

2012 VCAA Mathematical Methods (CAS)
Written Examination 2
MAV Suggested Answers and Solutions

SECTION 1

Answers

1. C 2. E 3. D 4. A 5. B 6. C
 7. D 8. A 9. D 10. C 11. E 12. A
 13. B 14. D 15. B 16. D 17. B 18. A
 19. B 20. E 21. A 22. B

Question 1

Answer C

$$f(x) = -3 \sin\left(\frac{\pi x}{5}\right)$$

$$\text{The period} = \frac{2\pi}{\left(\frac{\pi}{5}\right)} = 10$$

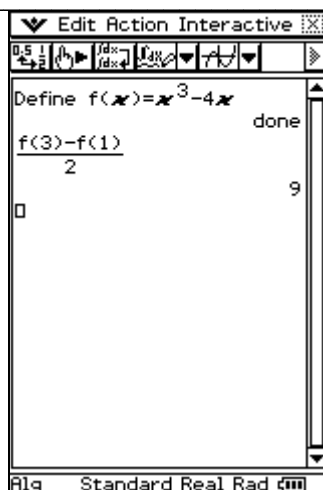
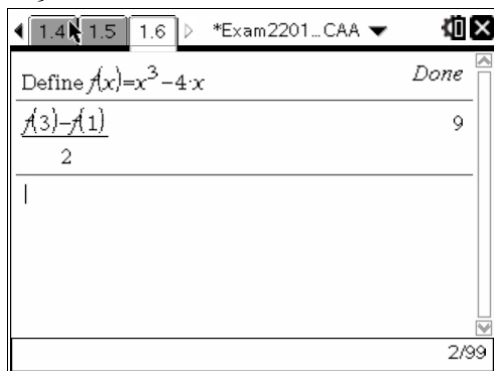
Question 2

Answer E

$$f(x) = x^3 - 4x$$

The average rate of change

$$= \frac{f(3) - f(1)}{3 - 1} = \frac{(27 - 12) - (1 - 4)}{2} = 9$$



Question 3

Answer D

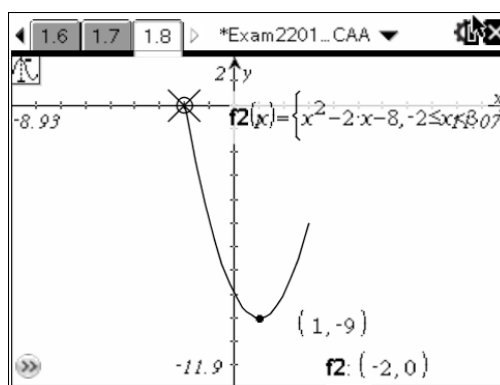
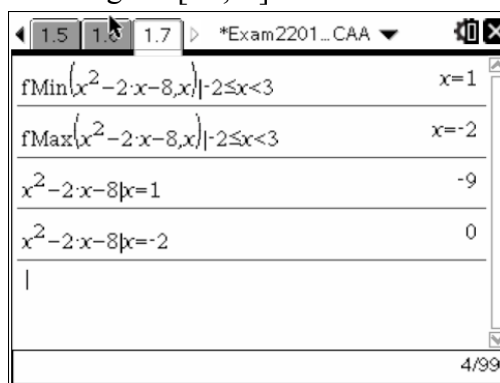
$$f: [-2, 3] \rightarrow \mathbb{R}, f(x) = x^2 - 2x - 8$$

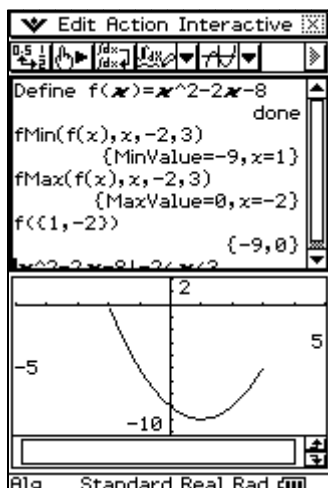
The x coordinate of the turning point

$$= \frac{-b}{2a} = \frac{2}{2} = 1$$

$$f(1) = -9, f(-2) = 0$$

The range is $[-9, 0]$.



**Question 4** **Answer A**

Using the chain rule

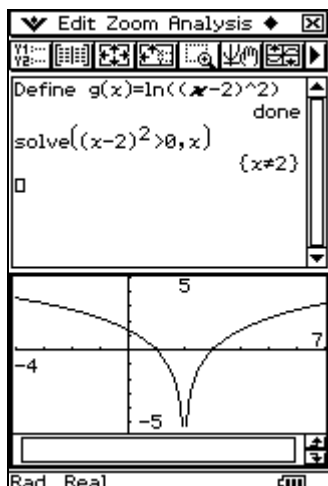
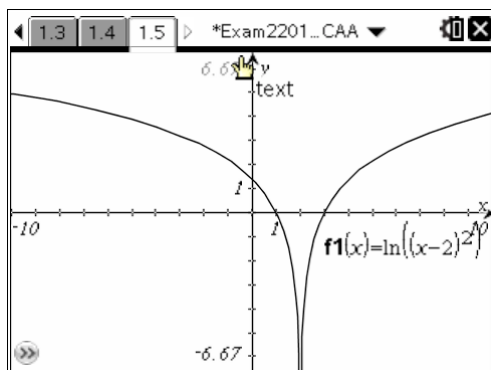
$$\frac{d}{dx}(g(e^{kx})) = ke^{kx} g'(e^{kx}) = kg'(e^{kx})e^{kx}$$

Question 5 **Answer B**

$$g(x) = \log_e((x-2)^2)$$

Maximal Domain

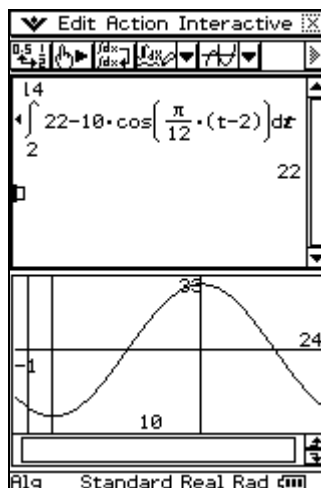
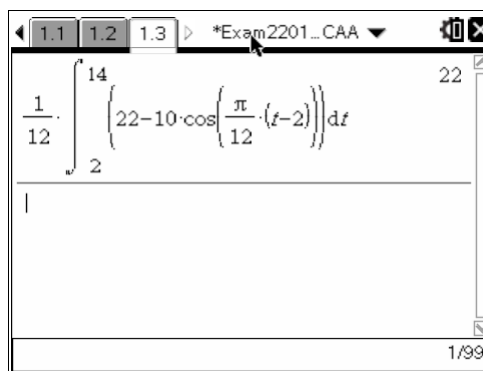
$$(x-2)^2 > 0, x \in \mathbb{R} \setminus \{2\}$$

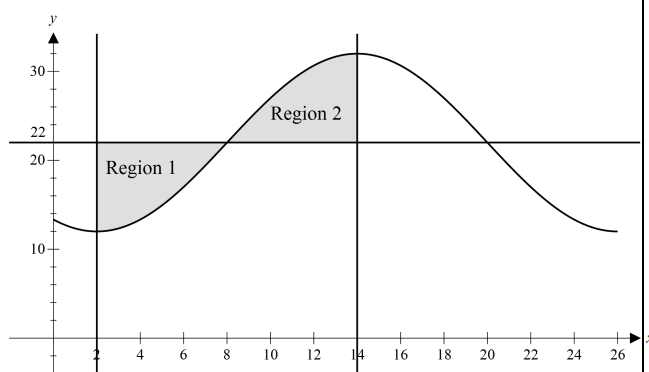
The range is $\mathbb{R}$.**Question 6** **Answer C**The period of the tan graph is $\frac{\pi}{2}$.The graph of f can be considered as the graph of $y = \tan(2x)$ translated $\frac{\pi}{4}$ units to the right.

$$\text{Hence } f(x) = \tan\left(2\left(x - \frac{\pi}{4}\right)\right).$$

Question 7 **Answer D**

$$\begin{aligned} \text{Average value} &= \frac{1}{b-a} \int_a^b (f(x)) dx \\ &= \frac{1}{14-2} \int_2^{14} (f(x)) dx \\ &= 22^\circ\text{C} \end{aligned}$$

**OR**The graph of the cosine function is translated up 22°C and has a period of $\frac{2\pi}{(\pi/12)} = 24$ hours. Therefore, the interval $[2, 14]$ covers one half of a complete cycle and, by symmetry, the average value must be 22°C .



That is, from the graph of the function, by symmetry, the area of Region 1 is equal to the area of Region 2, so the average is 22°C .

Question 8**Answer A**

The graph of f with rule $f(x) = ax^3 + bx^2 + cx$ has a local minimum at $x = m$ and a local maximum at $x = n$ as $a < 0$.

Hence the gradient of the graph of f will be negative for $(-\infty, m) \cup (n, \infty)$.

Question 9**Answer D**

$$y = \sqrt{b - x^2}$$

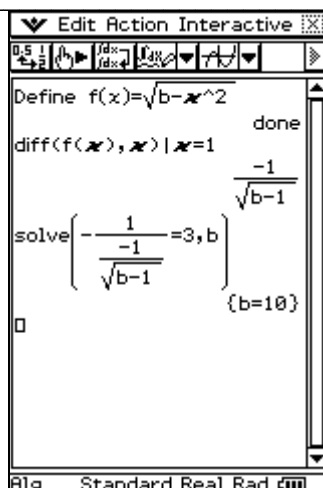
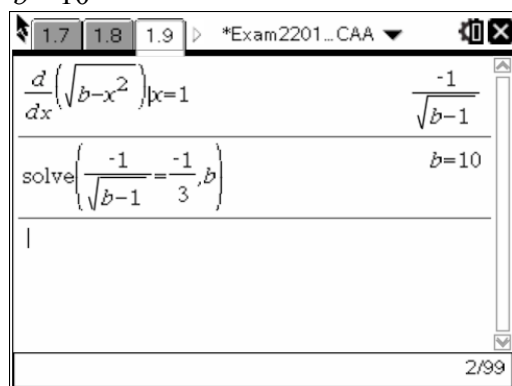
$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{b-x^2}} = \frac{-1}{\sqrt{b-1}}, \text{ when } x = 1$$

$$m_n \times m_t = -1$$

$$m_t = \frac{-1}{\sqrt{b-1}} = -\frac{1}{3}$$

$$\sqrt{b-1} = 3$$

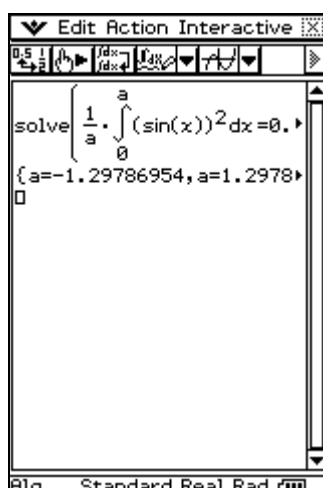
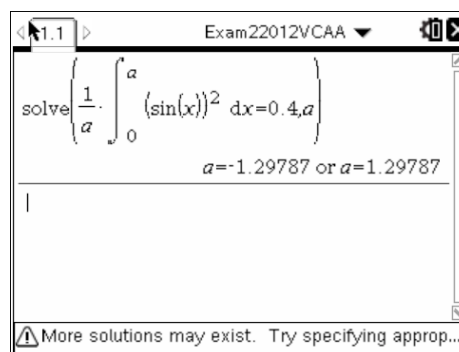
$$b = 10$$

**Question 10****Answer C**

$$\text{Average value} = \frac{1}{b-a} \int_a^b (f(x)) dx$$

$$= \frac{1}{a-0} \int_0^a (\sin^2(x)) dx$$

$$= 1.298 \text{ correct to three decimal places}$$

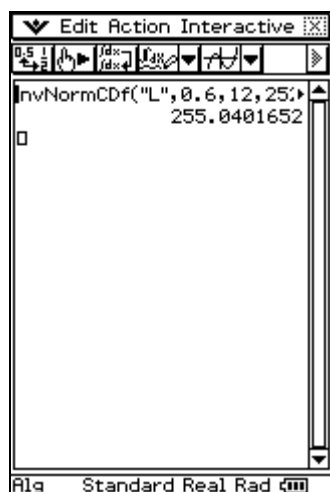
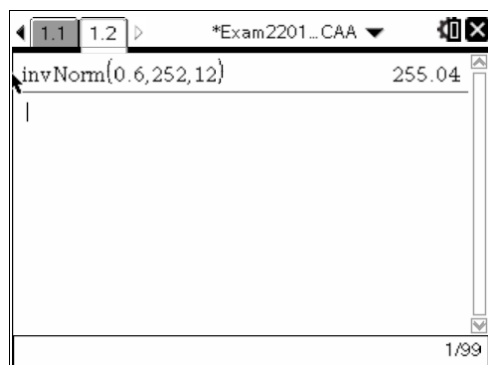


Question 11**Answer E**

$$X \sim N(252, 12^2)$$

$$\Pr(X < a) = 0.6$$

a is closest to 255.0

**Question 12****Answer A**

$$\Pr(W, W, L) + \Pr(W, L, W)$$

$$= 1 \times 0.7 \times 0.3 + 1 \times 0.3 \times 0.4$$

$$= 0.33$$

Question 13**Answer B**

$$\begin{aligned} \Pr(B' \setminus A) &= \frac{\Pr(B' \cap A)}{\Pr(A)} \\ &= \frac{\Pr(B' \cap A)}{\Pr(B' \cap A) + \Pr(A \cap B)} \\ &= \frac{\left(\frac{3}{7}\right)}{\left(\frac{3}{7} + \frac{2}{5}\right)} = \frac{15}{29} \end{aligned}$$

Question 14**Answer D**

Let $f(x) = x^2$, the inverse of $y = \sqrt{x}$.

Area of the rectangles

$$= f(1) + f(2) + f(3) + f(4)$$

$$= 1 + 4 + 9 + 16$$

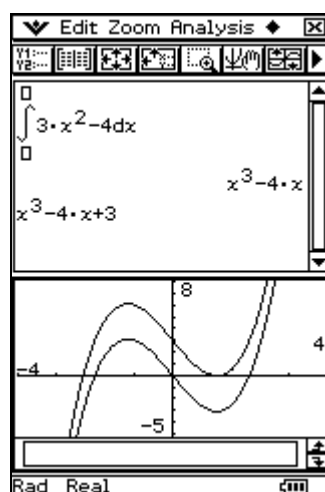
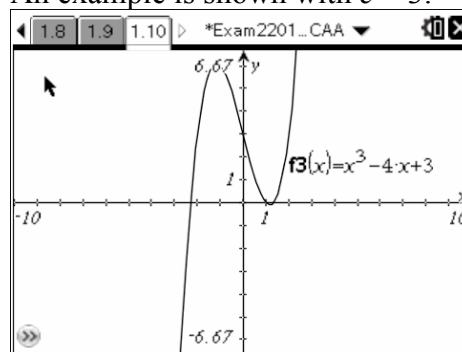
$$= 30$$

Question 15**Answer B**

$$f'(x) = 3x^2 - 4$$

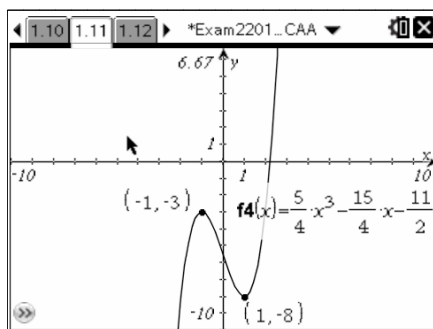
$$\begin{aligned} f(x) &= \int (3x^2 - 4) dx \\ &= x^3 - 4x + c \end{aligned}$$

An example is shown with $c = 3$.

**Question 16****Answer D**

The graph of f can be translated up so that the local maximum turning point is below the x -axis which means $c < 3$ or it can be translated up so that the local minimum is above the x -axis which means $c > 8$.

Hence $c < 0$ or $c > 8$.



Question 17

Answer B

$$\begin{bmatrix} m & 3 \\ 1 & m+2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ m \end{bmatrix}$$

Solve $\begin{vmatrix} m & 3 \\ 1 & m+2 \end{vmatrix} = 0$ for m

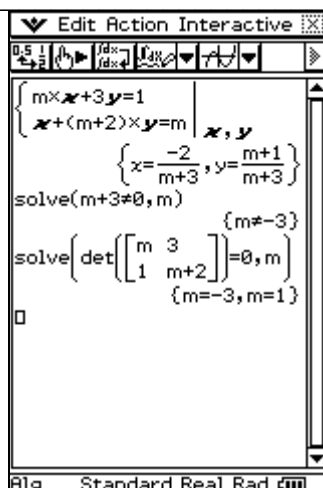
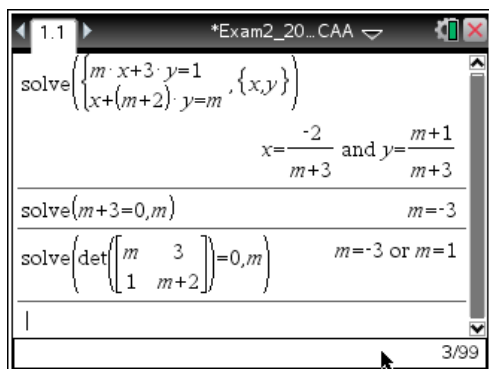
$$m = -3 \text{ or } m = 1$$

When $m = 1$, $x + 3y = 1$ and $x + 3y = 1$

The equations are the same. Infinite solutions.

When $m = -3$, $-3x + 3y = 1$ and $x - y = -3$

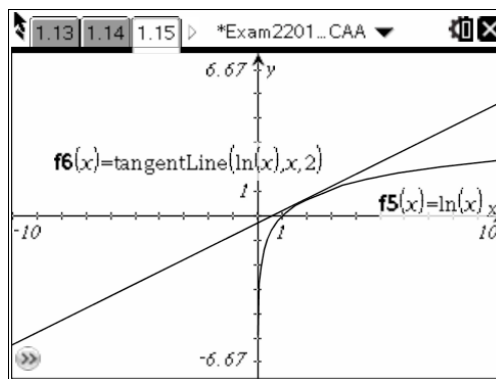
The lines are parallel. No solution.



Question 18

Answer A

Choose a value of a between 1 and e . The x -intercept for the tangent is positive which means A is false.



OR

The equation of the tangent is

$$y = \frac{1}{a}(x - a) + \log_e(a), \quad a > 0$$

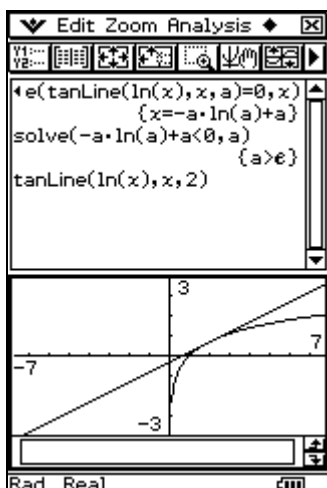
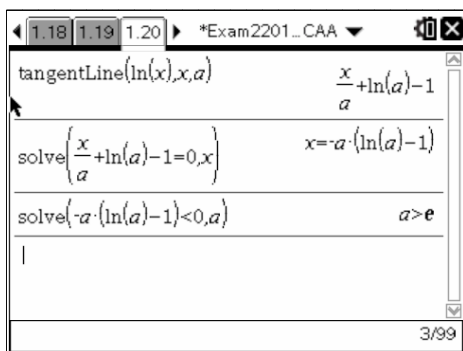
When $x = 0$

$$0 = \frac{1}{a}(x - a) + \log_e(a)$$

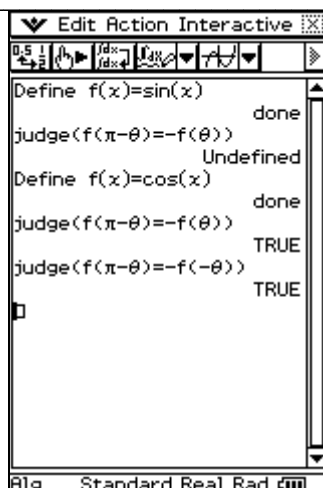
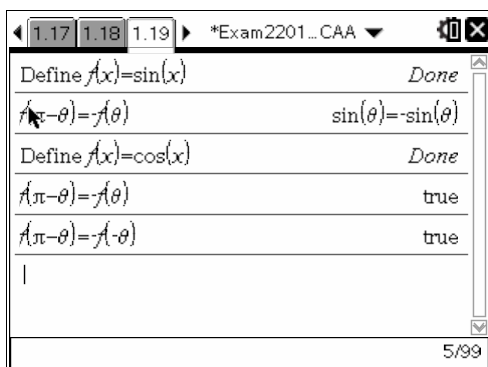
$$x = a - a \log_e(a) = b$$

If $b < 0$, $a \log_e(a) > a$

$$\log_e(a) > 1, \quad a > e$$



Question 19 **Answer B**
 $f(x) = \cos(x)$ is negative in the second quadrant and positive in the first and fourth quadrants.
Hence $f(\pi - \theta) = -f(\theta)$ and
 $f(\pi - \theta) = -f(-\theta)$.



Question 20 **Answer E**
 $\Pr(X > 1) = 1 - \Pr(X = 0) - \Pr(X = 1)$

$$= 1 - (1 - p)^0 p - (1 - p)p$$

$$= 1 - 2p + 2p^2 = (1 - p)^2$$

Question 21 **Answer A**

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$

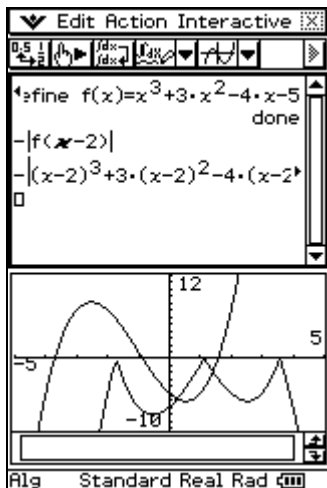
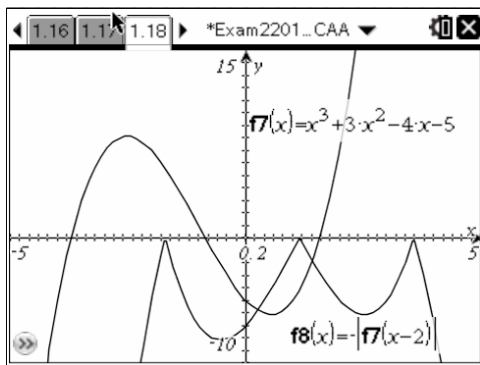
$$\frac{dV}{dt} = 300 \text{ cm}^3 / \text{min}$$

$$\frac{dV}{dh} = \pi h^2, \frac{dh}{dV} = \frac{1}{\pi h^2}, h = 5$$

$$\frac{dh}{dt} = 300 \times \frac{1}{\pi \times 5^2} = \frac{12}{\pi} \text{ cm/min}$$

Question 22 **Answer B**
The turning points of the graph of f are
 (a, b) local max and (c, d) local min.
Translate 2 units to the right.
 $(a + 2, b)$ local max and $(c + 2, d)$ local min
Make positive
 $(a + 2, b)$ local max and $(c + 2, -d)$ local max
Reflect in the x -axis
 $(a + 2, -b)$ local min and $(c + 2, d)$ local min
Local minima at $(a + 2, -b)$ and $(c + 2, d)$

An example is shown below.



SECTION 2

Question 1

a. $TSA = 2\left(xh + \frac{5x^2}{2} + \frac{5xh}{2}\right) = 6480$

$$2xh + 5x^2 + 5xh = 6480$$

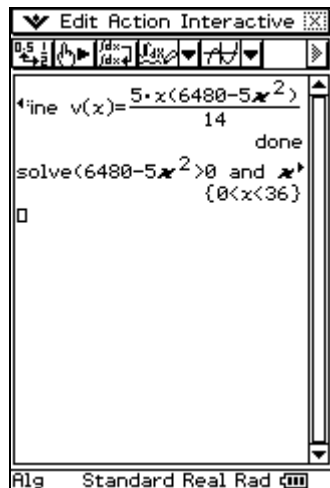
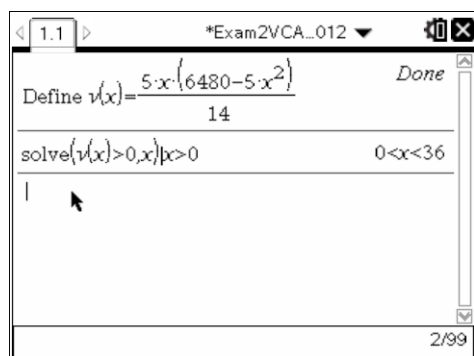
$$7xh + 5x^2 = 6480$$

$$7xh = 6480 - 5x^2$$

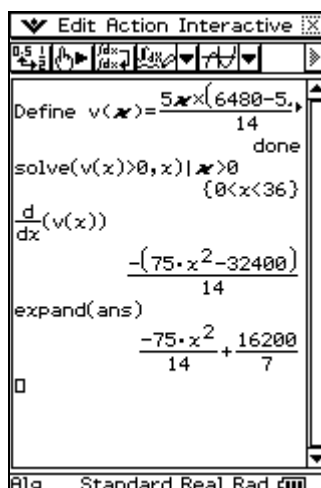
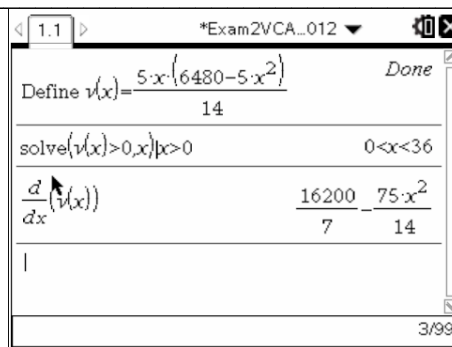
$$h = \frac{6480 - 5x^2}{7x}$$

b. Solve $V(x) > 0$ for x .

$$0 < x < 36$$

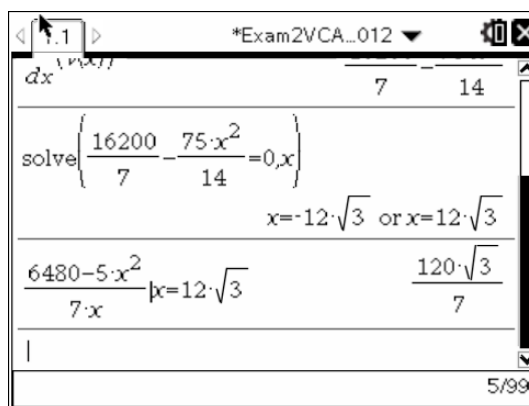


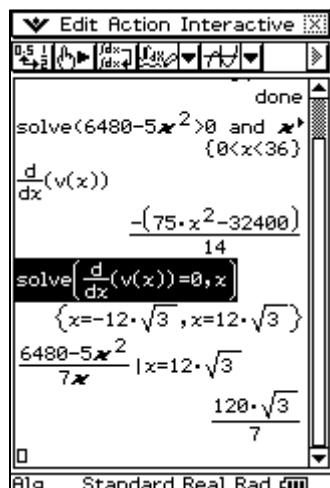
c. $\frac{dV}{dx} = -\frac{75x^2}{14} + \frac{16200}{7}$



d. Solve $\frac{dV}{dx} = 0$ for x .

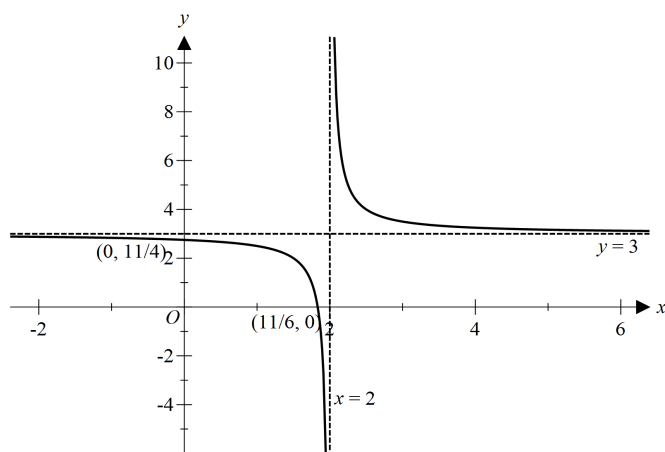
$$x = 12\sqrt{3}, h = \frac{120\sqrt{3}}{7}$$



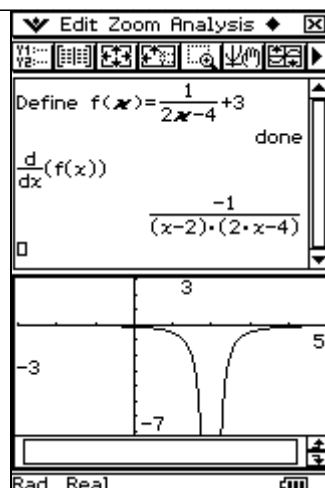
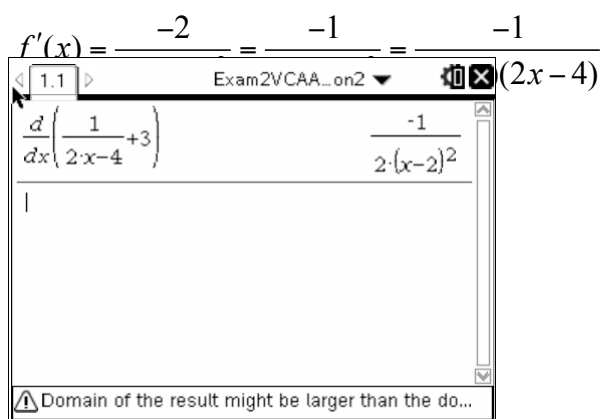


Question 2

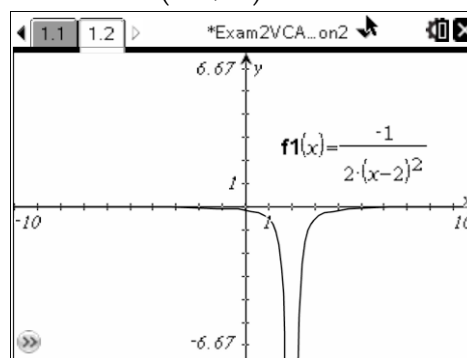
a. $y = f(x) = \frac{1}{2x-4} + 3$



b.i.



b.ii. R^- or $(-\infty, 0)$



b.iii. A stationary point occurs when the derivative is zero. $f'(x)$ never equals zero.

c. $m_t = f'(p) = \frac{-2}{(2p-4)^2}, (p, q)$

The equation of the tangent is

$$y - q = \frac{-2}{(2p-4)^2} (x - p).$$

$$f(q) = \frac{1}{2p-4} + 3$$

$$y - \frac{1}{2p-4} - 3 = \frac{-2}{(2p-4)^2} (x - p)$$

$$y - 3 = \frac{-2}{(2p-4)^2} (x - p) + \frac{1}{2p-4}$$

$$(2p-4)^2 (y-3) = -2(x-p) + 2p-4$$

$$(2p-4)^2 (y-3) = -2x + 4p - 4$$

d. Substitute $\left(-1, \frac{7}{2}\right)$ into

$$(2p-4)^2(y-3) = -2x+4p-4$$

and solve for p .

$$p=1 \text{ or } p=5$$

$$f(1) = \frac{5}{2}, f(5) = \frac{19}{6}$$

$$\left(1, \frac{5}{2}\right), \left(5, \frac{19}{6}\right)$$

e. $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$

If the graph of $f(x) = \frac{1}{2x-4} + 3$ is dilated by a factor of 2 from the y -axis, then

$$y' = \frac{1}{2\left(\frac{x}{2}\right) - 4} + 3 = \frac{1}{x-4} + 3.$$

This graph then needs to be translated 4 units to the left and 3 units down to obtain the graph of g where $g(x) = \frac{1}{x}$.

Hence $a=2$, $c=-4$, $d=-3$

OR

$$y' = \frac{1}{x'} \text{ and let } y = \frac{1}{2x-4} + 3, y-3 = \frac{1}{2x-4}$$

From the matrix equation

$$x' = ax + c \text{ and } y' = y + d$$

$$\text{Therefore } y + d = \frac{1}{ax + c}$$

$$\text{Equate coefficients with } y-3 = \frac{1}{2x-4}$$

Hence $a=2$, $c=-4$, $d=-3$

Question 3

a.i. $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$

a.ii. $X \sim \text{Bi}\left(20, \frac{1}{4}\right)$

$\Pr(X \geq 10) = 0.01386\dots = 0.0139$ correct to 4 decimal places.

$$\text{a.iii. } \text{Var}(X) = npq = \frac{75}{16}$$

$$n \times \frac{1}{4} \times \frac{3}{4} = \frac{75}{16}$$

$$n = \frac{75}{3} = 25$$

b.i. Let c be a correct answer.

There are two cases.

$c'cccc$ and $c'c'ccc$

$$\frac{1}{3} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} + \frac{2}{3} \times \frac{1}{3} \times \frac{3}{4} \times \frac{3}{4}$$

$$= \frac{9}{64} + \frac{1}{8} = \frac{17}{64}$$

OR

$$\begin{matrix} c_i & c'_i \\ c_{i+1} & c'_{i+1} \end{matrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{2}{3} \end{bmatrix}^2 = \begin{bmatrix} \frac{31}{48} & \frac{17}{36} \\ \frac{17}{48} & \frac{19}{36} \end{bmatrix}$$

$$\frac{17}{36} \times \frac{3}{4} \times \frac{3}{4} = \frac{17}{64}$$

$$\text{b.ii. } T^{24} \times S_0$$

$$\begin{matrix} c_i & c'_i \\ c_{i+1} & c'_{i+1} \end{matrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{2}{3} \end{bmatrix}^{24} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.57142... \\ 0.4285... \end{bmatrix}$$

The probability Q25 is answered correctly is 0.5714 correct to four decimal places.

c. $Y : \text{Bi}(25, p)$

$$\Pr(Y > 23) = \binom{25}{24} p^{24} (1-p) + p^{25}$$

$$6\Pr(Y = 25) = 6p^{25}$$

$$\binom{25}{24} p^{24} (1-p) + p^{25} = 6p^{25}, \quad p \neq 0$$

$$25(1-p) + p = 6p$$

$$25 - 30p = 0$$

$$p = \frac{5}{6}$$

d. $Y : \text{Bi}(25, p)$

$$\Pr(Y \geq 18) = 0.955268\dots$$

$$\Pr(Y \geq 22) = 0.381566\dots$$

$$W : N(a, b^2)$$

$$\frac{20-a}{b} = -1.69823\dots$$

$$\frac{25-a}{b} = 0.30137\dots$$

$$a = 24.2462\dots = 24.246 \text{ correct to three decimal places}$$

$$b = 2.50049\dots = 2.500 \text{ correct to three decimal places}$$

binomCdf(25, 5/6, 18, 25)	0.955268
binomCdf(25, 5/6, 22, 25)	0.381566
invNorm(1-0.38156649402957, 0, 1)	0.301369
invNorm(1-0.95526807234701, 0, 1)	-1.69823

invNorm(1-0.38156649402957, 0, 1)	0.301369
invNorm(1-0.95526807234701, 0, 1)	-1.69823
solve(20-a/b = -1.6982326770874 and 25-a/b = 0.301369)	a=24.24642702 and b=2.500497766

binomialCdf(18, 25, 25, 5/6)	0.9552680725
binomialCdf(22, 25, 25, 5/6)	0.3815664948
invNormCdf("L", 1-0.3815664948, 1, 0)	0.3013691848
invNormCdf("L", 1-0.9552680725, 1, 0)	-1.698232679
{ 20-a/b = -1.698232679, 25-a/b = 0.3013691848 }	a, b
{ a=24.24642703, b=2.500497766 }	

Question 4

a.i. $\frac{r}{h} = \frac{2}{10} = \frac{1}{5}, \quad r = \frac{h}{5}$

a.ii. $V = \frac{1}{3} \pi r^2 h$

$$= \frac{1}{3} \pi \left(\frac{h^2}{25} \right) h$$

$$= \frac{\pi h^3}{75}$$

b. $h = 10 + \frac{1}{1600} (t^3 - 1200t)$

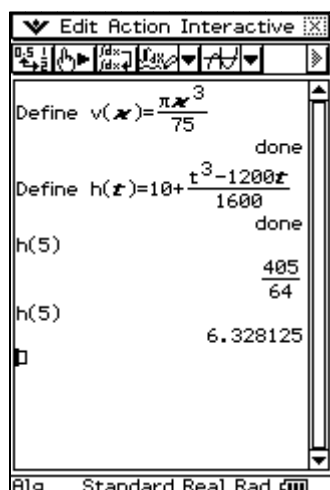
When $t = 20$

$$= 10 + \frac{1}{1600} (20^3 - 1200 \times 20)$$

$$= 10 - 10 = 0$$

c.i. $h(5) = \frac{405}{64} = 6.328125$

Define $v(x) = \frac{\pi x}{75}$	
Define $h(t) = 10 + \frac{t^3 - 1200t}{1600}$	Done
$h(5)$	405/64
$h(5)$	6.328125



c.ii. $\frac{dV}{dh} = \frac{\pi h^2}{25}$, $\frac{dh}{dt} = \frac{3}{1600}t^2 - \frac{3}{4}$

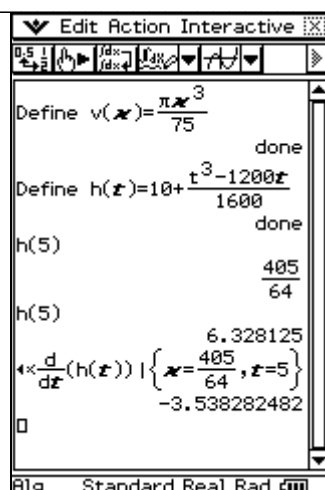
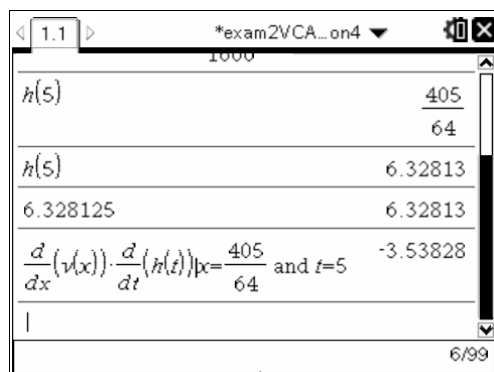
$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$= \frac{\pi h^2}{25} \times \left(\frac{3}{1600}t^2 - \frac{3}{4} \right), t = 5 \text{ and } h(5) = \frac{405}{64}$$

$$= -3.538...$$

$$= -3.5 \text{ correct to one decimal place}$$

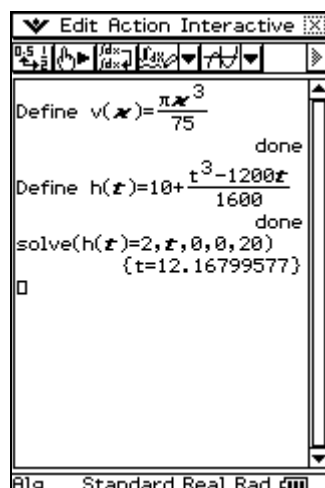
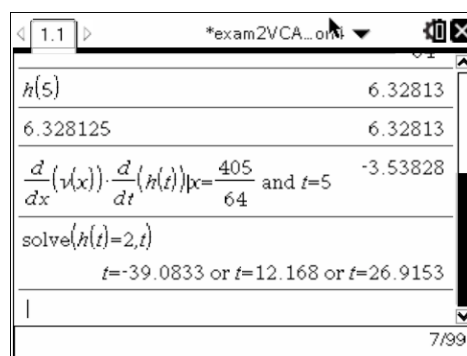
The the rate at which the volume is decreasing is 3.5 m³/min correct to one decimal place.



d. Solve $h(t) = 2$

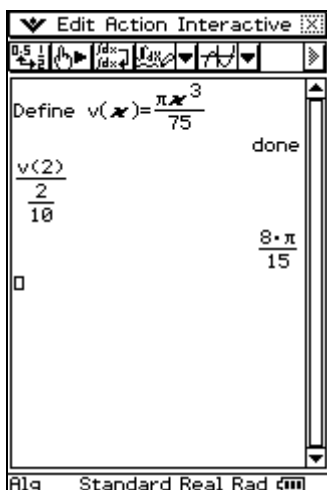
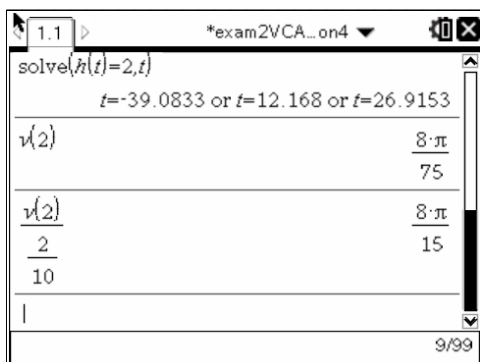
$t = 12.167...$

$= 12.2$ min correct to one decimal place.

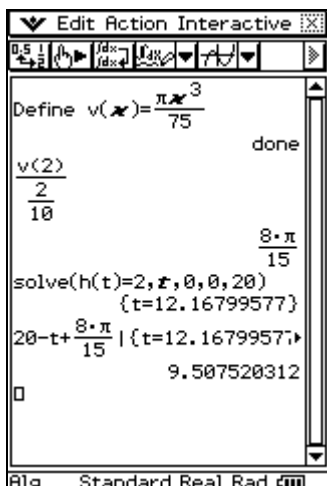
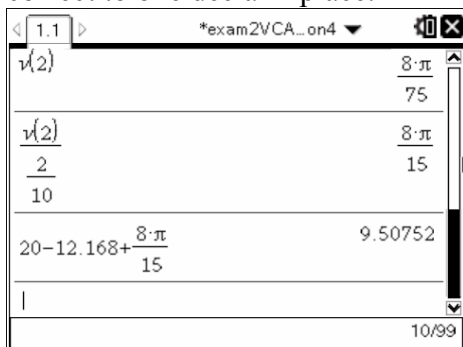


e. $V(2) = \frac{8\pi}{75} \text{ m}^3$, $\frac{dV}{dt} = 0.2 \text{ m}^3/\text{min}$

$$\text{time} = \frac{8\pi}{75 \times 0.2} = \frac{8\pi}{15} \text{ min}$$



f. time = $(20 - 12.167...) + \frac{8\pi}{15} = 9.5$ min
correct to one decimal place.

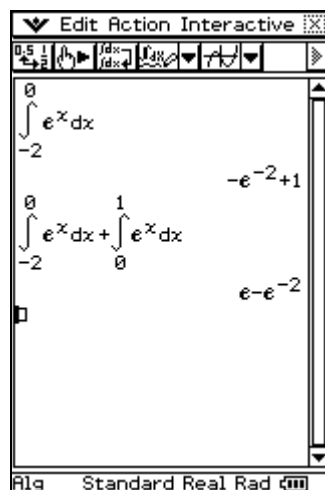
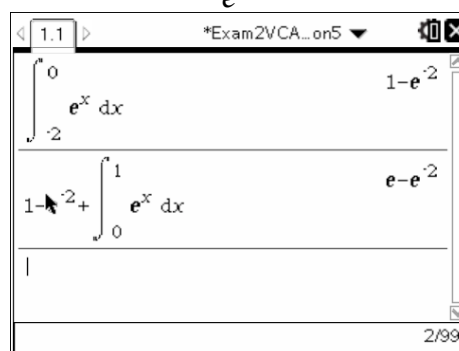


Question 5

a.i. $\int_{-2}^0 (e^x) dx = 1 - \frac{1}{e^2}$

a.ii. Area = $1 - \frac{1}{e^2}$, the same as a.i. because f and g are inverse functions. Therefore the region for a.ii. is a reflection in the line $y = x$ of the region for a.i.

a.iii. Area = $1 - \frac{1}{e^2} + \int_0^1 (e^x) dx$
 $= e - \frac{1}{e^2}$



b.i. Solve $g(x) = k(x)$ for x .

$$x = \frac{a \pm \sqrt{a^2 - 4}}{2}$$

1.1

$$1 - e^{-2} + \int_0^1 e^x dx = e - e^{-2}$$

$$\text{solve}(\ln(x) = -\ln(a-x), x)$$

$$x = \frac{\sqrt{a^2-4} - a}{2} \text{ and } -a \leq \sqrt{a^2-4} \leq a \text{ or } x = \frac{\sqrt{a^2}}{2}$$

3/99

Edit Action Interactive

$$\text{solve}(\ln(x) = -\ln(a-x), x)$$

$$\left\{ x = \frac{a - \sqrt{a^2-4}}{2}, x = \frac{a + \sqrt{a^2-4}}{2} \right\}$$

Alg Standard Real Rad

b.ii. $a^2 - 4 > 0$
 $a > 2$, as a is positive

1.1

$$1 - e^{-2} + \int_0^1 e^x dx = e - e^{-2}$$

$$\text{solve}(\ln(x) = -\ln(a-x), x)$$

$$x = \frac{\sqrt{a^2-4} - a}{2} \text{ and } -a \leq \sqrt{a^2-4} \leq a \text{ or } x = \frac{\sqrt{a^2}}{2}$$

$$\text{solve}(a^2 - 4 > 0, a) \quad a > 2$$

4/99

Edit Action Interactive

$$\text{solve}(\ln(x) = -\ln(a-x), x)$$

$$\left\{ x = \frac{a - \sqrt{a^2-4}}{2}, x = \frac{a + \sqrt{a^2-4}}{2} \right\}$$

$$\text{solve}(a^2 - 4 > 0, a) \quad a > 2$$

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c. Midpoint = $\frac{x_1 + x_2}{2} = \sqrt{2}$

$$\frac{1}{2} \left(\frac{a + \sqrt{a^2-4}}{2} + \frac{a - \sqrt{a^2-4}}{2} \right) = \sqrt{2}$$

$$a = 2\sqrt{2}$$

1.1

$$x = \frac{\sqrt{a^2-4} - a}{2} \text{ and } -a \leq \sqrt{a^2-4} \leq a \text{ or } x = \frac{\sqrt{a^2}}{2}$$

$$\text{solve}(a^2 - 4 > 0, a) \quad a > 2$$

$$\text{solve}\left(\frac{1}{2} \cdot \left(\frac{\sqrt{a^2-4} - a}{2} + \frac{\sqrt{a^2-4} + a}{2} \right) = \sqrt{2}, a\right)$$

$$a = 2\sqrt{2}$$

Domain of the result might be larger than the do...

Edit Action Interactive

$$\text{solve}(\ln(x) = -\ln(a-x), x)$$

$$\left\{ x = \frac{a - \sqrt{a^2-4}}{2}, x = \frac{a + \sqrt{a^2-4}}{2} \right\}$$

$$\text{solve}\left(\frac{\frac{a - \sqrt{a^2-4}}{2} + \frac{a + \sqrt{a^2-4}}{2}}{2} = \sqrt{2}, a\right)$$

$$\{a = 2\sqrt{2}\}$$

Alg Standard Real Rad

OR

The midpoint of AB is, by definition, $\frac{a}{2}$

$$\frac{a}{2} = \sqrt{2}$$

$$a = 2\sqrt{2}$$