y' \\ y' - x' \end{bmatrix} & \textcircled{\frac{1}{2}} \\ x &= y' \\ y &= y' - x' & \textcircled{\frac{1}{2}} \\ 2y' + 4(y' - x') &= 3 & \textcircled{\frac{1}{2}} \\ 6y' - 4x' &= 3 \\ \text{image } 4x - 6y + 3 &= 0 & \textcircled{1} \end{aligned}$$

OR

$$\begin{aligned} x' &= x - y & \textcircled{1} \\ y' &= x & \textcircled{2} \\ \text{Sub } \textcircled{2} \text{ into } \textcircled{1} \text{ then,} \\ x' &= y' - y \\ y &= y' - x' \text{ and } x = y' & \textcircled{1} \\ 2x + 4y &= 3 \text{ substitute } x \text{ and } y \\ 2y' + 4(y' - x') &= 3 \\ 2y' + 4y' - 4x' &= 3 \\ 6y' - 4x' &= 3 & \textcircled{1} \\ \text{Image } 4x - 6y + 3 &= 0 & \textcircled{1} \end{aligned}$$

(4 marks)

Question 11

In the entire college, a total of 100 students are studying year 11 Specialist Maths course. On a recent exam, the mean score was 73.4 with a standard deviation of 20. $\mu = 73.4$ $\sigma = 20$

- a) If there are 41 female and 59 male students taking the subject, what is the proportion of female students?

$$p = \frac{41}{100}$$

- b) A group of 15 students selected from this class and 3 of them were female. What is the sample proportion, $\hat{p}$?

$$\hat{p} = \frac{3}{15} = \frac{1}{5}$$

- c) The top 5 students comprise of 3 male and 2 female. If 3 students were selected from the top 5, what is the probability that only 1 female was selected?

$$\begin{aligned} P_r(X=1) &= \frac{{}^2C_1 {}^3C_2}{{}^5C_3} & \textcircled{1} \\ &= \frac{2 \times 3}{10} \\ &= 0.6 & \textcircled{1} \end{aligned}$$

$$\therefore P_r(X=1) = 0.6 \quad \textcircled{1}$$

- d) Samples of 4 students are selected from the year level. Find the mean and standard deviation of the distribution of $\bar{x}$.

$$\mu_{\bar{x}} = \mu = 73.4$$

$\textcircled{1}$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{4}} = \frac{20}{2} = 10 \quad \textcircled{1}$$

(1+1+2+2=6 marks)

Question 12

A particle moving in a straight line has acceleration of $(3 - 2t)$ m/s² at time t seconds ($t \geq 0$). If the particle starts at the origin O with a velocity of 4 m/s, find:

- a) the time when the particle comes to rest

$$v(t) = 3t - t^2 + C \quad v(0) = 4 \Rightarrow C = 4 \quad \left(\frac{1}{2}\right)$$

$$v(t) = -t^2 + 3t + 4 = 0$$

$$t^2 - 3t - 4 = 0 \quad \left(\frac{1}{2}\right)$$

$$(t-4)(t+1) = 0$$

$$t = 4 \text{ or } t = -1 \quad \left(\frac{1}{2}\right)$$

$$\text{As } t \geq 0 \quad \therefore t = 4 \text{ s.} \quad \left(\frac{1}{2}\right)$$

- b) the position of the particle at the instant it comes to rest

$$x(t) = -\frac{1}{3}t^3 + \frac{3}{2}t^2 + 4t + C \quad x(0) = 0 \Rightarrow C = 0 \quad \left(\frac{1}{2}\right)$$

$$x(4) = -\frac{1}{3}(4^3) + \frac{3}{2}(4^2) + (4 \times 4) \quad \left(\frac{1}{2}\right)$$

$$= -\frac{64}{3} + \frac{48}{2} + 16$$

$$= -\frac{128}{6} + \frac{144}{6} + 16$$

$$= \frac{16}{6} + 16$$

$$\therefore x(4) = 18\frac{2}{3} \text{ m} \quad (1)$$

- c) the acceleration at this instant

$$a(t) = 3 - 2t$$

$$a(4) = 3 - (2 \times 4)$$

$$= -5 \text{ m/s}^2$$

- d) the time when the acceleration is zero

$$3 - 2t = 0$$

$$t = \frac{3}{2} \text{ s}$$

- e) the velocity at this time.

$$v\left(\frac{3}{2}\right) = -\left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right) + 4 \quad \left(\frac{1}{2}\right)$$

$$= -\frac{9}{4} + \frac{18}{4} + 4$$

$$= \frac{19}{4} + 4$$

$$= 6\frac{1}{4} \text{ m/s} \quad \left(\frac{1}{2}\right)$$

(2+2+1+1+1=7 marks)

End of Exam