

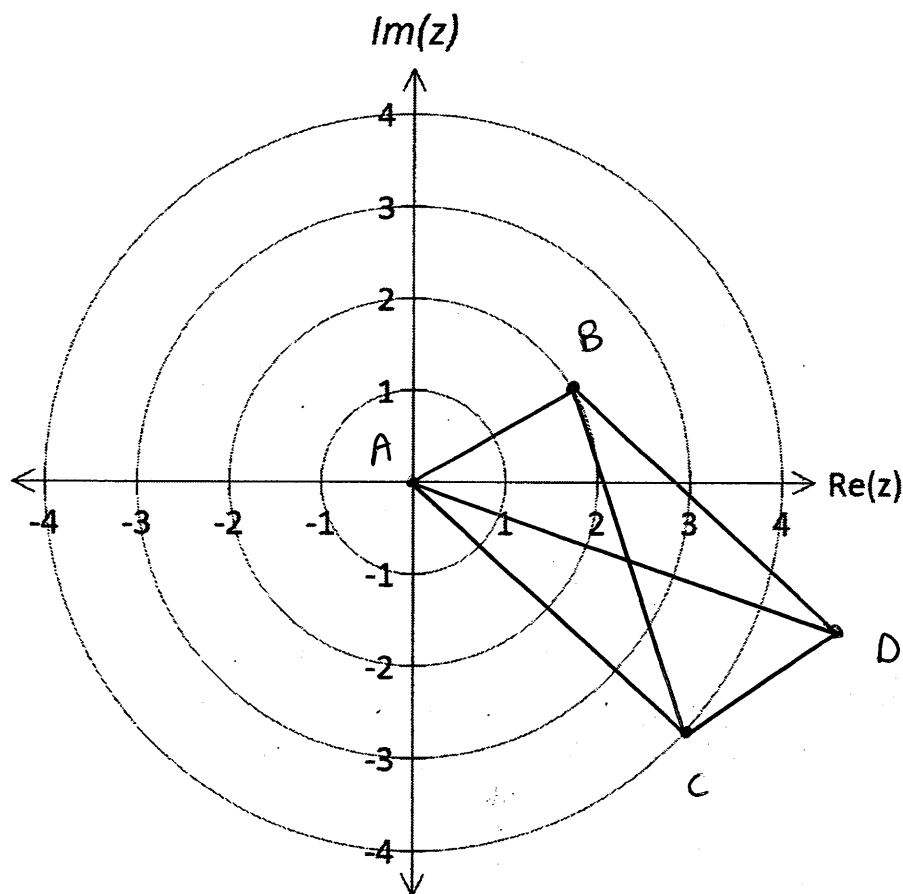
Application Task Part 1 – Complex Numbers & Trigonometry

Name: SOLUTIONS

Question 1 (17 Marks)

a) On the argand diagram below, draw accurately to scale the two complex numbers:

$$z_1 = \sqrt{3} + i \quad \text{and} \quad z_2 = 2\sqrt{2} - 2\sqrt{2}i$$



2 marks

b) Convert z_1 and z_2 to polar form.

$$z_1 = 2 \operatorname{cis}(\pi/6) \quad (1)$$

$$z_2 = 4 \operatorname{cis}(-\pi/4) \quad (1)$$

2 marks

c) Draw a triangle joining the origin, z_1 and z_2 and label the vertices "A", "B" and "C" where "A" is the origin, "B" is z_1 and "C" is z_2 .

Use your answer to (b) to find the angle BAC in degrees.

$$\angle BAC = 30^\circ + 45^\circ = 75^\circ$$

1 mark

d) Use the cosine rule to find the length BC. (Answer to four decimal places)

$$x^2 = 2^2 + 4^2 - 2 \times 2 \times 4 \times \cos 75^\circ \quad (1)$$

$$x^2 = 15.8589 \quad (1)$$

$$x = 3.9823 \quad (1)$$

3 marks

e) Find the area of the triangle ABC. (Answer to three decimal places)

$$A = \frac{1}{2} \times 2 \times 4 \times \sin 75^\circ \quad (1)$$

$$= 3.864 \text{ unit}^2 \quad (1)$$

2 marks

f) On the argand diagram accurately draw the point $z_1 + z_2$ and label it point "D". Construct a parallelogram ABDC. Use your CAS (in decimal mode) to convert $z_1 + z_2$ to polar form and hence state the length AD to four decimal places and the angle CAD in degrees to two decimal places.

$$z_1 + z_2 = (2\sqrt{2} + \sqrt{3}) + (1 - 2\sqrt{2})i \quad (1)$$

$$\approx 4.9134 \text{ cis } (-0.3813) \quad (1)$$

$$\therefore \overline{AD} = 4.9134 \quad (1)$$

$$\angle CAD = 45^\circ - 0.3813 \times \frac{180}{\pi}$$

(4 marks)

$$= 23.15^\circ \quad (1)$$

g) Hence find the area of triangle ADC to three decimal places. Comment on your result.

$$A = \frac{1}{2} \times 4.9134 \times 4 \times \sin 23.15^\circ \quad (1)$$

$$= 3.863 \text{ unit}^2 \quad (1)$$

SAME AS $\triangle ABC$ (1)

(3 marks)

Question 2 (9 Marks)

A complex number can be expressed as $z = \cos(\theta) + i \sin(\theta)$.

According to DeMoivre's Theorem, $z^2 = \cos(2\theta) + i \sin(2\theta)$

- a) Use the first equation to find z^2 in terms of $\cos(\theta)$ and $\sin(\theta)$ by expanding and simplifying the equation: $z^2 = (\cos(\theta) + i \sin(\theta))^2$

$$z^2 = (\cos \theta + i \sin \theta)^2 \quad (1)$$

$$= \cos^2 \theta + 2i \sin \theta \cos \theta - \sin^2 \theta$$

$$= (\cos^2 \theta - \sin^2 \theta) + 2i \sin \theta \cos \theta \quad (1)$$

2 marks

- b) By equating real and imaginary parts of both expressions for z^2 , find expressions for $\cos(2\theta)$ and $\sin(2\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta \quad (1)$$

$$\sin 2\theta = 2 \sin \theta \cos \theta \quad (1)$$

2 marks

- c) Using a similar method, two expressions for z^3 can be found:

$$z^3 = (\cos(\theta) + i \sin(\theta))^3 \quad \text{and} \quad z^3 = \cos(3\theta) + i \sin(3\theta)$$

By expanding the first equation and equating real and imaginary components, find expressions for $\cos(3\theta)$ and $\sin(3\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$.

$$z^3 = (\cos^2 \theta - \sin^2 \theta + 2i \sin \theta \cos \theta)(\cos \theta + i \sin \theta) \quad (1)$$

$$= \cos^3 \theta - \sin^2 \theta \cos \theta + 2i \sin \theta \cos^2 \theta + i \sin \theta \cos^2 \theta - i \sin^3 \theta - 2 \sin^2 \theta \cos \theta \quad (1)$$

$$= (\cos^3 \theta - 3 \sin^2 \theta \cos \theta) + i (3 \sin \theta \cos^2 \theta - \sin^3 \theta) \quad (1)$$

$$\cos 3\theta = \cos^3 \theta - 3 \sin^2 \theta \cos \theta \quad (1) \quad (5 \text{ marks})$$

$$\sin 3\theta = 3 \sin \theta \cos^2 \theta - \sin^3 \theta \quad (1)$$

Question 3 (8 marks)

Complex numbers can be used to analyse electronic circuits containing resistors and capacitors.

The admittance of a circuit (Y) is given by :

$$\frac{1}{R - \frac{i}{2\pi fC}}$$

Where R = Resistance, C = Capacitance, f = frequency

- a) For a particular circuit, $R = 100$, $f = \frac{50}{\pi}$ and $C = 2.0 \times 10^{-5}$.

Find the admittance of the circuit. Express your answer in the form $a + ib$

$$\begin{aligned} Y &= \frac{1}{100 - \frac{i}{2\pi \times \frac{50}{\pi} \times 2 \times 10^{-5}}} \quad (1) = \frac{1}{100 - \frac{i}{2 \times 10^{-3}}} \\ &= \frac{1}{100 - 500i} = \frac{1}{2600} + \frac{1}{520}i \quad (1) \end{aligned}$$

3 marks

- b) For the same circuit, it is desired to make the real and imaginary parts of the admittance equal. If only the frequency can be changed, find the required frequency.

NOTE : IF REAL & IMAGINARY PARTS ARE EQUAL ON DENOMINATOR THEN THE CONJUGATE ON THE NUMERATOR IS ALSO EQUAL.

$$\begin{aligned} \therefore \frac{1}{2\pi fC} &= R \quad (1) & f &= \frac{1}{2\pi RC} = \frac{1}{2\pi \times 100 \times 2 \times 10^{-5}} \quad (1) \\ & & f &= \frac{250}{\pi} \quad (1) \end{aligned}$$

3 marks

- c) Express the admittance in (b) in polar form.

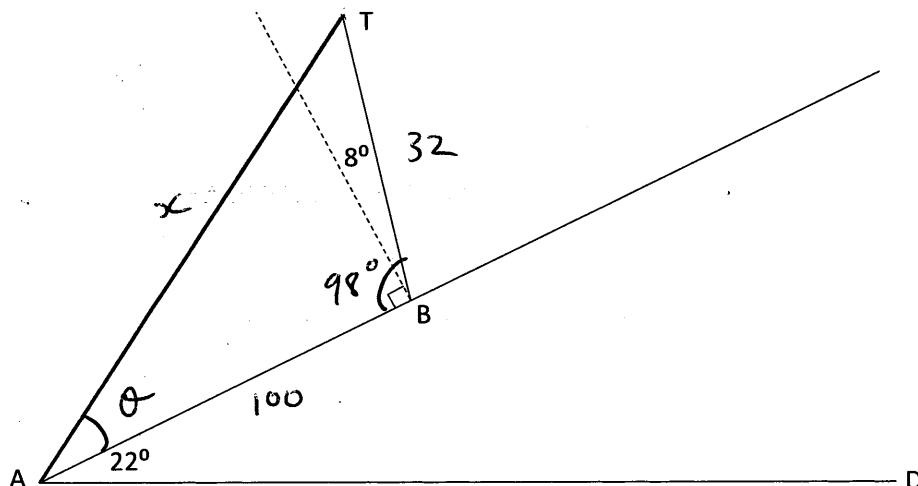
$$\begin{aligned} Y &= \frac{1}{100 - 100i} = \frac{100 + 100i}{20000} \quad (1/2) \\ &= \frac{1}{200} + \frac{1}{200}i \quad (1/2) \end{aligned}$$

$$Y = \frac{\sqrt{2}}{200} \cos\left(\frac{\pi}{4}\right) \quad (1)$$

2 marks

Question 4 (6 marks)

A flag pole is mounted on a sloping path. The path is inclined at 22° to the horizontal and the flag pole is at an angle of 8° to the normal line of the path. This is shown in the diagram below:



The base of the flag pole (B) is 100 metres from point A. The flag pole is 32 metres long.

- a) Find the length of the line AT. (Answer to three decimal places)

$$x^2 = 100^2 + 32^2 - 2 \times 100 \times 32 \times \cos 98^\circ \quad (1)$$

$$x^2 = 11914.708$$

$$x = 109.155 \text{ m} \quad (1)$$

2 marks

- b) Find the vertical height of T above the ground (AD). (Answer to the nearest metre)

$$\frac{\sin \theta}{32} = \frac{\sin 98^\circ}{109.155} \quad (1) \quad \sin \theta = 0.29031$$

$$\theta = 16.8765^\circ \quad (1)$$

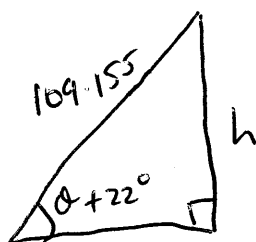
$$\theta + 22^\circ = 38.8765^\circ$$

$$h = 109.155 \times \sin 38.8765^\circ \quad (1)$$

4 marks

$$h = 68.51$$

$$h \approx 69 \text{ m} \quad (1)$$

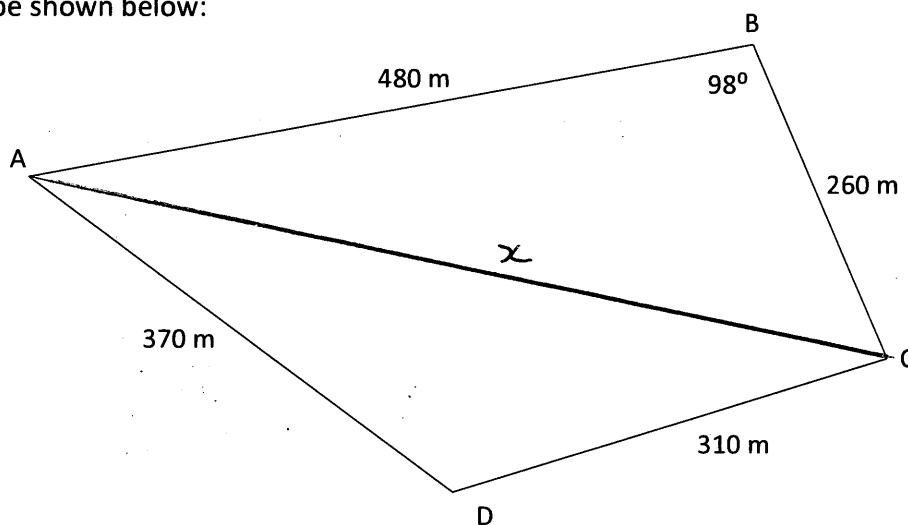


Application Task Part 2 – Complex Numbers & Trigonometry

Name: SOLUTIONS

Question 1 (13 Marks)

A farm is in the shape shown below:



2 marks

- a) Find the area of triangle ABC in square metres (answer to two decimal places).

$$A = \frac{1}{2} \times 480 \times 260 \times \sin 98^\circ \quad (1)$$

$$A = 61792.73 \text{ m}^2 \quad (1)$$

2 marks

- b) Find the length of the line AC in metres (answer to three decimal places).

$$x^2 = 480^2 + 260^2 - 2 \times 480 \times 260 \times \cos 98^\circ \quad (1)$$

$$x^2 = 332,737.606$$

$$x = 576.834 \text{ m} \quad (1)$$

2 marks

c) Use heron's formula to find the area of triangle ACD in square metres (answer to two decimal places).

$$s = \frac{1}{2} (370 + 310 + 576.834) = 628.417 \quad (1)$$

$$A = \sqrt{628.417 (628.417 - 576.834) (628.417 - 310) (628.417 - 370)}$$

$$= 51645.91 \text{ m}^2 \quad (1)$$

2 marks

d) Hence find the total area of the farm (ABCD) to two decimal places.

$$51645.91 + 61729.73 = 113,438.64 \text{ m}^2$$

1 mark

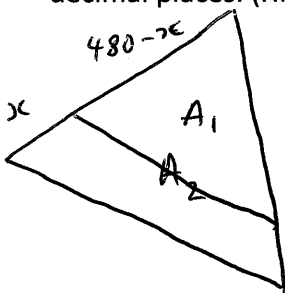
e) The farmer wishes to split the farm into two farms of equal area for each of his daughters. He decides to use a line parallel to AC to do this. Should the line be closer to point B or point D. Explain your answer.

• CLOSER TO B (1)

• AREA OF $\triangle ABC >$ AREA OF $\triangle ACD$ (1)

2 marks

f) If the line starts a distance of x metres from point A and is parallel to AC. Find the value of x to two decimal places. (Hint: use similar triangles)



$$A_1 = \frac{113438.64}{2} = 56,719.32 \text{ m}^2 \quad (1/2)$$

$$\frac{A_1}{A_2} = \frac{56719.32}{61729.73} = 0.9178963 \quad (1/2)$$

$$\left(\frac{480-x}{480} \right)^2 = 0.9178963 \quad (1)$$

4 marks

$$\frac{480-x}{480} = 0.9580691 \quad (1)$$

$$480-x = 459.87315 \quad \underline{x = 20.13 \text{ m}} \quad (1)$$

Question 2 (8 Marks)

Find all solutions to the following equations:

$$-\pi < \theta \leq \pi$$

a) $z^6 = -i$ (Express your answers in polar form)

$$-6\pi < 6\theta \leq 6\pi$$

$$r^6 \cos(6\theta) = \cos(-\pi/2)$$

$$r = 1$$

$$6\theta = -\frac{9\pi}{2}, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$$

$$\theta = -\frac{3\pi}{4}, -\frac{5\pi}{12}, -\frac{\pi}{12}, \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}$$

$$z = \cos\left(-\frac{3\pi}{4}\right), \cos\left(-\frac{5\pi}{12}\right), \cos\left(-\frac{\pi}{12}\right), \cos\left(\frac{\pi}{4}\right), \cos\left(\frac{7\pi}{12}\right), \cos\left(\frac{11\pi}{12}\right)$$

4 marks

b) $z^3 + 3z^2 + z - 5 = 0$ (Express your answers in cartesian form)

$$p(1) = 1 + 3 + 1 - 5 = 0 \quad (1/2)$$

$$\Rightarrow (z-1) \text{ is a factor} \quad (1/2)$$

$$\begin{array}{r} z^2 + 4z + 5 \\ z-1 \overline{) z^3 + 3z^2 + z - 5} \\ \underline{z^3 - z^2} \\ 4z^2 + z \\ \underline{4z^2 - 4z} \\ 5z - 5 \\ \underline{5z - 5} \\ 0 \end{array} \quad (1/2)$$

$$(z-1)(z^2 + 4z + 5) = 0 \quad (1/2)$$

$$(z-1)((z+2)^2 + 1) = 0 \quad (1/2)$$

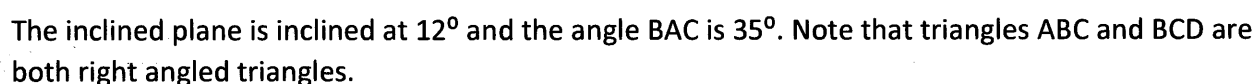
$$(z-1)(z+2+i)(z+2-i) = 0 \quad (1/2)$$

$$z = 1 \text{ OR } z = -2 \pm i \quad (1)$$

(CAN USE SHORT DIVISION)

(CAN USE QUADRATIC FORMULA)
4 marks

A road is to be built on an angle across an inclined plane as shown below:



- $$\cos 35^\circ = \frac{120}{AC} \quad (1)$$

$$AC = 146.493 \text{ m} \quad (1)$$

b) Find the angle of elevation of the road AC to Two decimal places.

$$\tan(35^\circ) = \frac{BC}{120} \quad BC = 84.0249 \text{ m}$$

$$\sin(12^\circ) = \frac{CD}{84.0249} \quad CD = 17.4698 \text{ m}$$

$$\sin \theta = \frac{17.4698}{146.49} \quad \theta = 6.85^\circ$$

c) It is desired to make the angle of elevation of the road exactly 10° . To achieve this the top of the road is moved x metres to the left along the line CE. Find the required value of x to the nearest metre.

$$BD = 84.0249 \times \cos 12^\circ = 82.1888 \text{ m} \quad (1/2)$$

$$\sin 10^\circ = \frac{17.4698}{y} \quad y = 100.6046 \text{ m} \quad (1)$$

$$z = \sqrt{100.6046^2 - 17.4698^2} = 99.0762 \text{ m} \quad \begin{matrix} (1) \\ 4 \text{ marks} \end{matrix}$$

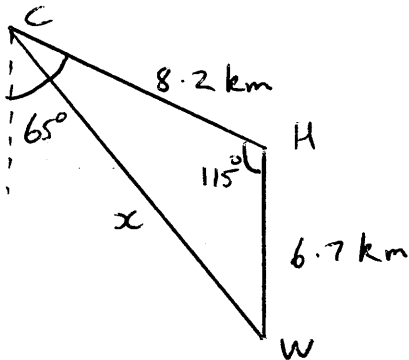
$$120 - x = \sqrt{99.0762^2 - 82.1888^2} = 55.3271 \text{ m} \quad (1)$$

$$x = 64.6729 \text{ m} \quad x \approx 65 \text{ m} \quad (1/2)$$

Question 4 (10 marks)

A hiker leaves their camp site and travels 8.2 km in the direction $115^\circ T$ and stops at a hut for lunch. She then heads due south for 6.7 km to reach a waterfall.

- a) Draw a triangle joining the camp site (C), hut (H) and waterfall (W) showing all relevant lengths and angles.



$$\frac{1}{2} M = \text{SHAPE}$$

$$\frac{1}{2} M = 8.2 \text{ km}$$

$$\frac{1}{2} M = 6.7 \text{ km}$$

$$\frac{1}{2} M = 115^\circ$$

2 marks

- b) How far is the waterfall from the camp site? (Answer to two decimal places)

$$x^2 = 8.2^2 + 6.7^2 - 2 \times 8.2 \times 6.7 \times \cos 115^\circ \quad (1)$$

$$x^2 = 158.567$$

$$x = 12.59 \text{ km} \quad (1)$$

2 marks

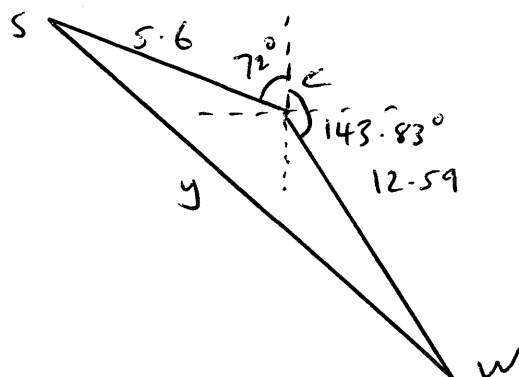
- c) What is the true bearing of the waterfall from the camp site to two decimal places?

$$\frac{\sin C}{6.7} = \frac{\sin 115^\circ}{12.59} \quad \left(\frac{1}{2}\right) \quad C = 28.83^\circ \quad (1)$$

$$\text{BEARING} = 115^\circ + 28.83^\circ = 143.83^\circ \quad (1)$$

2 marks

- d) If a second hiker leaves the campsite and walks on a bearing of $288^\circ T$ for 5.6 km to a shop, find the straight line distance between this shop and the waterfall. (Answer to two decimal places)



$$\angle C = 360 - 72 - 143.83 \quad (1)$$

$$= 144.17^\circ$$

4 marks

$$(1) \quad y^2 = 5.6^2 + 12.59^2 - 2 \times 5.6 \times 12.59 \times \cos(144.17^\circ)$$

$$(1) \quad y^2 = 304.19$$

$$(1) \quad y = 17.44 \text{ km}$$