



Glen Waverley Secondary College 2017

VCE Specialist Mathematics – Tech Free

Unit 1 Examination

Student Name: SOLUTIONS

Student ID Code: _____ Form: _____

Teacher: GMN

QUESTION AND ANSWER BOOK

Reading Time: 10 minutes
Writing Time: 60 minutes

Structure of Book

<i>Section</i>	<i>Number of questions</i>	<i>Number of questions to be answered</i>	<i>Number of marks</i>
Short Answer	13	13	50

- Students are permitted to bring into the room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the exam room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and Answer book.
- Formula sheet.

Instructions

- Write your **student name and student ID code** in the space provided above on this page.
- CAS calculators, a bound reference and/or summary notes are **NOT** allowed.
- Marks are allocated for neat and clear working as well as answers.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Question 1

Solve for x:

$$\frac{x^2-4}{x^2+3x+2} = \frac{x-3}{x+2}$$

$$(x-2)(x+2) = (x-3)(x+2)(x+1) \quad (1/2)$$

$$x^2-4 = x^2-2x-3 \quad (1/2)$$

$$2x = 1 \quad (1/2)$$

$$x = 1/2 \quad (1/2)$$

[2 marks]

Question 2

Express as partial fractions:

$$\frac{5x^2+5x+5}{(x^2+1)(x+2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+2} \quad (1/2)$$

$$5x^2+5x+5 = (Ax+B)(x+2) + C(x^2+1) \quad (1/2)$$

$$x = -2 \Rightarrow 20 - 10 + 5 = 5C$$

$$C = 3 \quad (1/2)$$

$$x = 0 \Rightarrow 5 = 2B + 3$$

$$B = 1 \quad (1/2)$$

$$x = 1 \Rightarrow 15 = (A+1) \times 3 + 6$$

$$A = 2 \quad (1/2)$$

$$\frac{5x^2+5x+5}{(x^2+1)(x+2)} = \frac{2x+1}{x^2+1} + \frac{3}{x+2} \quad (1/2)$$

[3 marks]

5

Question 3

For an arithmetic sequence the fourth term is 11 and the sum of the first six terms is 57. Find the second term of the sequence.

$$\begin{array}{rcl}
 a + 3d = 11 & - (1) & \\
 3(2a + 5d) = 57 & & \\
 6a + 15d = 57 & - (2) & \\
 \hline
 6a + 18d = 66 & - (3) & \\
 \hline
 (1) \times 6 \Rightarrow & & \\
 (3) - (2) \Rightarrow & 3d = 9 & (1/2) \\
 & d = 3 & \\
 \text{Sub in (1)} \Rightarrow & a + 9 = 11 & (1/2) \\
 & a = 2 & \\
 & t_2 = 2 + 3 = 5 & (1)
 \end{array}$$

[3 marks]

Question 4

The first three terms of a geometric sequence are 8, $8r$, $8r^2$ ($r \neq 1$). These terms can also be the first, third and ninth terms of an arithmetic sequence. Find the value of r .

$$\begin{array}{rcl}
 a = 8 & & \\
 8r = 8 + 2d & - (1) & \\
 8r^2 = 8 + 8d & - (2) & \\
 \hline
 (1) \times 4 \Rightarrow & 32r = 32 + 8d & - (3) \\
 (3) - (2) \Rightarrow & 32r - 8r^2 = 24 & \\
 & 8r^2 - 32r + 24 = 0 & \\
 & r^2 - 4r + 3 = 0 & \\
 & (r-3)(r-1) = 0 & (1) \\
 & r = 1, 3 &
 \end{array}$$

$r \neq 1 \Rightarrow r = 3 \quad (1)$

[3 marks]

Question 5

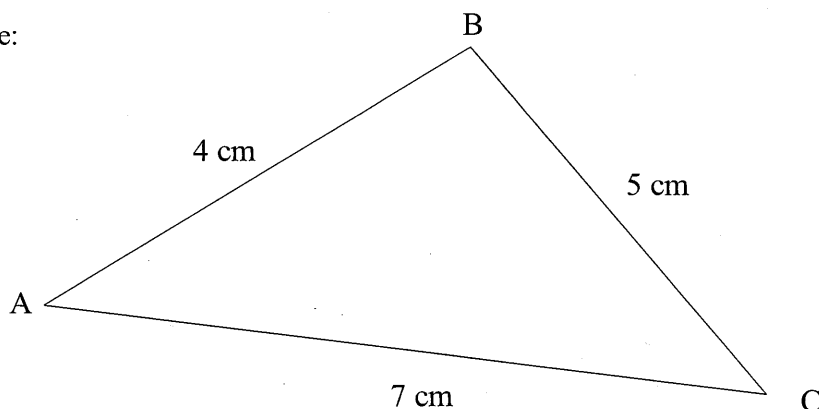
An infinite series of squares is drawn with the first square having a side length of one metre and each subsequent square having half the side length of the previous square. Find the sum of all the areas of the squares (infinite sum).

$$\begin{array}{rcl}
 \{ 1, 1/4, 1/16, \dots \} & a = 1, r = 1/4 & (1) \\
 S_{\infty} = \frac{1}{1 - 1/4} = 1/3/4 = 4/3 \text{ m}^2 & & (1)
 \end{array}$$

[2 marks]

Question 6

For the following triangle:



a) Use Heron's formula to find the area of the triangle

$$S = \frac{1}{2}(4+5+7) = 8 \quad (1/2)$$

$$A = \sqrt{8(8-4)(8-5)(8-7)} \quad (1/2)$$

$$= 4\sqrt{6} \text{ cm}^2 \quad (1)$$

b) Use the cosine rule to find $\cos(A)$.

$$5^2 = 4^2 + 7^2 - 2 \times 4 \times 7 \times \cos(A) \quad (1/2)$$

$$25 = 65 - 56 \cos(A) \quad (1/2)$$

$$\cos(A) = \frac{65-25}{56} = 40/56 \quad (1/2)$$

$$= 5/7 \quad (1/2)$$

[2 + 2 = 4 marks]

Question 7

A sector of a circle has a perimeter of 16 and an area of 16. Find the values of r and θ .

$$\left. \begin{aligned} 2r + r\theta &= 16 & \text{--- (1)} \\ \frac{1}{2}r^2\theta &= 16 & \text{--- (2)} \end{aligned} \right\} (1/2)$$

$$(2) \Rightarrow \theta = \frac{32}{r^2}$$

$$\text{SUB IN (1)} \Rightarrow 2r + r \times \frac{32}{r^2} = 16 \quad (1/2)$$

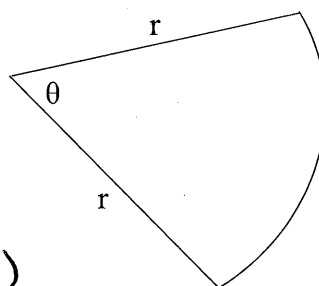
$$2r + \frac{32}{r} = 16$$

$$\times r \Rightarrow 2r^2 + 32 = 16r \quad (1/2)$$

$$r^2 - 8r + 16 = 0 \quad (1/2)$$

$$(r-4)^2 = 0 \quad (1/2)$$

$$r = 4, \theta = 2 \quad (1/2)$$



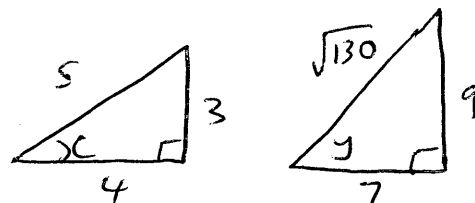
[3 marks]

$$\text{OR } \theta = \frac{360}{\pi}$$

7

Question 8

If $\sin(x) = \frac{3}{5}$, $0 \leq x \leq \frac{\pi}{2}$ and $\tan(y) = \frac{9}{7}$, $\pi \leq y \leq \frac{3\pi}{2}$



Find:

$$\begin{aligned} \text{a) } \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \quad (1/2) \\ &= \frac{4}{5} \times \frac{-7}{\sqrt{130}} - \frac{3}{5} \times \frac{-9}{\sqrt{130}} \quad (1/2) \\ &= \frac{-28}{5\sqrt{130}} + \frac{27}{5\sqrt{130}} = \frac{-1}{5\sqrt{130}} \quad \text{or} \quad \frac{-\sqrt{130}}{650} \quad (1/2) \end{aligned}$$

b) $\tan(x-y)$

$$= \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)} \quad (1/2)$$

$$\begin{aligned} &= \frac{3/4 - 9/7}{1 + 3/4 \times 9/7} = \frac{\frac{21-36}{28}}{\frac{28+27}{28}} = \frac{-15}{55} = \frac{-3}{11} \quad (1/2) \end{aligned}$$

c) $\operatorname{cosec}(2y)$

$$\begin{aligned} &= \frac{1}{\sin(2y)} \quad (1/2) \\ &= \frac{1}{2\sin(y)\cos(y)} = \frac{1}{2 \times \frac{-9}{\sqrt{130}} \times \frac{-7}{\sqrt{130}}} = \frac{65}{63} \quad (1/2) \end{aligned}$$

[2+2+2 = 6 marks]

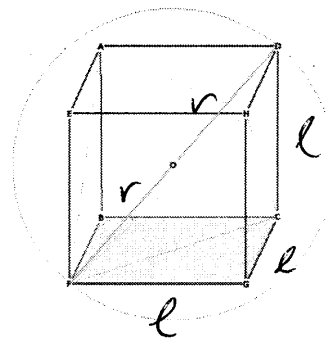
Question 9.

A sphere has radius r . Find (in terms of r) the side length of the largest possible cube which can fit inside the sphere.

$$(1) \quad \sqrt{l^2 + l^2 + l^2} = 2r$$

$$(1) \quad 3l^2 = 4r^2$$

$$(1) \quad l = \frac{2}{\sqrt{3}}r \quad \text{OR} \quad l = \frac{2\sqrt{3}}{3}r$$



[3 marks]

Question 10.

If $z = 3 + 3\sqrt{3}i$ and $w = 1 + i$, find:

a) z and w in polar form.

$$z = 6 \operatorname{cis}(\pi/3) \quad (1)$$

$$w = \sqrt{2} \operatorname{cis}(\pi/4) \quad (1)$$

b) w^6 in cartesian form.

$$w^6 = (\sqrt{2})^6 \operatorname{cis}(6 \times \pi/4) \quad (1/2)$$

$$= 8 \operatorname{cis}(-\pi/2) \quad (1/2)$$

$$= -8i \quad (1)$$

c) $\frac{z}{w}$ in Cartesian form.

$$\frac{z}{w} = \frac{3+3\sqrt{3}i}{1+i} \times \frac{1-i}{1-i} = \frac{3-3i+3\sqrt{3}i+3\sqrt{3}}{2} \quad (1/2)$$

$$= \frac{3+3\sqrt{3}}{2} + \frac{3\sqrt{3}-3}{2}i \quad (1)$$

d) $\frac{z}{w}$ in polar form.

$$\frac{z}{w} = \frac{6 \operatorname{cis}(\pi/3)}{\sqrt{2} \operatorname{cis}(\pi/4)} = 3\sqrt{2} \operatorname{cis}(\pi/3 - \pi/4) = 3\sqrt{2} \operatorname{cis}(\pi/12) \quad (1/2) \quad (1/2) \quad (1)$$

e) Use your answer to parts c) and d) to find the exact value of $\sin(\frac{\pi}{12})$

EQUATING IMAGINARY COMPONENTS: $(1/2)$

$$3\sqrt{2} \sin(\pi/12) = \frac{3\sqrt{3}-3}{2} \quad (1/2)$$

$$\sin(\pi/12) = \frac{3\sqrt{3}-3}{6\sqrt{2}} \quad (1/2)$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \quad \text{OR} \quad \frac{\sqrt{6}-\sqrt{2}}{4} \quad (1/2) \quad [2+2+2+2+2 = 10 \text{ marks}]$$

Question 11.

a) Show that $z = i$ is a solution to the equation: $z^3 - iz^2 - 4z + 4i = 0$

$$\text{Sub } z=i \Rightarrow \text{LHS} = i^3 - i^2 - 4i + 4i \quad (1)$$

$$= 0 = \text{RHS} \quad (1)$$

$\therefore z=i$ is a solution

b) Find all solutions to the equation.

$$z^2(z-i) - 4(z-i) = 0 \quad (1/2)$$

$$(z^2 - 4)(z-i) = 0 \quad (1/2)$$

$$(z-2)(z+2)(z-i) = 0 \quad (1/2)$$

$$z = \pm 2, i \quad (1/2)$$

[2 + 2 = 4 marks]

Question 12.

A polar equation is defined by: $r = 2 \sin(\theta)$

Find the equivalent Cartesian equation.

$$\sqrt{x^2 + y^2} = \frac{2y}{\sqrt{x^2 + y^2}} \quad (1/2)$$

$$x^2 + y^2 = 2y \quad (1/2)$$

$$x^2 + y^2 - 2y + 1 = 1 \quad (1/2)$$

$$x^2 + (y-1)^2 = 1 \quad (1/2)$$

[2 marks]

6

Question 13.

An ellipse is defined by the parametric equations:

$$x = 5 \cos(2t)$$

$$y = 3 \sin(2t)$$

a) Using a trigonometric identity find the Cartesian equation of the ellipse.

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \quad (1)$$

$$\frac{x^2}{25} + \frac{y^2}{9} = 1 \quad (1)$$

b) Show that the ellipse can also be defined by the locus definition :

$F_1P + F_2P = 10$, where $F_1 = (-4,0)$ and $F_2 = (4,0)$

$$\sqrt{(x+4)^2 + y^2} + \sqrt{(x-4)^2 + y^2} = 10 \quad (1/2)$$

$$\sqrt{(x+4)^2 + y^2} = 10 - \sqrt{(x-4)^2 + y^2}$$

$$\cancel{x^2} + 8x + \cancel{16} + y^2 = 100 - 20\sqrt{(x-4)^2 + y^2} + \cancel{x^2} - 8x + \cancel{16} + y^2 \quad (1/2)$$

$$20\sqrt{(x-4)^2 + y^2} = 100 - 16x \quad (1/2)$$

$$5\sqrt{(x-4)^2 + y^2} = 25 - 4x$$

$$25(x^2 - 8x + 16 + y^2) = 625 - 200x + 16x^2 \quad (1/2)$$

$$25x^2 - 200x + 400 + 25y^2 = 625 - 200x + 16x^2$$

$$9x^2 + 25y^2 = 225 \quad (1/2)$$

$$\frac{x^2}{25} + \frac{y^2}{9} = 1 \quad (1/2)$$

[2+3 = 5 marks]

End of Exam