



The Mathematical Association of Victoria

SPECIALIST MATHEMATICS

2007 WRITTEN EXAMINATIONS

SOLUTIONS

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2007 Specialist Mathematics
Written Examination 2
Suggested answers and solutions

Section 1 (Multiple-choice) Answers and Solutions

1. B	2. E	3. D	4. B	5. E
6. B	7. A	8. C	9. D	10. C
11. D	12. A	13. E	14. C	15. A
16. C	17. D	18. E	19. B	20. C
21. A	22. B			

Question 1

$$\frac{(x-2)^2}{a^2} - \frac{4(y+3)^2}{a^2} = 1$$

$$\frac{(x-2)^2}{a^2} - \frac{4(y+3)^2}{\left(\frac{a}{2}\right)^2} = 1$$

Asymptotes occur at:

$$y+3 = \pm \frac{\frac{a}{2}}{a}(x-2)$$

$$y+3 = \pm \frac{1}{2}(x-2)$$

Product of gradients:

$$-\frac{1}{2} \times \frac{1}{2} = -\frac{1}{4}$$

Question 2

Equation of Ellipse:

$$\frac{(x-1)^2}{3^2} + \frac{(y-3)^2}{2^2} = 1$$

$$\text{Let } \cos t = \frac{x-1}{3}$$

$$\text{and } \sin t = \frac{y-3}{2}$$

$$y = 2 \sin t + 3$$

$$x = 3 \cos t + 1$$

Question 3

D

$$\text{Range of } \arctan(x) \text{ is } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\text{Range of } b \arctan(x) \text{ is } \left(-\frac{b\pi}{2}, \frac{b\pi}{2}\right)$$

$$\text{Range of } a + b \arctan(x) \text{ is } \left(a - \frac{b\pi}{2}, a + \frac{b\pi}{2}\right)$$

For $a + b \arctan(x - c)$ the c will translate the graph to the right but will not affect the range. Thus Range will be

$$\left(a - \frac{b\pi}{2}, a + \frac{b\pi}{2}\right)$$

Question 4

B

An approach to this question is to eliminate correct alternatives until you reach the incorrect response.

Alternative A

$$\sin^2 x + \cos^2 x = 1$$

$$\frac{\sin^2 x}{\cos^2 x} + 1 = \frac{1}{\cos^2 x}$$

$$\sec^2 x - 1 = \tan^2 x$$

Alternative B

$$\sin^2 x + \cos^2 x = 1$$

$$\frac{\cos^2 x}{\sin^2 x} + 1 = \frac{1}{\sin^2 x}$$

$$\cot^2 x + 1 = \operatorname{cosec}^2 x$$

$$\cot^2 x - \operatorname{cosec}^2 x = -1$$

If we let $x = 4\theta$, we notice that

Alternative B has 1 where the result should be -1.

Question 5**E**

The graph has a period length of

$$T = \frac{5a}{2} - \frac{a}{2} = \frac{4a}{2} = 2a$$

$$T = \frac{2\pi}{n}$$

$$n = \frac{2\pi}{T} = \frac{2\pi}{2a} = \frac{\pi}{a}$$

Disregarding any translation a possible equation

$$\text{would be } y = \cos ec \frac{\pi}{a} x$$

The graph has also be translated by $\frac{a}{2}$ to the

left, which suggests that a possible function for

$$\text{the graph would be } y = \cos ec \frac{\pi}{a} \left(x + \frac{a}{2} \right)$$

Question 6**B**

$z^5 + z^2 - z + c = 0$ has real coefficients and therefore its roots must be in conjugate pairs, one of which would be of the form $z = x + 0i$. The remaining roots would be $x_1 \pm y_1 i$ and $x_2 \pm y_2 i$. Thus the complex plane which could represent this is given by alternative B.

Question 7**A**Given $z = 3 - 4i$

$$\begin{aligned} \frac{1}{1-z} &= \frac{1}{1-3+4i} \\ &= \frac{1}{-2+4i} \times \frac{-2-4i}{-2-4i} \\ &= \frac{-2-4i}{4+16} \\ &= \frac{2(-1-2i)}{20} \\ &= \frac{-1-2i}{10} \end{aligned}$$

Question 8**C**

$$z = 2cis\left(-\frac{\pi}{3}\right)$$

$$z^4 = 2^4 cis\left(-\frac{4\pi}{3}\right)$$

$$= 16cis\left(\frac{2\pi}{3}\right)$$

Question 9**D**

Algebraic Fractions of the form

$$\frac{ax+b}{(x+c)^2} \text{ have partial fractions}$$

$$\frac{A}{x+c} + \frac{B}{(x+c)^2}$$

$$\text{In the case of } \frac{x}{3(x+c)^2} = \frac{\frac{x}{3}}{(x+c)^2}$$

$$\frac{\frac{x}{3}}{(x+c)^2} = \frac{A}{x+c} + \frac{B}{(x+c)^2}$$

Question 10**C**

$$V = \pi \int_b^a x^2 dy$$

$$y = \sin^{-1}(2x)$$

$$2x = \sin y$$

$$x = \frac{\sin y}{2}$$

$$x^2 = \frac{\sin^2 y}{4}$$

$$= \frac{2\sin^2 y}{8}$$

$$= \frac{1 - \cos 2y}{8}$$

Considering the terminals

$$\text{When } x = 0 \quad y = \sin^{-1}(0) = 0$$

$$\text{When } x = \frac{1}{2} \quad y = \sin^{-1}(1) = \frac{\pi}{2}$$

$$V = \frac{\pi}{8} \int_0^{\frac{\pi}{2}} 1 - \cos 2y dy$$

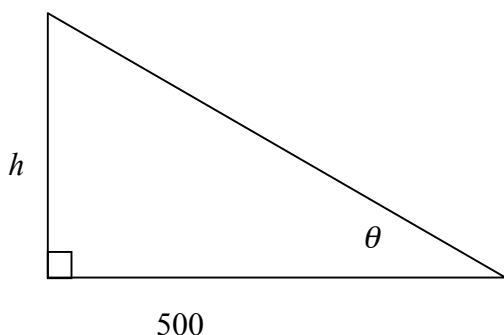
Question 11**D**

The numeric solution of $\frac{dy}{dx} = f(x)$ with $y = b$

when $x = a$ is $y = \int_a^x f(x)dx + b$.

In this case $f(x) = \sqrt{\sin(x)}$ and $a = 0$ and $b = 1$

$$y = \int_0^{\frac{\pi}{3}} \sqrt{\sin(x)} dx + 1 \text{ when } x = \frac{\pi}{3}$$

Question 12**A**

$$\tan \theta = \frac{h}{500}$$

$$h = 500 \tan \theta$$

$$\frac{dh}{dt} = 500 \sec^2 \theta \times \frac{d\theta}{dt}$$

We are told that when $\theta = \frac{\pi}{6}$, $\frac{d\theta}{dt} = 0.5$

$$\therefore \frac{dh}{dt} = 500 \sec^2 \left(\frac{\pi}{6} \right) \times 0.5$$

$$= 500 \left(\frac{2}{\sqrt{3}} \right)^2 \times 0.5$$

$$= 500 \times \frac{4}{3} \times \frac{1}{2} = \frac{1000}{3} = 333$$

Question 13**E**

$$\int_1^{e^x} \frac{(\log_e(x))^3}{x} dx$$

Let $u = \log_e(x)$

$$\frac{du}{dx} = \frac{1}{x}$$

$$dx = x du$$

When $x = 1$ $u = \log_e(1) = 0$

and when $x = e^3$ $u = \log_e e^3 = 3$

$$\begin{aligned} \int_1^{e^x} \frac{(\log_e(x))^3}{x} dx &= \int_0^3 \frac{u^3}{x} \times x du \\ &= \int_0^3 u^3 du \end{aligned}$$

Question 14**C**

$$\begin{aligned} \frac{dN}{dt} &\propto N(1000 - N) \\ &= kN(1000 - N) \end{aligned}$$

Question 15**A**

$$3x + 2y + 1 = 0$$

$$y = -\frac{3}{2}x - 1$$

$$m_p = -\frac{3}{2}$$

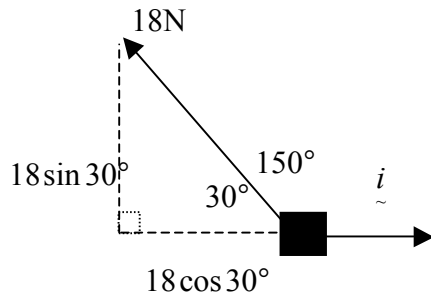
$$m_p \times m_n = -1$$

$$m_n = \frac{2}{3}$$

Gradient of Vector given by $\frac{y}{x}$

$\therefore$ Vector is $3\hat{i} + 2\hat{j}$

Question 16**C**



$$\begin{aligned} \vec{F} &= -18 \cos 30^\circ \vec{i} + 18 \sin 30^\circ \vec{j} \\ &= -18 \frac{\sqrt{3}}{2} \vec{i} + 18 \frac{1}{2} \vec{j} \\ &= -9\sqrt{3} \vec{i} + 9 \vec{j} \end{aligned}$$

Question 17

$$\begin{aligned} \vec{a} &= \vec{i} - 2\vec{j} - 2\vec{k} \\ \vec{b} &= 2\vec{i} + \vec{j} + 2\vec{k} \end{aligned}$$

$$\begin{aligned} \cos(\theta) &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \\ &= \frac{2 - 2 - 4}{\sqrt{1 + 4 + 4} \times \sqrt{4 + 1 + 4}} \\ &= \frac{-4}{\sqrt{9} \times \sqrt{9}} \\ &= \frac{-4}{9} \\ \theta &= \cos^{-1}\left(\frac{-4}{9}\right) \\ &= \pi - \cos^{-1}\left(\frac{4}{9}\right) \end{aligned}$$

Question 18

D

E

$$\vec{u} = 2\vec{i} - \vec{j} - 2\vec{k}$$

$$|\vec{u}| = \sqrt{4 + 1 + 4} = 3$$

$$\hat{\vec{u}} = \frac{1}{3} (2\vec{i} - \vec{j} - 2\vec{k})$$

Scalar resolute is

$$\hat{\vec{u}} \cdot \vec{v} = \frac{1}{3} (2\vec{i} - \vec{j} - 2\vec{k}) \cdot (a\vec{i} + 2\vec{j} - \vec{k}) = 1$$

$$\frac{1}{3} (2a - 2 + 2) = 1$$

$$a = \frac{3}{2}$$

Question 19

B

$$p = mv$$

To find the mass (m)

$$F = ma$$

$$F = 18N$$

$$a = 1.5m/s^2$$

$$18 = 1.5m$$

$$m = \frac{18}{1.5} = 12$$

To find the velocity (v)

$$v = u + at$$

$$u = 5m/s$$

$$a = 1.5m/s^2$$

$$t = 4$$

$$v = 5 + 1.5 \times 4$$

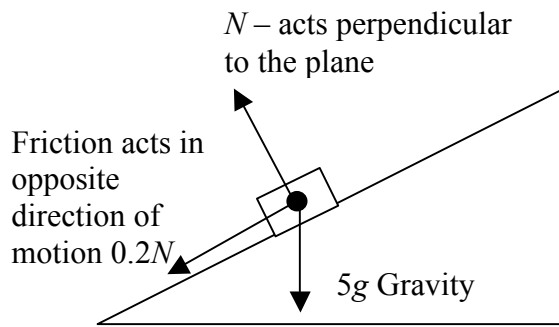
$$v = 11m/s$$

$$p = mv$$

$$p = 12 \times 11 = 132$$

Question 20

C



F is not included in the correct diagram because you are asked to consider the situation after the push by F

Question 21

A

$$F = ma$$

$$m = 12kg$$

$$a = \frac{d(\frac{1}{2}v^2)}{dx}$$

$$v = \sqrt{3x^2 - x^3 + 16}$$

$$v^2 = 3x^2 - x^3 + 16$$

$$\frac{v^2}{2} = \frac{3x^2}{2} - \frac{x^3}{2} + 8$$

$$\frac{d(\frac{1}{2}v^2)}{dx} = 3x - \frac{3x^2}{2}$$

$$F = 12\left(3x - \frac{3x^2}{2}\right)$$

Question 22

B

$$F = ma$$

$$= m \frac{dv}{dt}$$

$$\text{Resultant Force} = P - \text{Resistance}$$

$$= P - mkv^2$$

$$m \frac{dv}{dt} = P - mkv^2$$

Section 2

Question 1

a.

$$z_1 = -\sqrt{3} + i$$

$$= r \operatorname{cis} \theta$$

$$r = \sqrt{3+1}$$

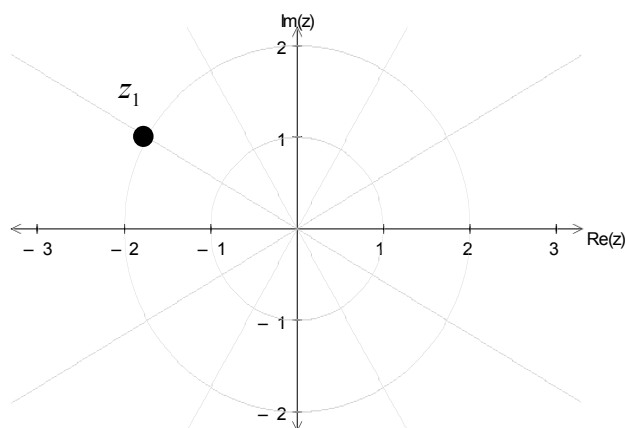
$$= \sqrt{4} = 2$$

$$\theta = \tan^{-1} \frac{y}{x}$$

$$= \tan^{-1} \frac{1}{-\sqrt{3}} = \frac{5\pi}{6}$$

$$z_1 = 2 \operatorname{cis} \left(\frac{5\pi}{6} \right)$$

b.



c.

$$z^2 - 2\sqrt{3}z + 4$$

$$z^2 - 2\sqrt{3}z + 3 - 3 + 4$$

$$(z - \sqrt{3})^2 + 1$$

$$(z - \sqrt{3})^2 - i^2$$

$$(z - \sqrt{3} + i)(z - \sqrt{3} - i)$$

$$(z - \sqrt{3} + i)(z - \sqrt{3} - i) = 0$$

$$z = \sqrt{3} - i$$

$$z = \sqrt{3} + i$$

d.

$$z = \sqrt{3} + i$$

$$z_1 = -\sqrt{3} + i$$

$$\text{For } z = \sqrt{3} - i$$

$$= -1(-\sqrt{3} + i)$$

$$= -1 \times z_1$$

$$\text{For } z = \sqrt{3} + i$$

$$= -1(-\sqrt{3} - i)$$

$$= -1 \times \bar{z}_1$$

e.

$$|z| = |z - z_1|$$

$$|x + iy| = |x + iy + \sqrt{3} - i|$$

$$|x + iy| = |x + \sqrt{3} + i(y - 1)|$$

$$\sqrt{x^2 + y^2} = \sqrt{(x + \sqrt{3})^2 + (y - 1)^2}$$

$$x^2 + y^2 = (x + \sqrt{3})^2 + (y - 1)^2$$

$$x^2 + y^2 = x^2 + 2\sqrt{3}x + 3 + y^2 - 2y + 1$$

$$0 = 2\sqrt{3}x - 2y + 4$$

$$y = \sqrt{3}x + 2$$

f.

$$z_1 = -\sqrt{3} + i$$

$$\bar{z}_1 = -\sqrt{3} - i$$

$$\Rightarrow x = -\sqrt{3} \text{ and } y = -1$$

From Question 1 e. we know

$$y = \sqrt{3}x + 2$$

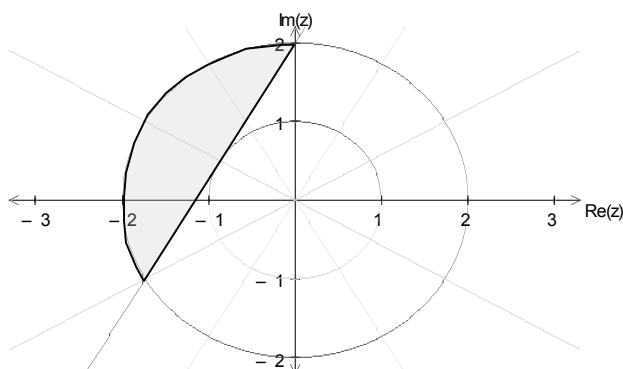
When $x = -\sqrt{3}$

$$y = \sqrt{3} \times -\sqrt{3} + 2$$

$$= -3 + 2 = -1$$

$$\therefore \bar{z}_1 \text{ satisfies } |z| = |z - z_1|$$

g.



Question 2

a.

$$f(x) = x \arctan(x)$$

$$f'(x) = \arctan(x) + \frac{x}{1+x^2}$$

$$f'(0) = \arctan(0) + \frac{0}{1+0^2} = 0$$

b.



c. i.

$$\int_0^1 x \arctan(x) dx$$

c. ii.

Using graphics calculator

$$\int_0^1 x \arctan(x) dx = 0.285$$

d.

From **part a.**

$$\frac{d(x \arctan(x))}{dx} = \arctan(x) + \frac{x}{1+x^2}$$

$$\int \frac{d(x \arctan(x))}{dx} dx = \int \arctan(x) dx + \int \frac{x}{1+x^2} dx$$

$$x \arctan(x) = \int \arctan(x) dx + \frac{1}{2} \int \frac{2x}{1+x^2} dx$$

$$x \arctan(x) = \int \arctan(x) dx + \frac{1}{2} \log_e(1+x^2)$$

$$\int \arctan(x) dx = x \arctan(x) - \frac{1}{2} \log_e(1+x^2)$$

e.

$$\begin{aligned} \int_0^1 \int \arctan(x) dx &= \left[x \arctan(x) - \frac{1}{2} \log_e(1+x^2) \right]_0^1 \\ &= 1 \arctan(1) - \frac{1}{2} \log_e(1+1^2) - 0 \\ &= \frac{\pi}{4} - \frac{1}{2} \log_e(2) \end{aligned}$$

f.

$$\text{Area enclosed} = \frac{\pi}{4} - \frac{1}{2} \log_e(2) - 0.285 \approx 0.15$$

Question 3

a.

$$F_r = ma = 3g - g$$

$$4a = 2g$$

$$a = \frac{2g}{4} = \frac{g}{2}$$

b.

Considering the 1 kg mass only

$$F_r = -1 \times \frac{g}{2} = g - T_1$$

$$T_1 = g + \frac{g}{2} = \frac{3g}{2}$$

c.

$$F_r = 4b$$

Force down the Plane (disregarding 1 kg mass):

$$F_p = 3g \sin(30^\circ)$$

Force on 1 kg mass (disregarding 3 kg mass):

$$\begin{aligned} F_1 &= g \\ F_r &= 3g \sin(30^\circ) - g \\ 4b &= 3g \sin(30^\circ) - g \\ b &= \frac{3g \sin(30^\circ) - g}{4} \\ &= \frac{g(\frac{3}{2} - 1)}{4} \\ &= \frac{g}{8} \end{aligned}$$

d.

$$b = \frac{3g \sin(\theta) - g}{4}$$

When $b = 0$

$$\begin{aligned} 0 &= \frac{3g \sin(\theta) - g}{4} \\ 0 &= 3g \sin(\theta) - g \\ \sin(\theta) &= \frac{1}{3} \\ \theta &= 19.5^\circ \end{aligned}$$

e.

Normal reaction of plane to 3 kg mass is given by $N = 3g \cos \theta$

Friction (F) is given by $F = \mu N = 0.6g \cos \theta$

On verge of sliding down the plane

$$\begin{aligned} F_r &= 3g \sin B - F - g \\ &= 3g \sin B - 0.6g \cos B - g = 0 \end{aligned}$$

Using a Calculator $B = 30.4^\circ$

On verge of sliding down the plane

$$\begin{aligned} F_r &= 3g \sin A + F - g \\ &= 3g \sin A + 0.6g \cos A - g = 0 \end{aligned}$$

Using a Calculator $A = 7.8^\circ$

Question 4**a.**

$$\vec{v}(t) = 30\vec{i} - 40\vec{j} - 4\vec{k}$$

$$\begin{aligned} \frac{d\vec{r}(t)}{dt} &= 30\vec{i} - 40\vec{j} - 4\vec{k} \\ \vec{r}(t) &= 30t\vec{i} - 40t\vec{j} - 4t\vec{k} + \vec{c} \end{aligned}$$

$$\begin{aligned} \vec{r}(10) &= 300\vec{i} - 400\vec{j} - 40\vec{k} + \vec{c} \\ &= -500\vec{i} - 2500\vec{j} + 200\vec{k} \\ \vec{c} &= -800\vec{i} + 2900\vec{j} + 240\vec{k} \end{aligned}$$

$$\begin{aligned} \vec{r}(t) &= 30t\vec{i} - 40t\vec{j} - 4t\vec{k} - 800\vec{i} + 2900\vec{j} + 240\vec{k} \\ &= (30t - 800)\vec{i} + (2900 - 40t)\vec{j} + (240 - 4t)\vec{k} \end{aligned}$$

b.

Landing occurs when $\vec{k}$ component of

$\vec{r}(t)$ is equal to zero

$$\therefore 240 - 4t = 0$$

$$\Rightarrow t = 60$$

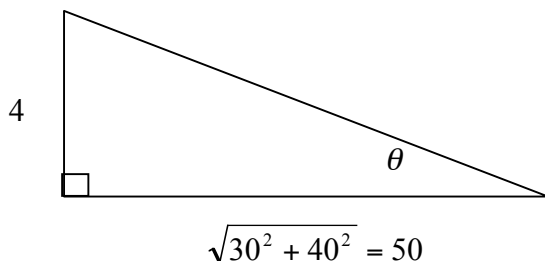
Plane lands at $t = 60$

$$\begin{aligned} \vec{r}(60) &= (1800 - 800)\vec{i} + (2900 - 1600)\vec{j} + (240 - 240) \\ &= 1000\vec{i} + 500\vec{j} \end{aligned}$$

$$\begin{aligned} |\vec{r}(60)| &= \sqrt{1000^2 + 500^2} \\ &\approx 1118 \text{ metres from tower} \end{aligned}$$

c.

The velocity vector $\vec{v}(t) = 30\vec{i} - 40\vec{j} - 4\vec{k}$ tells us both the rate and the angle of descent. We can consider this diagrammatically.



$$\tan^{-1} \theta = \frac{5}{50}$$

$$\theta \approx 4.6^\circ$$

d.

The plane is closest to the tower when the position vector $\vec{r}(t)$ is perpendicular to the velocity vector $\vec{v}$.

$$\vec{v}(t) \cdot \vec{r}(t) = 0$$

$$\Rightarrow \left((30t - 800)\vec{i} + (2900 - 40t)\vec{j} + (240 - 4t)\vec{k} \right) \cdot$$

$$\left(30\vec{i} - 40\vec{j} - 4\vec{k} \right) = 0$$

$$\Rightarrow 30(30t - 800) - 40(2900 - 40t) - 4(240 - 4t) = 0$$

$$900t - 24000 - 116000 + 1600t - 960 + 16t$$

$$2516t = 140960$$

$$t = 56$$

e.

$$\text{Speed} = |\vec{v}|$$

$$= \sqrt{30^2 + (-40)^2 + (-4)^2}$$

$$= \sqrt{900 + 1600 + 16}$$

$$= \sqrt{2516} = 50.1597$$

$$\text{Distance} = \text{Speed} \cdot \text{Time}$$

$$= 50.1597 \times 60$$

$$\approx 3010$$

Question 5**a.**

$$v = 20 - 2 \tan^{-1}(t)$$

$$17 = 20 - 2 \tan^{-1}(t)$$

$$2 \tan^{-1}(t) = 3$$

$$t = \tan\left(\frac{3}{2}\right) \approx 14.1$$

b.

For large values of t , $\tan^{-1}(t)$ approaches

$$\frac{\pi}{2},$$

$$\therefore v \rightarrow \left(20 - 2 \times \frac{\pi}{2} \right) = 20 - \pi$$

$$= 20 - 3.142 = 16.858$$

$$\therefore v < 16.858 \text{ for all values of } t$$

c.

$$D_{\text{polluting}} = \int_0^T (20 - 2 \tan^{-1}(t)) dt$$

d.

$$D_{\text{police}} = \int_3^8 \left(13 \cos^{-1} \left(\frac{13 - 2t}{7} \right) \right) dt$$

$$\text{Distance Ahead} = D_{\text{polluting}} - D_{\text{police}}$$

$$= \int_0^8 (20 - 2 \tan^{-1}(t)) dt - \int_3^8 \left(13 \cos^{-1} \left(\frac{13 - 2t}{7} \right) \right) dt$$

$$= 60.7 \text{ via a calculator}$$

e.

For the police car when

$$t \geq 8, v = 13 \cos^{-1} \left(\frac{13 - 2 \times 8}{7} \right) = 13 \cos^{-1} \left(\frac{-3}{7} \right)$$

The distance travel by the police car for values
 $8 \leq t \leq T_c$

$$\text{Distance} = (T_c - 8) 13 \cos^{-1} \left(\frac{-3}{7} \right)$$

$$\text{Distance of polluting Car} = \int_0^{T_c} (20 - 2 \tan^{-1}(t)) dt$$

$$\therefore (T_c - 8) 13 \cos^{-1} \left(\frac{-3}{7} \right) = 60.7 + \int_8^{T_c} (20 - 2 \tan^{-1}(t)) dt$$

f.

From Question 5 b. we know that

$$20 - 2 \tan^{-1}(t) \approx 16.848$$

$$\therefore \int_8^{T_c} (20 - 2 \tan^{-1}(t)) dt \approx \int_8^{T_c} (16.848) dt$$

$$= [16.848t]_8^{T_c} = 16.848(T_c - 8)$$

From Question 5 e.

$$\therefore (T_c - 8) 13 \cos^{-1} \left(\frac{-3}{7} \right) = 60.7 + 16.848(T_c - 8)$$

$$T_c \approx 15$$