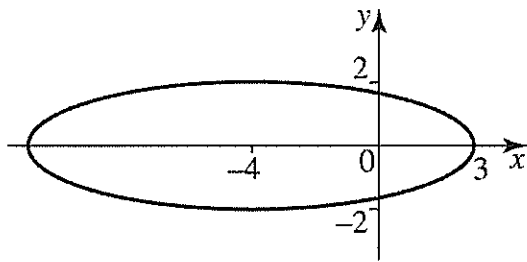


Question 20

A possible equation for the graph below is:



- A. $(x - 4)^2 + y^2 = 49$
- B. $\frac{(x + 4)^2}{49} + \frac{y^2}{4} = 1$
- C. $\frac{(x - 4)^2}{49} + \frac{y^2}{4} = 1$
- D. $\frac{(x - 4)^2}{49} - \frac{y^2}{4} = 1$
- E. $\frac{(x + 4)^2}{49} - \frac{y^2}{4} = 1$

Multiple Choice Answer Table

1	2	3	4	5	6	7	8	9	10
E	C	B	A	E	D	E	E	C	A
11	12	13	14	15	16	17	18	19	20
C	E	A	B	A	C	A	E	E	B

SHORT ANSWER/EXTENDED RESPONSE QUESTIONS

Question 1 (2 marks)

Using your CAS calculator, solve the following simultaneous equations for x and y

$$(p - q)x + (p + q)y = (p + q)^2$$

$$qx - py = q^2 - pq$$

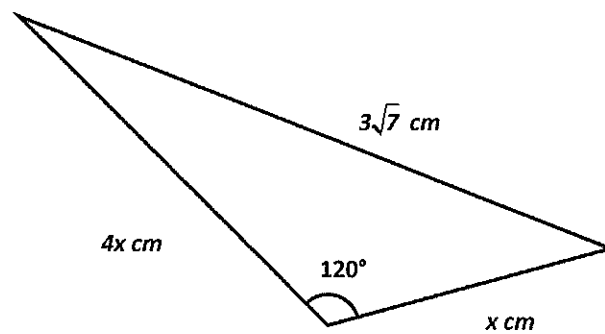
$$x = p + q$$

$$y = 2q$$

(2 marks)

Question 2 (3 marks)

Calculate the exact value of x in the following diagram:



$$(3\sqrt{7})^2 = (4x)^2 + x^2 - 2(4x)(x)\cos 120^\circ \quad (1)$$

$$63 = 17x^2 - 8x^2 \times -\frac{1}{2}$$

$$63 = 21x^2$$

$$x = \sqrt{\frac{63}{21}}$$

$$= \sqrt{3} \text{ cm. as } x > 0$$

(1)

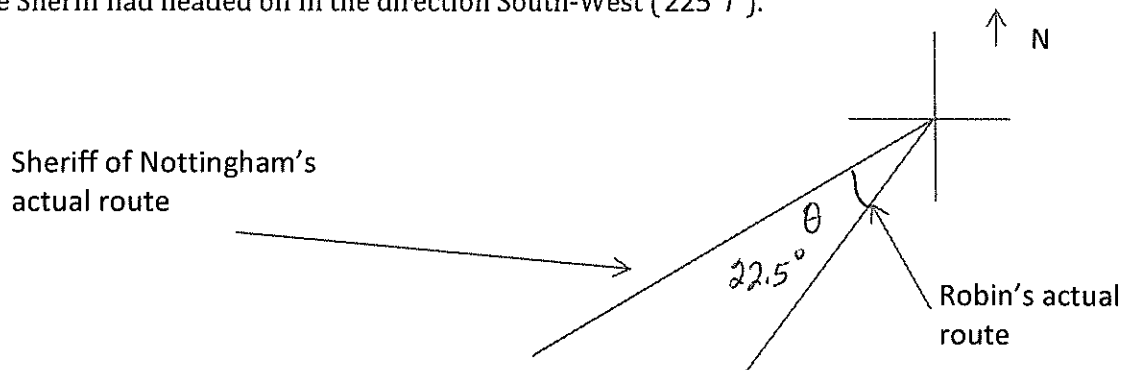
(3 marks)

(1)

Question 3 (9 marks)

Robin Hood was chasing the Sheriff of Nottingham who had kidnapped Maid Marion. Robin headed off on a direction South-South-West ($202.5^\circ T$).

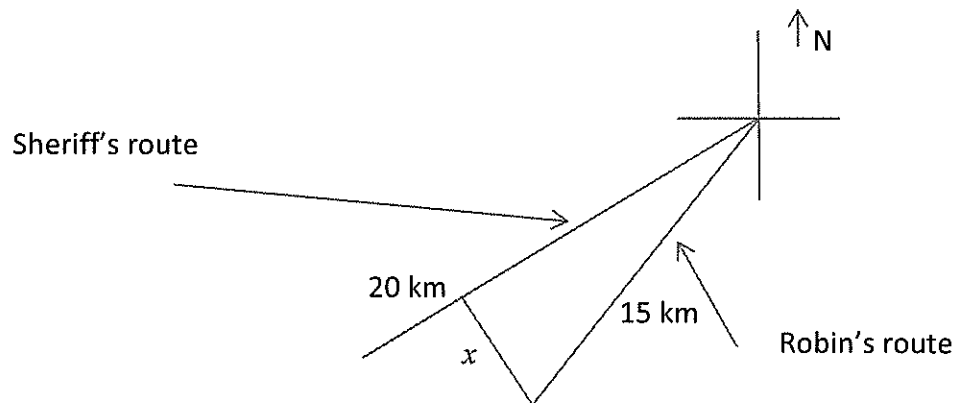
In fact, the Sheriff had headed off in the direction South-West ($225^\circ T$).



- (a) Calculate the angle between the Sheriff's route and Robin's route.

$$\theta = 225^\circ - 202.5^\circ = 22.5^\circ$$

- (b) The sheriff travelled 20km after one hour. Robin travelled 15km but in the wrong direction (see diagram). Calculate the **shortest distance** (x) Robin must travel to intersect with the Sheriff's route. (Express your answer in *km* correct to two decimal places.)



$$\textcircled{1} \quad \sin \theta = \frac{x}{15}$$

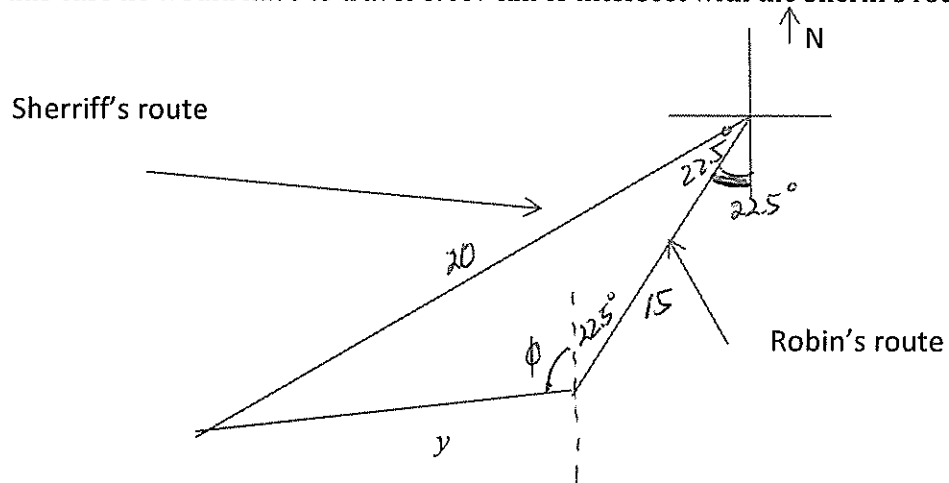
$$\begin{aligned} x &= 15 \sin \theta \\ &= 15 \sin 22.5^\circ \\ &\approx 5.74 \text{ km.} \end{aligned}$$

$\textcircled{1}$

Note -
shortest distance
at right angle.

(c) If on the other hand Robin heads in the direction of where the Sheriff is at that time.

(i) Show that in this case he would have to travel 8.407 km to intersect with the Sheriff's route.

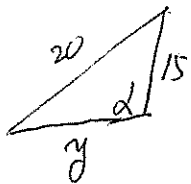


$$y^2 = 15^2 + 20^2 - 2(15)(20)\cos 22.5^\circ \quad (1)$$

$$y = \sqrt{70.67228} \quad (1)$$

$$y = 8.407 \text{ as required}$$

(ii) Calculate the true bearing (correct to the nearest minute) Robin would need to take in this case.



$$\cos \alpha = \left(\frac{15^2 + y^2 - 20^2}{2 \times 15 \times y} \right) \quad (1)$$

$$\cos \alpha = -0.4137$$

$$\alpha = 114.44^\circ \quad (1)$$

$$\text{Bearing} = 360^\circ - \alpha + 22.5^\circ \quad (1)$$

$$= 268.06^\circ$$

$$\approx 268^\circ 4' T \quad (1)$$

$$\text{OR} \quad \frac{\sin \phi}{20} = \frac{\sin 22.5^\circ}{y} \quad (1)$$

$$\phi = \sin^{-1} \left(\frac{20 \sin 22.5^\circ}{y} \right)$$

$$= 65.56^\circ \quad (1)$$

or 114.44°
(ambiguous case)

$$\text{Bearings} = 360^\circ + 22.5^\circ - \phi \quad (1)$$

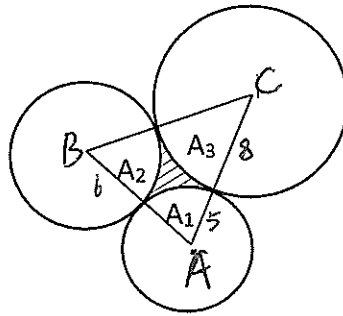
$$= 268.06^\circ$$

$$\approx 268^\circ 4' T \quad (1)$$

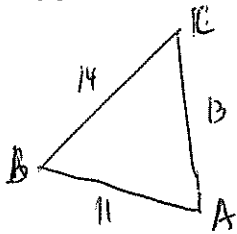
(1 + 2 + 2 + 4 = 9 marks)

Question 4 (7 marks)

Three circles of radii 5 cm, 6 cm and 8 cm are all touching each other as shown in the diagram below. A triangle is drawn by connecting the centres of the three circles.



(a) Find the area of the triangle to three decimal places.



Heron's Formula

$$A = \sqrt{19 \times 5 \times 6 \times 8} \\ \approx 67.528 \text{ cm}^2$$

$$S = \frac{1}{2} (11 + 13 + 14) = 19 \quad \textcircled{\frac{1}{2}}$$

$$\textcircled{\frac{1}{2}}$$

$$\textcircled{1}$$

(b) Find the area of each of the three sectors (A_1 , A_2 , A_3) inside the triangle to three decimal places.

$$A = \cos^{-1} \left(\frac{11^2 + 13^2 - 14^2}{2 \times 11 \times 13} \right)$$

$$= 70.81^\circ$$

$$B = \cos^{-1} \left(\frac{11^2 + 14^2 - 13^2}{2 \times 11 \times 14} \right)$$

$$= 61.28^\circ$$

$$C = \cos^{-1} \left(\frac{13^2 + 14^2 - 11^2}{2 \times 13 \times 14} \right)$$

$$= 47.91^\circ$$

$$A_1 = \frac{1}{2} \times 5^2 \times \frac{70.81}{180} \times \pi$$

$$= 15.449 \quad (\text{using exact values})$$

$$A_2 = \frac{1}{2} \times 6^2 \times \frac{61.28}{180} \times \pi$$

$$= 19.252$$

$$A_3 = \frac{1}{2} \times 8^2 \times \frac{47.91}{180} \times \pi$$

$$= 26.757$$

(all using exact values)

①

①

①

$A_1 = 15.449 \text{ cm}^2$	$A_2 = 19.252 \text{ cm}^2$	$A_3 = 26.757 \text{ cm}^2$
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deduct $\frac{1}{2}$ m if exact values not used.

(c) Hence, find the area of the shaded region between the circles to two decimal places.

$$\text{Shaded Area} = \triangle ABC - (A_1 + A_2 + A_3) \\ = 6.07 \text{ cm}^2$$

(2 + 3 + 2 = 7 marks)

Question 5 (6 marks)

A farmer decides to build up a flock of sheep. He starts with a flock of 10 000. Each year 3% of the sheep die through old age, while 20% produce one lamb. The farmer sells 1000 sheep each year at the spring market sale. (Both percentages are calculated on the sheep population at the **beginning of the year**.)

- (a) Find the sheep population after one year

$$\begin{aligned} P_1 &= P_0 + 0.2P_0 - 0.03P_0 - 1000 \\ &= 10000 + 2000 - 300 - 1000 \\ &= 10700 \text{ sheep} \end{aligned}$$

(1)^M

(1)^A

- (b) Let P_n be the sheep population at the end of the n th year. A difference equation that can be used to determine the number of sheep at the end of the n th year is: $P_n = aP_{n-1} - b$ where $P_0 = 10000$. Find the values of a and b .

$$\begin{aligned} P_n &= P_{n-1} + 0.2P_{n-1} - 0.03P_{n-1} - 1000 \\ &= P_{n-1}(1 + 0.2 - 0.03) - 1000 \\ &= 1.17P_{n-1} - 1000 \end{aligned}$$

$$\therefore a = 1.17 \\ b = 1000$$

(1)^A

(1)^A

- (c) When the sheep population reaches 20 341, the farmer decides that he has to sell enough sheep each year to keep the sheep population stable. How many sheep per year must he sell to keep the sheep population at approximately this size? Give your answer to the nearest sheep.

$$P_n = 20341$$

$$P_{n-1} = 20341$$

$$P_n = 1.17P_{n-1} - x$$

$$20341 = (1.17 \times 20341) - x$$

$$x = 3457.97$$

$$\therefore 3458 \text{ sheep}$$

(1)^M

(1)^A

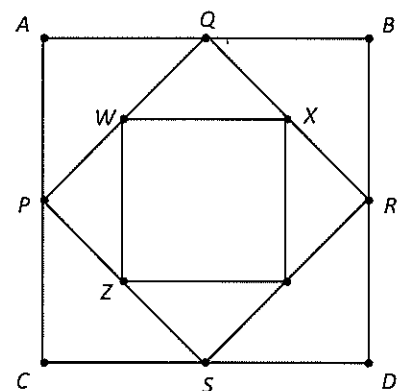
(2+2+2=6 marks)

Question 6 (5 marks)

The diagram shows a square $ABCD$ of side length 4 cm. The midpoints P, Q, R and S of the sides are joined to form a second square. The midpoints W, X, Y and Z of the sides of the square $PQRS$ are joined to form a third square as shown.

- (a) Show that the areas of the three squares $ABCD$, $PQRS$ and $WXYZ$ form a geometric sequence and find the common ratio, r , of this sequence.

$$\left. \begin{aligned} A_1 &= 4 \times 4 = 16 \\ A_2 &= 2\sqrt{2} \times 2\sqrt{2} = 8 \\ A_3 &= 2 \times 2 = 4 \end{aligned} \right\} \begin{aligned} \frac{A_2}{A_1} &= \frac{1}{2} \\ \frac{A_3}{A_2} &= \frac{1}{2} \end{aligned} \quad (1)^M$$



$$\therefore \text{Geometric Sequence with } a = 16 \text{ and } r = \frac{1}{2}. \quad (1)^A$$

(3 marks)

The process of forming smaller and smaller squares (by joining the midpoints) is continued.

(b) Find the area of the 11th square.

$$a = 16$$

$$r = \frac{1}{2}$$

$$A_n = ar^{10} = 16 \times \left(\frac{1}{2}\right)^{10} = \frac{1}{64}$$

(1 mark)

(c) If the process is continued indefinitely, find the sum of the areas of all the squares.

$$S_{\infty} = \frac{a}{1-r}$$

$$= \frac{16}{1-\frac{1}{2}}$$

$$= 32$$

(1 mark)

Question 7 (12 marks)

Let $z = \cos \theta + i \sin \theta$.

(a) Using De Moivre's Theorem, find z^n and z^{-n} , $n \in \mathbb{Z}^+$, expressing both answers with positive angles.

$$z^n = \cos(n\theta) + i \sin(n\theta) \quad (1)$$

$$z^{-n} = \cos(-n\theta) + i \sin(-n\theta) \quad (1)$$

$$= \cos(n\theta) - i \sin(n\theta) \quad (1)$$

(3 marks)

(b) Hence show that $z^n + \frac{1}{z^n} = 2 \cos(n\theta)$.

$$z^n + \frac{1}{z^n} = z^n + z^{-n} \quad (1)$$

$$= \cos(n\theta) + i \sin(n\theta) + \cos(n\theta) - i \sin(n\theta) \quad (1)$$

$$= 2 \cos(n\theta) \quad (1)$$

(3 marks)

(c) (i) Use your CAS to expand the equation $\left(z^2 + \frac{1}{z^2}\right)^4$.

$$\left(z^2 + \frac{1}{z^2}\right)^4 = z^8 + 4z^4 + \frac{4}{z^4} + \frac{1}{z^8} + 6$$

(ii) Use the formula from part (b) to expand $\left(z^2 + \frac{1}{z^2}\right)^4$.

$$\begin{aligned} \text{As } z^n + z^{-n} &= 2\cos(n\theta) \text{ therefore } \left(z^2 + \frac{1}{z^2}\right)^4 = \left[2\cos(2\theta)\right]^4 \\ &= 16\cos^4(2\theta) \end{aligned}$$

(iii) Hence, using parts (i) and (ii), show that $\cos^4 2\theta = A\cos 8\theta + B\cos 4\theta + C$ where A , B and C are rational numbers.

$$16\cos^4(2\theta) = z^8 + 4z^4 + \frac{4}{z^4} + \frac{1}{z^8} + 6 \quad (1)$$

$$= \left(z^8 + \frac{1}{z^8}\right) + 4\left(z^4 + \frac{1}{z^4}\right) + 6 \quad (1)$$

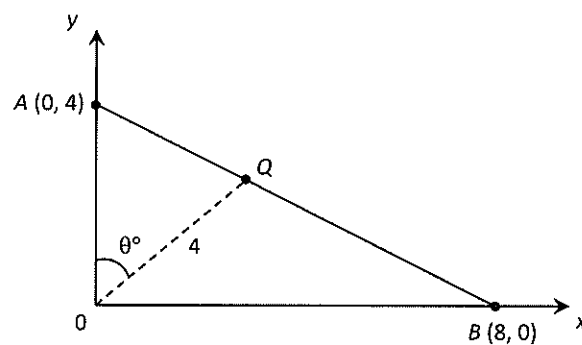
$$= 2\cos(8\theta) + 4(2\cos(4\theta)) + 6$$

$$\cos^4(2\theta) = \frac{1}{8}\cos(8\theta) + \frac{1}{2}\cos(4\theta) + \frac{3}{8} \quad \left. \vphantom{\cos^4(2\theta)} \right\} (1)$$

$$\therefore \left. \begin{aligned} A &= \frac{1}{8} \\ B &= \frac{1}{2} \\ C &= \frac{3}{8} \end{aligned} \right\} (1)$$

(1 + 1 + 4 = 6 marks)

Question 8 (8 marks)



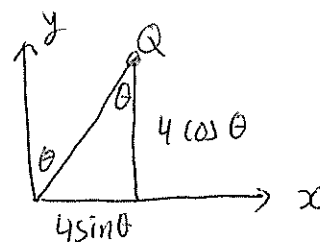
For the diagram above:

(a) Find the equation of line AB.

$$y = -\frac{1}{2}x + 4$$

(b) Find the coordinates of Q in terms of θ .

$$\text{Point } Q = (4 \sin \theta, 4 \cos \theta)$$



(c) Show that $2 \cos \theta + \sin \theta = 2$

$$y = -\frac{1}{2}x + 4$$

$$4 \cos \theta = -\frac{1}{2}(4 \sin \theta) + 4 \quad (1)$$

$$4 \cos \theta + 2 \sin \theta = 4 \quad (1)$$

$$2 \cos \theta + \sin \theta = 2 \quad \text{as required.}$$

(d) Write $2 \cos \theta + \sin \theta$ in the form $r \sin(\theta + \alpha)$. Give α degrees correct to 2 decimal places.

$$r = \sqrt{4+1} = \sqrt{5} \quad (1/2)$$

$$\tan \alpha = 2$$

$$\alpha = 63.43^\circ \quad (1/2)$$

$$\therefore \sqrt{5} \sin(\theta + 63.43^\circ) \quad (1)$$

(e) Find angle θ . Give your answer correct to 2 decimal places.

$$\sqrt{5} \sin(\theta + \tan^{-1}(2)) = 2$$

Solve for θ ,

$$\theta = 0 \quad \text{or} \quad 53.13^\circ \quad (1)$$

(1+2+2+2+1=8 marks)

Question 9 (3 marks)

Find the Cartesian equation of the smallest ellipse that passes through the point $(6, 0)$ and has the same centre as the circle with the polar equation $r = 4 \cos(\theta)$. It may be helpful to use a CAS calculator to first plot the graph of the circle.

$$r = 4 \cos \theta$$

$$r^2 = 4r \cos \theta$$

$$x^2 + y^2 = 4x \quad (1)$$

$$(x-2)^2 + y^2 = 4$$

Centre $(2, 0)$

$$r = 2.$$

} (1)

Ellipse go through $(6, 0)$

Centre $(2, 0) \therefore a = 4, b = 2$

$$\frac{(x-2)^2}{16} + \frac{y^2}{4} = 1$$

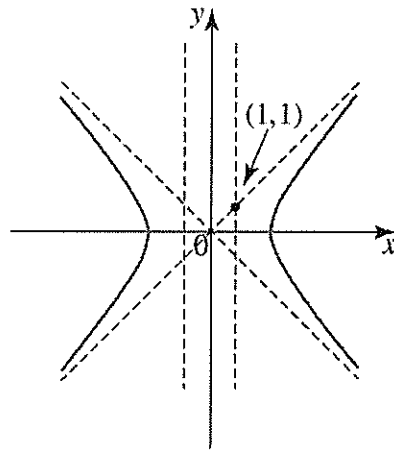
(1)

(3 marks)

OR using CAS to sketch.

Question 10 (5 marks)

For the hyperbola shown in the figure below:



- (a) Find the equations of each asymptote

Gradient of asymptotes $m = \pm 1$

$$y = x \quad \text{and} \quad y = -x$$

① ①

- (b) Given that the equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, state the relationship between a and b

$$m = \pm \frac{b}{a} = \pm 1$$

$$\therefore a = b$$

- (c) Find the equation of the hyperbola if it goes through point $(2, \sqrt{2})$.

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$$

$$\frac{x^2 - y^2}{a^2} = 1 \quad \text{①}$$

$$\frac{4 - 2}{a^2} = 1$$

$$a^2 = 2 \quad \text{②}$$

$$a = \sqrt{2}$$

$$\therefore \frac{x^2}{2} - \frac{y^2}{2} = 1 \quad \text{①}$$

(2+1+2=5 marks)

END OF EXAM