



The Mathematical Association of Victoria

MATHS METHODS (CAS)

2007 WRITTEN EXAMINATIONS

SOLUTIONS

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"Cliveden", 61 Blyth Street, Brunswick, 3056

Phone: (03) 9380 2399 Fax: (03) 9389 0399

E-mail: office@mav.vic.edu.au website: <http://www.mav.vic.edu.au>

2007 Mathematical Methods (CAS)
Exam 2 Suggested Answers and Solutions
Part 1 Multiple Choice

Answers

1. A 2. D 3. C 4. C 5. C 6. B
 7. A 8. B 9. E 10. B 11. E 12. C
 13. D 14. A 15. B 16. C 17. E 18. D
 19. A 20. A 21. A 22. E

Question 1**A**

$$f(x) = 6 - 2x$$

Range is $[-4, 12]$

$$-4 = 6 - 2x$$

$$-2x = -10$$

$$x = 5$$

$$12 = 6 - 2x$$

$$-2x = 6$$

$$x = -3$$

Domain, D , is $[-3, 5]$ **Question 2****D**

$$g(x) = x^2 + 2x - 3 \text{ and } f(x) = e^{2x+3}$$

$$f(g(x)) = e^{2(x^2+2x-3)+3}$$

$$= e^{2x^2+4x-6+3}$$

$$= e^{2x^2+4x-3}$$

F1+ Tools	F2+ Algebra	F3+ Calc	F4+ Other	F5 Pr3mID	F6+ Clean Up
■ Define $g(x) = x^2 + 2 \cdot x - 3$ Done					
■ Define $f(x) = e^{2 \cdot x + 3}$ Done					
■ $f(g(x))$ $e^{2 \cdot x^2 + 4 \cdot x - 3}$					
$f(g(x))$					
MAIN DEG AUTO FUNC 3/99					

Question 3**C**

$$y = \log_a(7x - b) + 3$$

$$y - 3 = \log_a(7x - b)$$

$$a^{y-3} = 7x - b$$

$$7x = a^{y-3} + b$$

$$x = \frac{1}{7}(a^{y-3} + b)$$

Question 4**C**

$$f(x) = x^3 - \sqrt{x+1}$$

$$f(0) = -1$$

$$f(3) = 25$$

$$\begin{aligned} \text{Average rate} &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ &= \frac{f(3) - f(0)}{3 - 0} \\ &= \frac{25 - (-1)}{3} \end{aligned}$$

$$\text{Average rate} = \frac{26}{3}$$

F1+ Tools	F2+ Algebra	F3+ Calc	F4+ Other	F5 Pr3mID	F6+ Clean Up
■ Define $f(x) = x^3 - \sqrt{x+1}$ Done					
■ $\frac{f(3) - f(0)}{3 - 0}$ 26/3					
$(f(3) - f(0)) / (3 - 0)$					
MAIN DEG AUTO FUNC 2/99					

Question 5**C**

$$mx + 12y = 24$$

$$3x + my = m$$

Since the equations are different there will be

$$\text{no solution if } \begin{vmatrix} m & 12 \\ 3 & m \end{vmatrix} = 0$$

$$m^2 - 36 = 0$$

$$m = -6 \text{ or } m = 6$$

A unique solution exists for $m \in \mathbb{R} \setminus \{-6, 6\}$ **OR**

Solving the system of equations gives the result

$$x = \frac{12m}{m^2 - 36}$$

$$y = \frac{m^2 - 72}{m^2 - 36}$$

F1+ Tools	F2+ Algebra	F3+ Calc	F4+ Other	F5 Pr3mID	F6+ Clean Up
[[3 m m]]					
$\begin{bmatrix} 1 & 0 & \frac{12 \cdot m}{m^2 - 36} \\ 0 & 1 & \frac{m^2 - 72}{m^2 - 36} \end{bmatrix}$					
rref([m, 12, 24; 3, m, m])					
MAIN DEG AUTO FUNC 1/99					

The solution will be undefined when

$$m^2 - 36 = 0, \text{ that is, when } m = \pm 6.$$

A unique solution exists for $m \in \mathbb{R} \setminus \{-6, 6\}$

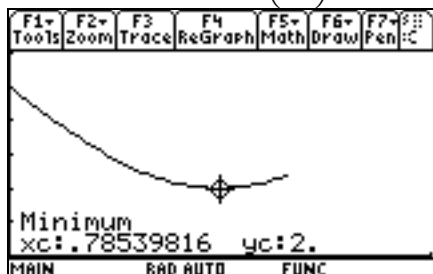
Question 6**B**

Graphing the function, including the domain restriction, shows that the range is $[2, 5]$.

The maximum value occurs at $x = 0$, $f(0) = 5$

The minimum value occurs at the turning point

where $x = \frac{\pi}{4}$ and $f\left(\frac{\pi}{4}\right) = 2$

**Question 7****A**

For X , $\mu = 11$ and $\sigma = 0.25$

For Z , $\mu = 0$ and $\sigma = 1$

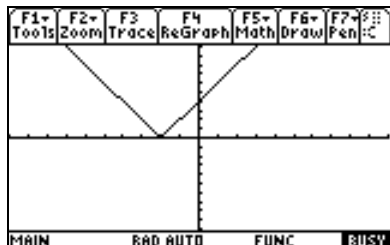
$$\begin{aligned}\Pr(X < 10.5) &= \Pr(X < \mu - 2\sigma) \\ &= \Pr(Z < -2)\end{aligned}$$

By symmetry,

$$\Pr(Z < -2) = \Pr(Z > 2)$$

Question 8**B**

The graph of f has a cusp at $x = -2$, and is therefore not differentiable at this point.



The graph of f' is **not** continuous at $x = -2$

Question 9**E**

$$\begin{aligned}k &= \int_{-2}^{-1} \frac{1}{x} dx \\ &= [\log_e(|x|)]_{-2}^{-1} \\ &= \log_e(1) - \log_e(2) \\ &= \log_e\left(\frac{1}{2}\right) \\ e^k &= \frac{1}{2}\end{aligned}$$

Question 10**B**

At the point(s) of intersection,

$$x^2 + 8x = kx - 3$$

$$x^2 + (8 - k)x + 3 = 0$$

There will be two distinct solutions when the discriminant ($\Delta = b^2 - 4ac$) is greater than zero.

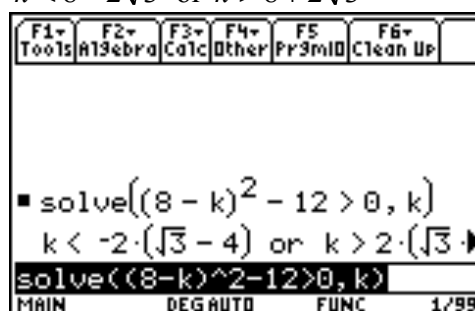
That is,

$$(8 - k)^2 - 4 \times 1 \times 3 > 0$$

$$(8 - k)^2 - 12 > 0$$

Solving for k (by hand or using CAS) gives the result

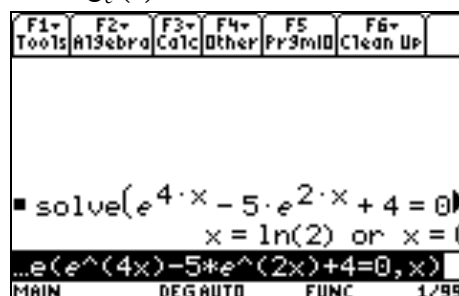
$$k < 8 - 2\sqrt{3} \text{ or } k > 8 + 2\sqrt{3}$$

**Question 11****E**

$$e^{4x} - 5e^{2x} + 4 = 0$$

Using CAS,

$$x = \log_e(2) \text{ or } x = 0$$



Alternatively,

$$(e^{2x})^2 - 5e^{2x} + 4 = 0$$

$$\text{Let } e^{2x} = a$$

$$a^2 - 5a + 4 = 0$$

$$(a - 4)(a - 1) = 0$$

$$e^{2x} = 4 \text{ or } e^{2x} = 1$$

$$2x = \log_e(4) \text{ or } 2x = \log_e(1)$$

$$x = \frac{1}{2} \log_e(4) \text{ or } 2x = 0$$

$$x = \log_e\left(4^{\frac{1}{2}}\right)$$

$$x = \log_e(2) \text{ or } x = 0$$

Question 12**C**

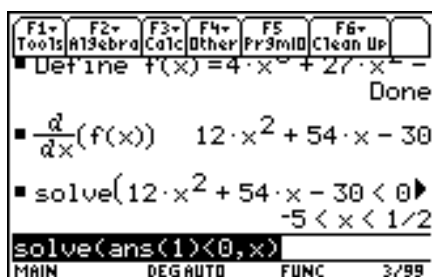
The graph of f has a stationary point at $x = 3$ and the gradient of the graph is negative on both sides of the stationary point. The stationary point is therefore also a point of inflection.

Question 13**D**

The graph has a negative gradient for $\frac{dy}{dx} < 0$.

Solving for x ,

$$x \in \left(-5, \frac{1}{2}\right)$$

**Question 14****A**

$f(x) = \log_e(|x-3|) + 6$ is undefined only at $x = 3$. The maximal domain is $R \setminus \{3\}$.

Question 15**B**

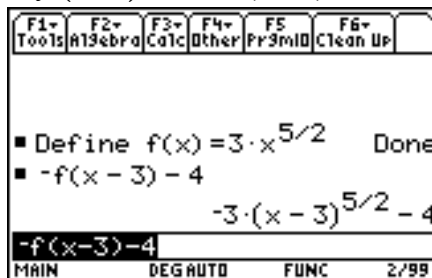
The transformation of $f(x) = 3x^{\frac{5}{2}}$

Reflection in x -axis: $-f(x) = -3x^{\frac{5}{2}}$

Translate 3 units right: $-f(x-3) = -3(x-3)^{\frac{5}{2}}$

Translate 4 units down:

$$-f(x-3) - 4 = -3(x-3)^{\frac{5}{2}} - 4$$

**Question 16****C**

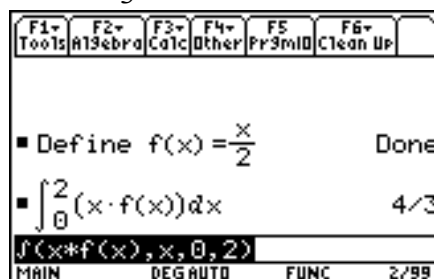
$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0, 2] \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} (x f(x)) dx$$

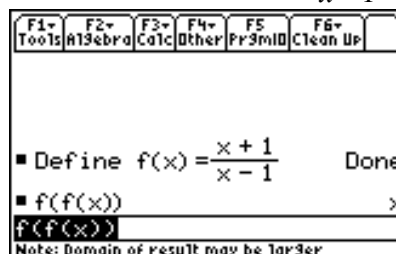
$$= 0 + \int_0^2 \left(\frac{x^2}{2}\right) dx$$

$$= \left[\frac{x^3}{6}\right]_0^2$$

$$= \frac{4}{3}$$

**Question 17****E**

$$f(f(x)) = x \text{ for } f(x) = \frac{x+1}{x-1}$$

**Question 18****D**

Let X be the height of the children.

$$\mu = 130 \text{ and } \sigma = 2.7$$

$$\Pr(X < x) = 0.35$$

$$x = 129$$



Question 19**A**

$$a + 0.2 + 0.2 + 0.3 + b = 1$$

$$a + b = 0.3 \quad \text{equation 1}$$

$$\mu = \sum x p(x)$$

$$5 = 0 + 0.4 + 0.8 + 1.8 + 8b$$

$$8b = 2$$

$$b = 0.25 \quad \text{equation 2}$$

Substitute equation 2 in equation 1

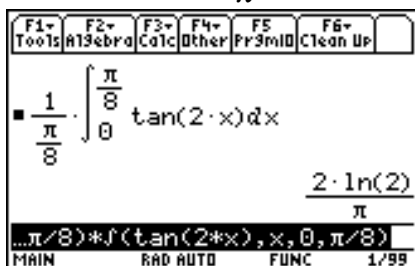
$$a = 0.05$$

Question 20**A**

$$\text{Average value} = \frac{1}{b-a} \int_a^b f(x) dx$$

$$= \frac{1}{\frac{\pi}{8} - 0} \int_0^{\frac{\pi}{8}} \tan(2x) dx$$

$$= \frac{2}{\pi} \log_e(2)$$

**Question 21****A**

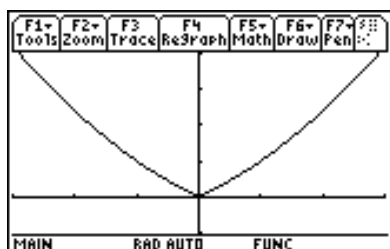
$$\cos^2(x) + 2\cos(x) = 0$$

$$\cos(x)(\cos(x) + 2) = 0$$

$$\cos(x) = 0$$

Note that $\cos(x) + 2$ cannot equal zero (i.e. $\cos(x) \neq -2$) because $-1 \leq \cos(x) \leq 1$.**Question 22****E**

$$f(g(0)) = f(0) = 0$$

The point $(0, 0)$ belongs to the curveElsewhere $g(x)$ is negative $f(x)$ is positive when x is negativeHence $f(g(x))$ is positive elsewhere**OR**A possible rule for f could be $f(x) = x(x - 4)$.A possible rule for g could be $g(x) = -|x|$.The graph of $f(g(x))$ is most similar to **E**.

Exam 2 Part 2

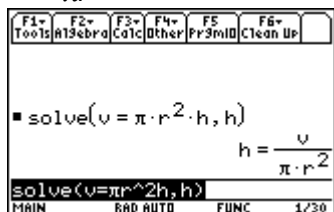
Suggested Solutions to Extended Answer Questions - with a suggested marking scheme.

Question 1

a. $V = \pi r^2 h$

$$h = \frac{V}{\pi r^2}$$

1A



b. $A = 2\pi r h + 2\pi r^2$

$$= 2\pi r \times \frac{V}{\pi r^2} + 2\pi r^2$$

1M

$$= \frac{2V}{r} + 2\pi r^2$$

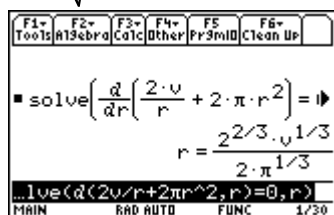
1A

c. $\frac{dA}{dr} = 0$

1M

$$r = \sqrt[3]{\frac{V}{2\pi}}$$

1A



OR

$$\frac{dA}{dr} = -\frac{2V}{r^2} + 4\pi r$$

$$\frac{dA}{dr} = 0 \Rightarrow -\frac{2V}{r^2} + 4\pi r = 0$$

1M

$$-2V + 4\pi r^3 = 0$$

$$r = \sqrt[3]{\frac{V}{2\pi}}$$

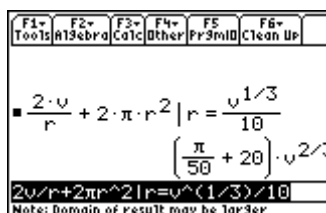
1A

d. $A_{\max} = A\left(\frac{V^{1/3}}{10}\right)$

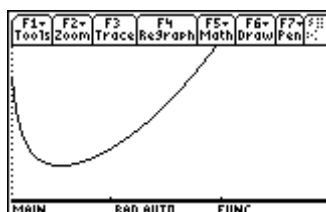
1M

$$= \left(\frac{\pi}{50} + 20\right) V^{2/3}$$

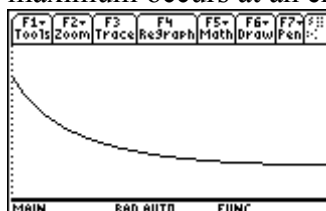
1A



The graph of $A = \frac{2V}{r} + 2\pi r^2$ where $V = 1$ is shown below.



Restrict the domain $\frac{V^{1/3}}{10} \leq r \leq \frac{5V^{1/3}}{9}$. The maximum occurs at an end point.



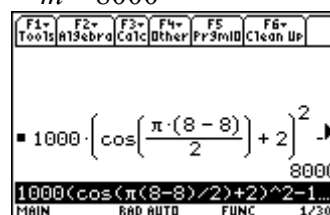
Question 2

a. $C(8) = 1000(\cos(0) + 2)^2 - 1000$
 $= 9000 - 1000 = 8000$

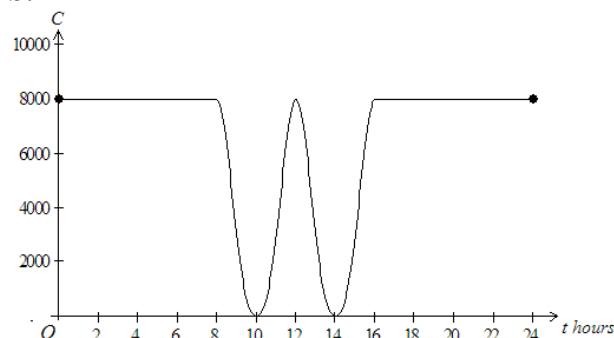
$$C(8) = C(16) = 8000$$

$$m = 8000$$

1A

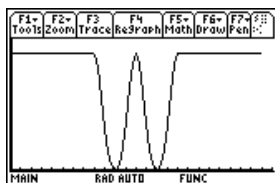


b.



The graph should show:

correct minimum turning points **1A**
 correct lines over the restricted domain **1A**
 continuous smooth curve with the correct maximum turning point **1A**



c. From the graph, the minimum concentration is zero **1A**
 Minimum occurs at $t=10$ or $t=14$ **1A**

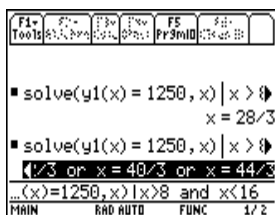
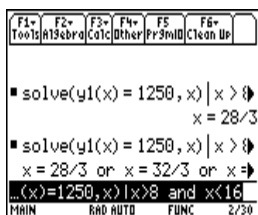
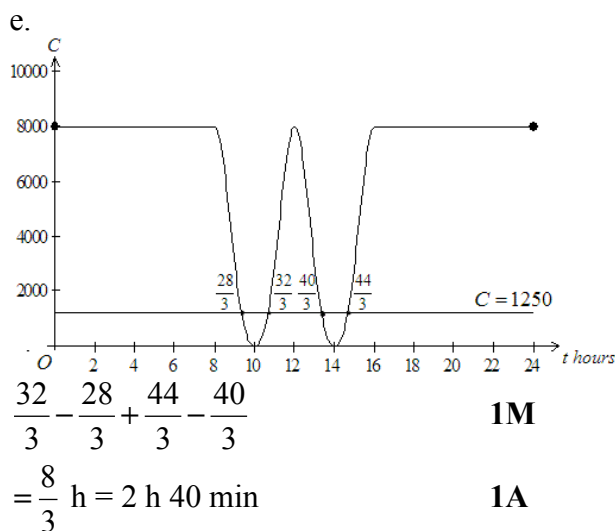
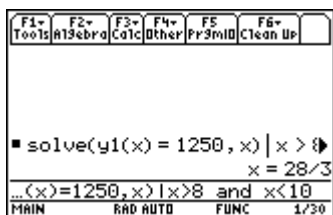
d. Solve $C(t)=1250$ for t

$$t = \frac{28}{3} = 9\frac{1}{3}$$

$9\frac{1}{3}$ hours after midnight

9:20 am

1A



f. i. $T = p(q^x - 1)$

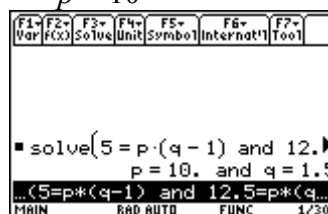
$$5 = p(q - 1) \text{ and } 12.5 = p(q^2 - 1)$$

$$q = 1.5$$

$$p = 10$$

1A

1A



ii. $T = 10(1.5^4 - 1) = 40.625$ min **1A**

g. $40.625 + \frac{40.625}{2} + 19 = 79.9375$ min **1M**

$$80 - 79.9375$$

$$= 0.0625 \text{ min}$$

1M

1A

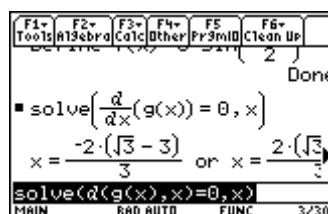
Question 3

a. Solve $g'(x) = 0$

1M

$$x = \frac{6 - 2\sqrt{3}}{3}$$

1A



OR

$$\frac{d(g(x))}{dx} = 2(3x^2 - 12x + 8) = 0$$

1M

$$6x^2 - 24x + 16 = 0$$

$$x = \frac{12 \pm \sqrt{144 - 96}}{6}$$

Maximum value of $g(x)$ is at $x = \frac{6 - 2\sqrt{3}}{3}$ **1A**

b. i. $2 \left(\int_0^1 (g(x) - f(x)) dx \right) +$ **1A**

$$2 \left(\int_0^1 (f(x) - g(x)) dx \right)$$

1A

OR

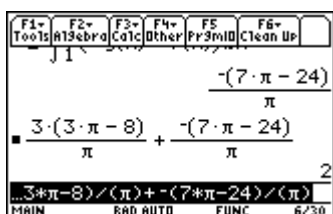
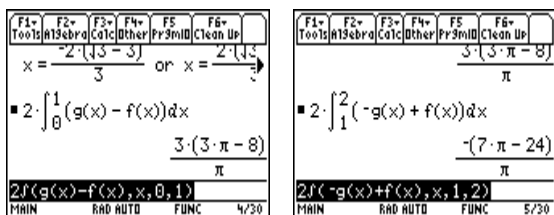
$$\int_0^1 (g(x) - f(x))dx + \int_1^2 (f(x) - g(x))dx + \int_2^3 (g(x) - f(x))dx + \int_3^4 (f(x) - g(x))dx$$

2A

$$\text{ii. } \frac{3(3\pi - 8)}{\pi} + \frac{24 - 7\pi}{\pi} = 2 \text{ units}^2$$

1M

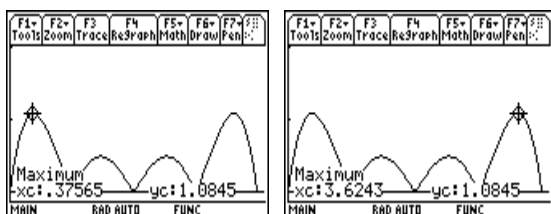
1A



c. Maximum value of 1.08 when $x = 0.38$ or $x = 3.62$

1A

1A



d. i. A translation of 3 units to the left and a reflection in the x -axis (order does not matter)

1A

1A

OR

A translation of 1 unit to the left and a reflection in the y -axis (must be in this order)

1A

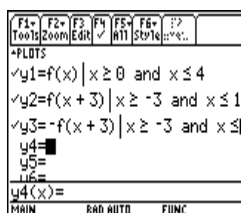
1A

OR

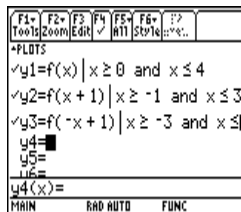
A reflection in the y -axis and a translation 1 unit to the right (must be in this order)

1A

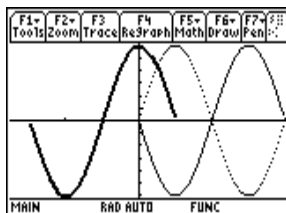
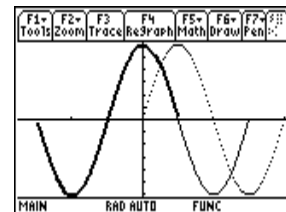
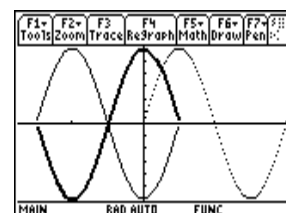
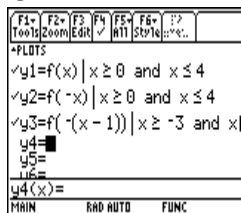
1A



OR



OR



ii. Translate the graph of g 3 units to the left

1M

$$y = 2(x+3)(x+1)(x-1) = 2x^3 + 6x^2 - 2x - 6$$

1A

OR

2M for applying each of the transformations in d i

$$y = -2(x+3)(x+1)(x-1) = -2x^3 - 6x^2 + 2x + 6$$

OR

$$y = 2(-x+1)(-x-1)(-x-3) = -2(x-1)(x+1)(x+3)$$

OR

$$y = -2(x-1)(-x-1-2)(-x-1-4) = -2(x-1)(x+1)(x+3)$$

Question 4

a. $a = 1$ because the equation of the asymptote is $y = 1$

1A

$b = -1$ because the graph of f passes through the origin

1A

$$f(0) = 1 + be^0 = 0$$

$$b = -1$$

b. $h(0) = 1 - e^0 = 0$

$$h(2) = 1 - e^{-2}$$

$$\text{range} = [0, 1 - e^{-2}]$$

2A

c. i. Let $y = 1 - e^{-x}$

Inverse: swap x and y

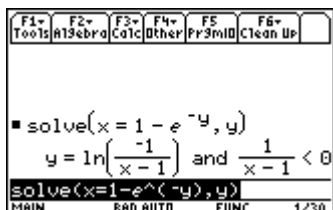
Solve $x = 1 - e^{-y}$ for y

$$y = \log_e \left(\frac{-1}{x-1} \right) \quad 1A$$

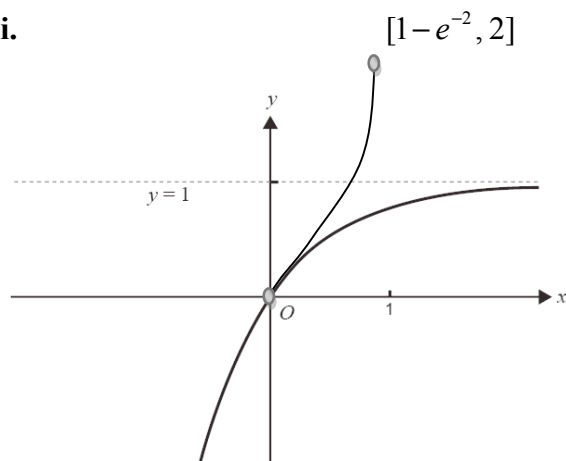
$$= -\log_e(1-x)$$

$$h^{-1}: [0, 1 - e^{-2}] \rightarrow \mathbb{R}, \text{ where } h^{-1}(x) = -\log_e(1-x)$$

1A



ii.



Correct end points

1A

Correct shape

1A

d. i. $f(u)f(v) = (1 - e^{-u})(1 - e^{-v})$ 1M

$$= 1 - e^{-v} - e^{-u} + e^{-u}e^{-v}$$

$$= f(u) + f(v) - 1 + e^{-u}e^{-v} \quad 1M$$

$$= f(u) + f(v) - (1 - e^{-(u+v)})$$

$$= f(u) + f(v) - f(u+v)$$

ii. $f(u)f(-u) = (1 - e^{-u})(1 - e^u)$

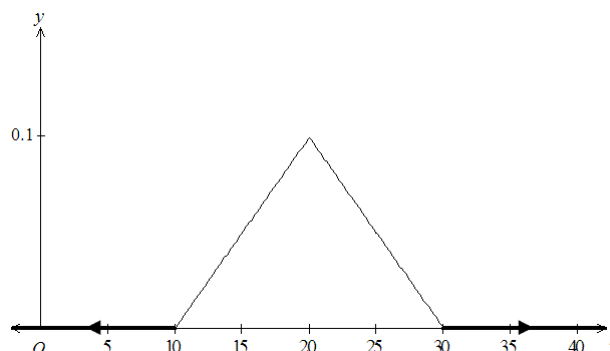
$$= 1 - e^u - e^{-u} + 1$$

$$= f(u) + (1 - e^{-(-u)}) \quad 1M$$

$$= f(u) + f(-u)$$

Question 5

a.



Correct oblique lines

1A

Everything else correct

1A

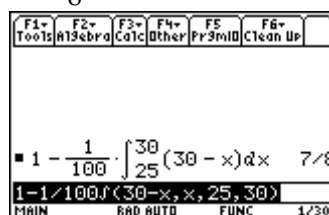
b. $\Pr(f(t) < 25)$

$$= 1 - \frac{1}{100} \int_{25}^{30} (30 - t) dt$$

1M

$$= \frac{7}{8}$$

1A



c. $\Pr(T \leq 15 \mid T \leq 25)$

$$= \frac{\Pr(T \leq 15) \cap \Pr(T \leq 25)}{\Pr(T \leq 25)}$$

$$= \frac{1}{8}$$

1M

$$= \frac{1}{7}$$

1A

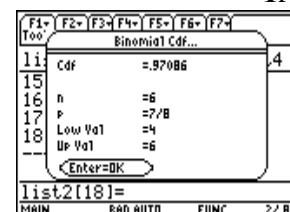
d. $X \sim \text{Bi}(6, \frac{7}{8})$

1M

$$\Pr(X > 4)$$

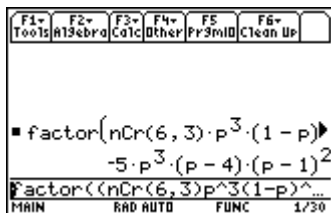
$$\approx 0.9709$$

1A



$$\text{e. } \binom{6}{3} p^3 (1-p)^3 + \binom{6}{4} p^4 (1-p)^2 \quad \mathbf{2M}$$

$$= 5p^3 (1-p)^2 (4-p)$$

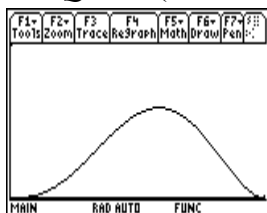


f. i. The maximum is at the turning point

Solve $Q'(p) = 0$ for p

$$p = 2 - \sqrt{2} \quad \mathbf{1A}$$

$$Q = 20(17 - 12\sqrt{2}) \quad \mathbf{1A}$$



The calculator screen shows the command: `solve(d/dp(-5*p^3*(p-4)*(p-1)^2)=0,p)`. The result is: `p = 2 - sqrt(2) or p = 1`. The status bar at the bottom indicates 'MAIN', 'RAD AUTO', 'FUNC', and '2/30'.

$$\text{ii. } \frac{1}{100} \int_b^{30} (30-t) dt = 1 - (2 - \sqrt{2}) \quad \mathbf{1M}$$

$$b \approx 20.9 \quad \mathbf{1A}$$

The calculator screen shows the command: `solve(1/100 * integral(30-t, t, b, 30) = 1 - (2 - sqrt(2)), b)`. The result is: `b = 20.898 or b = 39.102`. The status bar at the bottom indicates 'MAIN', 'RAD AUTO', 'FUNC', and '1/30'.