

SOLUTIONS



Glen Waverley Secondary College
2016 Semester 1 Technology Free Exam

UNIT 1 Specialist Maths

Teacher:

Please circle



CAL



JLS



GMN

Student Name:

Student Code: Form Group:

Instructions:

Reading Time:	10 min
Writing Time:	60 min

Total Marks:	50 marks
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This examination consists of 14 short answer questions.
CAS calculators, a bound reference and/or summary notes are **NOT** allowed.
A formula sheet is provided.

Marks are allocated for neat and clear working as well as answers.

Question 1Solve the following simultaneous equations to find x and y in terms of m and n :

$$\begin{array}{rcll}
 2x - y = 3n & - \textcircled{1} & & \\
 x + y = 3m & - \textcircled{2} & \textcircled{1} + \textcircled{2} \Rightarrow & 3x = 3m + 3n & - (\frac{1}{2}) \\
 & & & x = m + n & - (\frac{1}{2}) \\
 & & & y = 3m - (m + n) & - (\frac{1}{2}) \\
 & & & y = 2m - n & - (\frac{1}{2})
 \end{array}$$

[2 marks]**Question 2**

Express the following as partial fractions:

$$\begin{aligned}
 \frac{-2x^2 - 3x - 5}{(x+1)(x^2 + 2x + 3)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+3} & (\frac{1}{2}) \\
 &= \frac{A(x^2+2x+3) + (x+1)(Bx+C)}{(x+1)(x^2+2x+3)} & (\frac{1}{2})
 \end{aligned}$$

$$x = -1 \Rightarrow -2 + 3 - 5 = A(1 - 2 + 3)$$

$$-4 = 2A$$

$$A = -2 \quad (\frac{1}{2})$$

$$x = 0 \Rightarrow -5 = 3x - 2 + C$$

$$C = 1 \quad (\frac{1}{2})$$

$$\begin{aligned}
 x = 1 \Rightarrow -10 &= -12 + 2B + 2 \Rightarrow B = 0 \\
 \frac{-2x^2 - 3x - 5}{(x+1)(x^2+2x+3)} &= \frac{-2}{x+1} + \frac{1}{x^2+2x+3} & (1)
 \end{aligned}$$

[3 marks]

Question 3

If $z_1 = 1 + \sqrt{2}i$ and $z_2 = 2 + \sqrt{5}i$ calculate:

$$\begin{aligned} \text{a) } z_1 \times z_2 &= (1 + \sqrt{2}i)(2 + \sqrt{5}i) = 2 + \sqrt{5}i + 2\sqrt{2}i - \sqrt{10} \quad (1) \\ &= (2 - \sqrt{10}) + (\sqrt{5} + 2\sqrt{2})i \quad (1) \end{aligned}$$

b) $\frac{z_1}{z_2}$ in the form $a + bi$ where a, b are real numbers.

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{1 + \sqrt{2}i}{2 + \sqrt{5}i} \times \frac{2 - \sqrt{5}i}{2 - \sqrt{5}i} \quad (1/2) \\ &= \frac{2 + 2\sqrt{2}i - \sqrt{5}i + \sqrt{10}}{4 + 5} \quad (1/2) \\ &= \frac{(2 + \sqrt{10}) + (2\sqrt{2} - \sqrt{5})i}{9} \\ &= \frac{2 + \sqrt{10}}{9} + \frac{2\sqrt{2} - \sqrt{5}}{9}i \quad (1) \end{aligned}$$

[2 + 2 = 4 marks]

Question 4

Find all solutions over the complex field in cartesian form for the equation:

$$z^3 = -1$$

$$r^3 \cos(3\theta) = \cos(\pi) \quad (1/2) \quad -\pi < \theta \leq \pi$$

$$r = 1 \quad (1/2) \quad -3\pi < 3\theta \leq 3\pi$$

$$3\theta = -\pi, \pi, 3\pi \quad (1/2)$$

$$\theta = -\pi/3, \pi/3, \pi \quad (1/2)$$

$$z = \cos(-\pi/3), \cos(\pi/3), \cos(\pi) \quad (1/2)$$

$$z = \frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i, -1 \quad (1/2)$$

[3 marks]

Question 5

For the complex equation: $z^3 - 2z^2 + z - 2 = 0$

a) Find $P(2)$

$$P(2) = 8 - 8 + 2 - 2 = 0$$

(1/2) (1/2)

b) Hence fully factorise the equation and find all possible solutions.

$$(z-2)(z^2+1) = 0 \quad (1/2)$$

$$(z-2)(z+i)(z-i) = 0 \quad (1/2)$$

$$z = 2, \pm i \quad (1)$$

[1 + 2 = 3 marks]

Question 6

An arithmetic sequence has a tenth term of 23 and the sum of the first six terms is 21. Find the third term.

$$t_{10} = a + 9d = 23 \quad \text{--- (1)} \quad (1/2)$$

$$S_6 = 3(a + 5d) = 21$$

$$3a + 15d = 21 \quad \text{--- (2)} \quad (1/2)$$

$$\textcircled{1} \times 3 \Rightarrow 3a + 27d = 69 \quad \text{--- (3)}$$

$$\textcircled{3} - \textcircled{2} \Rightarrow 12d = 48$$

$$d = 4 \quad (1/2)$$

$$\Rightarrow a + 36 = 23$$

$$a = -13 \quad (1/2)$$

$$t_3 = -13 + 2 \times 4$$

$$t_3 = -5 \quad (1)$$

[3 marks]

Question 7

The numbers $a, a+12, 9a$ form a geometric series. Find all possible values of a .

$$\frac{a+12}{a} = \frac{9a}{a+12} \quad (1/2)$$

$$a^2 + 24a + 144 = 9a^2 \quad (1/2)$$

$$8a^2 - 24a - 144 = 0 \quad (1/2)$$

$$a^2 - 3a - 18 = 0 \quad (1/2)$$

$$(a-6)(a+3) = 0 \quad (1/2)$$

$$a = -3, 6 \quad (1/2)$$

[3 marks]

Question 8

For the geometric sequence: $2, \sqrt{2}, 1, \dots$

a) Find the common ratio.

$$r = \frac{\sqrt{2}}{2} \quad (1)$$

b) Find the sum to infinity. Express as an exact value in simplest form.

$$S_{\infty} = \frac{2}{1 - \sqrt{2}/2} = \frac{4}{2 - \sqrt{2}} \times \frac{2 + \sqrt{2}}{2 + \sqrt{2}} \quad (1)$$

$$= \frac{8 + 4\sqrt{2}}{4 - 2}$$

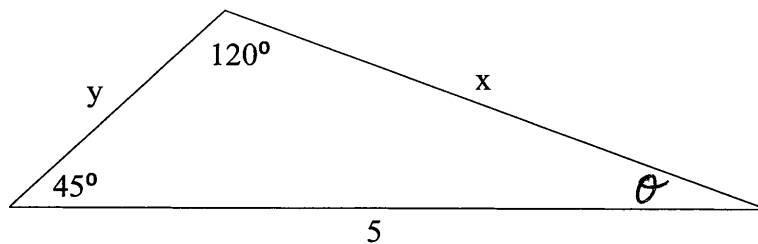
$$= 4 + 2\sqrt{2}$$

(1)

[1+2 = 3 marks]

(c) : $\frac{2}{1 - (a)}$

Question 9



a) Find x . Express your answer in exact form with a rational denominator.

$$\frac{x}{\sin 45^\circ} = \frac{5}{\sin 120^\circ} \quad (1/2)$$

$$x = \frac{5 \times \frac{1}{\sqrt{2}}}{\frac{\sqrt{3}}{2}} \quad (1/2)$$

$$x = \frac{10}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} \quad (1/2)$$

$$= \frac{5\sqrt{6}}{3} \quad (1/2)$$

b) Use the cosine rule to find an equation which could be used to solve for y . Simplify your equation as much as possible but do NOT solve this equation.

$$\theta = 180 - 120 - 45 = 15^\circ \quad (1/2)$$

$$y^2 = \left(\frac{5\sqrt{6}}{3}\right)^2 + 5^2 - 2 \times \frac{5\sqrt{6}}{3} \times 5 \times \cos 15^\circ \quad (1/2)$$

$$y^2 = \frac{50}{3} + 25 - \frac{50\sqrt{6}}{3} \cos 15^\circ \quad (1/2)$$

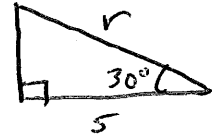
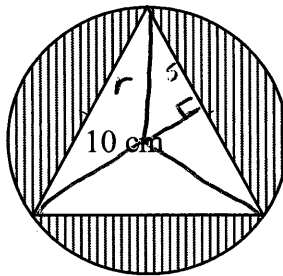
$$y^2 = \frac{125}{3} - \frac{50\sqrt{6}}{3} \cos 15^\circ \quad (1/2)$$

[2 + 2 = 4 marks]

(c) : $(a)^2 + 25 - 10(a) \cos 15^\circ$

Question 10.

An equilateral triangle of side length 10 cm is placed inside a circle such that each vertex is on the circumference of a circle.



a) Find the radius of the circle

$$\cos 30^\circ = \frac{5}{r} \quad (1)$$

$$r = \frac{5}{\sqrt{3}/2} = \frac{10}{\sqrt{3}} \text{ cm} \quad (1)$$

b) Find the shaded area in the diagram above.

$$A = 3 \times \frac{1}{2} \times \left(\frac{10}{\sqrt{3}}\right)^2 \times \left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right) \quad (1)$$

$$= 50 \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \quad (1)$$

$$= \left(\frac{100\pi}{3} - \frac{50\sqrt{3}}{2}\right) \text{ cm}^2 \quad (1)$$

$$\textcircled{C}: (10)^2 \times \frac{3}{2} \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

[2 + 3 = 5 marks]

Question 11.

A locus of points is defined as being equidistant from points A(1,-4) and B(3,2). Find the Cartesian equation of the locus. Express your answer in the form $y = mx + c$

$$\sqrt{(x-1)^2 + (y+4)^2} = \sqrt{(x-3)^2 + (y-2)^2} \quad (1)$$

$$\cancel{x^2} - 2x + 1 + \cancel{y^2} + 8y + 16 = \cancel{x^2} - 6x + 9 + \cancel{y^2} - 4y + 4$$

$$4x + 12y = -4 \quad (1)$$

$$12y = -4x - 4$$

$$y = -\frac{1}{3}x - \frac{1}{3} \quad (1)$$

[3 marks]

Question 12.

A graph is defined by the parametric equations:

$$x = 3 \cos(t) - 2 \quad \text{and} \quad y = 2 \sin(t) + 1$$

a) Find the Cartesian equation of the graph

$$\frac{(x+2)^2}{9} + \frac{(y-1)^2}{4} = 1 \quad (1)$$

b) Sketch the graph showing all relevant points (including x- and y- intercepts)

CENTRE $(-2, 1)$ $a = 3, b = 2$

x-INT : $y = 0$ $\frac{(x+2)^2}{9} + \frac{1}{4} = 1$

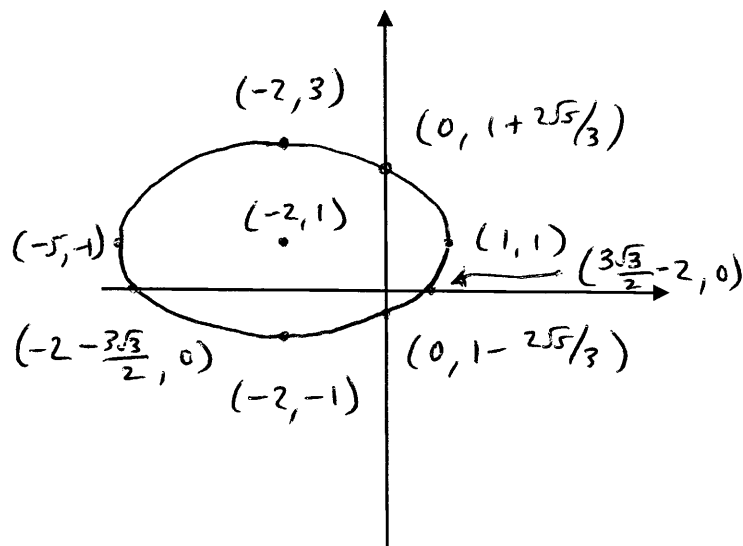
$$(x+2)^2 = \frac{27}{4}$$

$$x = -2 \pm \frac{3\sqrt{3}}{2}$$

y-INT : $x = 0$ $\frac{4}{9} + \frac{(y-1)^2}{4} = 1$

$$(y-1)^2 = \frac{20}{9}$$

$$y = 1 \pm \frac{2\sqrt{5}}{3}$$



NOTE : ALL NINE POINTS SHOULD BE SHOWN

Question 13.

Find the cartesian form of the equation:

$$r = \sin(\theta) \cdot \cos(\theta)$$

NOTE: y does not have to be the subject of the equation.

$$\sqrt{x^2 + y^2} = \frac{y}{\sqrt{x^2 + y^2}} \times \frac{x}{\sqrt{x^2 + y^2}} \quad (1)$$

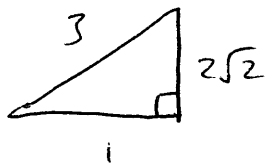
$$(x^2 + y^2)^{3/2} = xy \quad (1)$$

OR $(x^2 + y^2)^3 = x^2 y^2$

[2 marks]

Question 14.

If $\cos \theta = -\frac{1}{3}$ and $\pi < \theta < \frac{3\pi}{2}$ calculate the exact value of $\operatorname{cosec} 2\theta$



$$\sin \theta = -\frac{2\sqrt{2}}{3}$$



$$\begin{aligned}\operatorname{cosec}(2\theta) &= \frac{1}{\sin(2\theta)} \\ &= \frac{1}{2 \sin(\theta) \cos(\theta)} \\ &= \frac{1}{2 \times \frac{-2\sqrt{2}}{3} \times \frac{-1}{3}}\end{aligned}$$

$$= \frac{9}{4\sqrt{2}} \quad \text{OR} \quad \frac{9\sqrt{2}}{8}$$

(3 marks)

Question 15

Prove that $\sin \theta + \sin\left(\theta + \frac{2\pi}{3}\right) + \sin\left(\theta + \frac{4\pi}{3}\right) = 0$

$$\begin{aligned}\text{LHS} &= \sin \theta + \sin \theta \cos \frac{2\pi}{3} + \cos \theta \sin \frac{2\pi}{3} + \sin \theta \cos \frac{4\pi}{3} + \cos \theta \sin \frac{4\pi}{3} \\ &= \sin \theta - \frac{1}{2} \sin \theta + \frac{\sqrt{3}}{2} \cos \theta - \frac{1}{2} \sin \theta - \frac{\sqrt{3}}{2} \cos \theta \\ &= \sin \theta - \sin \theta + \frac{\sqrt{3}}{2} \cos \theta - \frac{\sqrt{3}}{2} \cos \theta \\ &= 0 \\ &= \text{RHS}\end{aligned}$$

AS REQUIRED

(3 marks)

End of Exam