



The Mathematical Association of Victoria

MATHS METHODS

2007 WRITTEN EXAMINATIONS

SOLUTIONS

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VCAA MATHEMATICAL METHODS 2007: PAPER 2 (mav)**SECTION 1 (Multiple-choice) Answers and Suggested Solutions**

1. A	2. D	3. A	4. C	5. D
6. B	7. A	8. B	9. E	10. E
11. E	12. C	13. D	14. A	15. B
16. E	17. C	18. D	19. A	20. A
21. A	22. E			

Question 1**A**Substituting the end-point values for y into $f(x)$:

$$-4 = 6 - 2x \text{ and so } x = 5$$

$$12 = 6 - 2x \text{ and so } x = -3$$

The domain is $[-3, 5]$ **Question 2****D**Substituting $g(x)$ for x in $f(x)$ gives:

$$f(g(x)) = e^{2(x^2+2x-3)+3}$$

$$= e^{2x^2+4x-6+3}$$

$$= e^{2x^2+4x-3}$$

Question 3**A**

$$\int \left(\frac{1}{x^2} - \frac{1}{\cos^2(x)} \right) dx$$

$$= \int (x^{-2} - \sec^2(x)) dx$$

$$= -\frac{1}{x} - \tan(x) + c$$

Note: “an” anti-derivative requires no constant.

Question 4**C**

$$f(x) = x^3 - \sqrt{x+1}$$

$$f(0) = 0^3 - \sqrt{0+1} = -1$$

$$f(3) = 3^3 - \sqrt{3+1} = 27 - 2 \text{ or } 25$$

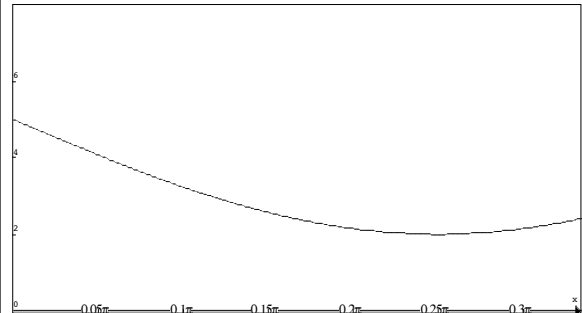
$$\begin{aligned} \text{Average rate of change} &= \frac{f(3) - f(0)}{3 - 0} \\ &= \frac{25 - (-1)}{3} \text{ or } \frac{26}{3} \end{aligned}$$

Question 5**D**

$$\begin{aligned} \int (\sin(2x) + 24x^3) dx &= -\frac{1}{2} \cos(2x) + \frac{24}{4} x^4 + c \\ &= -\frac{1}{2} \cos(2x) + 6x^4 + c \end{aligned}$$

Question 6**B**

The graph of the function between 0 and $\frac{\pi}{3}$ is shown below:



Its range is from 5 to 2 inclusive.

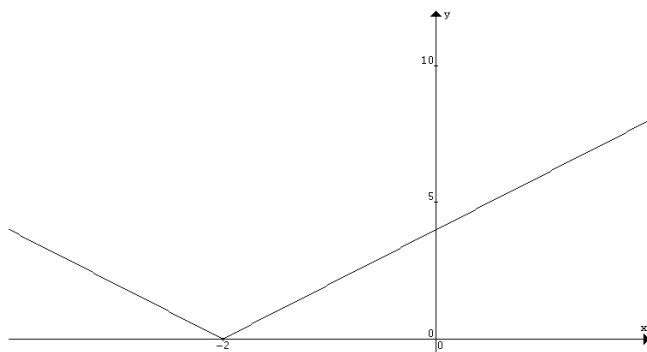
Answer: $[2, 5]$.**Question 7****A**Substitute $X = 10.5$, $\mu = 11$ and $\sigma = 0.25$ into

$$Z = \frac{X - \mu}{\sigma}$$

$$Z = \frac{10.5 - 11}{0.25} = -2$$

By symmetry, $Z < -2$ is the same as $Z > 2$.**Question 8****B**The graph of $f(x) = |2x + 4|$ is shown below.

The graph is continuous everywhere and is also non-negative.



Its derivative exists at all values of x except -2 because the derivative does not exist at a sharp point. Its gradient is $+2$ for all $x > -2$ and is -2 for $x < -2$ (and non-existent at $x = -2$). The graph of $f'(x)$ is therefore *discontinuous*.

Question 9**E**

$$\begin{aligned}\int_{-2}^{-1} \frac{1}{x} dx &= [\log_e |x|]_{-2}^{-1} \\ &= \log_e |-1| - \log_e |-2| \\ &= \log_e 1 - \log_e 2 \\ &= \log_e 0.5 \\ &= k\end{aligned}$$

$$\text{So } e^k = e^{\log_e 0.5} = 0.5$$

Question 10**E**

$$f(x) = e^{2x} - 3$$

$$\text{If } x = 0 \text{ then } f(0) = -2$$

$$\text{If } x = 1 \text{ then } f(1) = e^2 - 3$$

Hence the range of $f(x)$ is $[-2, e^2 - 3)$ which becomes the domain of the inverse function.

$$f(x) = e^{2x} - 3$$

Switch x for $f(x)$ and vice-versa:

$$x = e^{2y} - 3$$

$$e^{2y} = x + 3$$

$$2y = \log_e(x + 3)$$

$$f^{-1}(x) = \frac{1}{2} \log_e(x + 3)$$

Combining this rule and domain gives answer E.

Question 11**E**

$$e^{4x} - 5e^{2x} + 4 = 0$$

$$\text{Let } m = e^{2x} \text{ so } m^2 - 5m + 4 = 0$$

$$(m - 4)(m - 1) = 0$$

$$m = 4 \text{ or } m = 1$$

$$e^{2x} = 4 \text{ or } e^{2x} = 1$$

$$2x = \log_e 4 \text{ or } 2x = 0$$

$$x = \frac{1}{2} \log_e 4 \text{ or } x = 0$$

$$x = \log_e \sqrt{4} \text{ or } x = 0$$

$$\text{Answer: } \{\log_e 2, 0\}$$

Question 12**C**

If the gradient is zero at $x = 3$ and does not change sign on either side of this value then there must be a stationary point of inflection at $x = 3$.

Question 13**D**

$$\text{If } y = 4x^3 + 27x^2 - 30x + 10 \text{ then}$$

$$\frac{dy}{dx} = 12x^2 + 54x - 30$$

$$= 6(2x^2 + 9x - 5)$$

$$= 6(2x - 1)(x + 5)$$

This is negative if $-5 < x < 0.5$ or $(-5, 0.5)$

Question 14**A**

$|x - 3|$ is never negative. The logarithm of zero is undefined and so $f(x) = \log_e(|x - 3|) + 6$ is defined for all values of x except $x = 3$.

Answer: $R \setminus \{3\}$.

Question 15
B
 $f : [0, \infty) \rightarrow R$ where $f(x) = 3x^{\frac{5}{2}}$

Reflecting in the x -axis it becomes $-3x^{\frac{5}{2}}$;

Translating 3 units right: $-3(x-3)^{\frac{5}{2}}$

Translating 4 units down: $-3(x-3)^{\frac{5}{2}} - 4$

So the rule is $y = -3(x-3)^{\frac{5}{2}} - 4$
Question 16
E
 $f(x) = (x-a)^2 g(x)$ is the product of two functions of x and so the Product Rule must be applied:

$$\begin{aligned} f'(x) &= g(x) \times \frac{d}{dx}(x-a)^2 + (x-a)^2 \frac{d}{dx}g(x) \\ &= 2(x-a)g(x) + (x-a)^2 g'(x) \\ &= (x-a)(2g(x) + (x-a)g'(x)) \end{aligned}$$

Question 17
C

$$\begin{aligned} E(X) &= \int_0^2 x f(x) dx \\ &= \int_0^2 \left(\frac{x^2}{2} \right) dx \\ &= \left[\frac{x^3}{6} \right]_0^2 \\ &= \frac{8}{6} \text{ or } \frac{4}{3}. \end{aligned}$$

Question 18
D
 $\text{Invnorm}(0.35, 130, 2.7) = 128.9596347$

This is 129 to the nearest centimeter.

Question 19
A

Sum of the probabilities = 1.

$$a + b + 0.2 + 0.2 + 0.3 = 1$$

$$a + b = 0.3 \quad \textbf{i.}$$

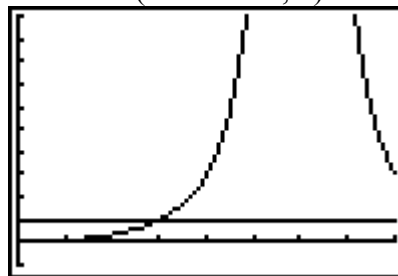
$$\text{Mean} = \sum x \Pr(x)$$

$$0 \times a + 2 \times 0.2 + 4 \times 0.2 + 6 \times 0.3 + 8 \times b = 5 \quad \textbf{ii.}$$

$$8b = 5 - 0.4 - 0.8 - 1.8$$

$$8b = 2 \text{ and so } b = 0.25$$

Substituting for b in **i.** gives $a = 0.05$
Question 20
A

The graphs of $y_1 = \tan^2\left(\frac{\theta}{3}\right)$ and $y_2 = 1$ over the domain $[0, 2\pi]$ are shown. The intersection point can be found: $(2.3561945, 1)$

 $\frac{3\pi}{4}$ is the exact value for this x -value approximation.

Question 21
A

$$\cos^2(x) + 2\cos(x) = 0$$

$$\text{Factorising: } \cos(x)(\cos(x) + 2) = 0$$

$$\text{Therefore } \cos(x) = 0 \text{ or } \cos(x) + 2 = 0.$$

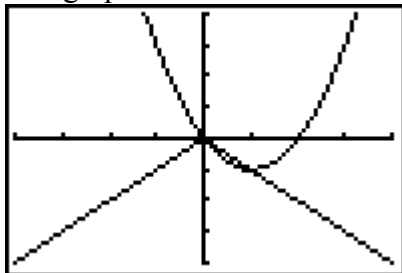
This second equation is not possible as $-1 \leq \cos(x) \leq 1$ and so the only valid equation is $\cos(x) = 0$.

Question 22

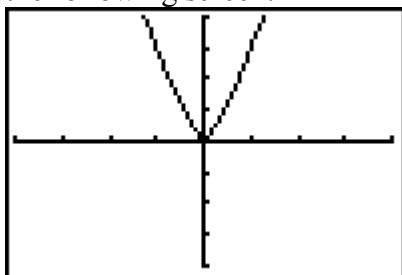
E

The two functions shown on the examination paper resemble $g(x) = -|x|$ and $f(x) = x(x-2)$.

The graphics calculator shows these two curves:



The graph of $f(g(x)) = -|x|(-|x|-2)$ appears as the following screen:



This closely resembles alternative E.

SECTION 2 - Suggested Solutions

Question 1

a. $1000 = \pi r^2 h$ and so $h = \frac{1000}{\pi r^2}$

b. $A = 2\pi r^2 + 2\pi r h$ (area of the two circular ends and the curved surface area)
 $= 2\pi r^2 + 2\pi r \times \frac{1000}{\pi r^2}$ (substituting for h from a.)
 $= 2\pi r^2 + \frac{2000}{r}$ as required.

c. $A = 2\pi r^2 + 2000r^{-1}$

$$\frac{dA}{dr} = 4\pi r - 2000r^{-2}$$

$= 0$ for a minimum value.

$$4\pi r - 2000r^{-2} = 0$$

Multiplying right through by r^2 gives $4\pi r^3 = 2000$

$$r^3 = \frac{2000}{4\pi}$$

$$r = \sqrt[3]{\left(\frac{500}{\pi}\right)}$$

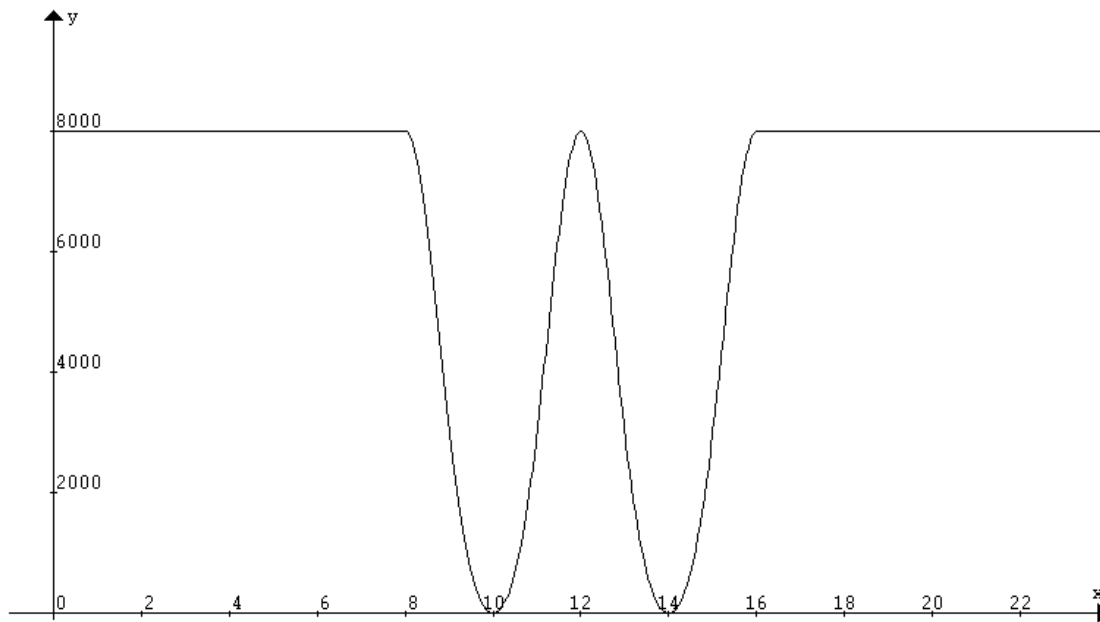
d. Substituting $r = \sqrt[3]{\left(\frac{500}{\pi}\right)}$ into A gives 553.58 correct to two decimal places.

(Note: This last answer can also be found from finding the minimum value from the curve on the graphics calculator rather than complicated arithmetic.)

Question 2.

a. $m = 8000$ if the function is to be continuous

b. (Note: the joins of the two straight lines to the cosine curve must be smooth and continuous, so ensure that there are no sharp points).



- c. Minimum concentration $C = 0$ at $t = 10$ and $t = 14$ (from graph).
- d. Solve $C(t) = 1250$ on the graphics calculator to find the first solution between $t = 8$ and $t = 16$.
It is $t = 9.33333\dots$ hours.
Answer: 9:20 am
- e. By symmetry from the graph, the next value of t where $C(t) = 1250$ is $t = 10.666666\dots$ or 10:40 am.
The total length of time $= 2 \times (10\frac{2}{3} - 9\frac{1}{3})$.
Answer: $2\frac{2}{3}$ hours.
- f. Substituting the data into $T = p(q^x - 1)$ gives the two equations:
- $5 = p(q^1 - 1)$
 - $12.5 = p(q^2 - 1)$
- Divide equation ii. by i. : $\frac{12.5}{5} = \frac{p(q^2 - 1)}{p(q - 1)}$
- $$2.5 = \frac{p(q - 1)(q + 1)}{p(q - 1)} = q + 1 \text{ by cancelling common factors.}$$
- Hence $q = 1.5$ and $p = 10$.
- g. $T = 10(1.5^4 - 1) = 40.625$ minutes.
- h. If Tasmania leaves at 9:20 am: It takes him 40.625 minutes to run in, 19 minutes to dig and $\frac{1}{2} \times 40.625$ minutes to run out.
Total time $= 1.5 \times 40.625 + 19$
 $= 79.9375$ minutes.
He has $1\frac{1}{3}$ hours to do this in, that is 80 minutes and so he can just do it.

Time to spare = $80 - 79.9375$ or 0.0625 minutes

Question 3

a. $g(x) = 2x^3 - 12x^2 + 16x$

$$g'(x) = 6x^2 - 24x + 16$$

Now this derivative is zero for a maximum value.

$$6x^2 - 24x + 16 = 0$$

$$\begin{aligned} \text{Using the quadratic formula: } x &= \frac{-(-24) \pm \sqrt{(24)^2 - 4 \times 6 \times 16}}{12} \\ &= \frac{24 \pm \sqrt{192}}{12} \text{ or } \frac{24 \pm 8\sqrt{3}}{12} \text{ which cancels to } \frac{6 \pm 2\sqrt{3}}{3} \end{aligned}$$

From the graph, clearly $x = \frac{6 - 2\sqrt{3}}{3}$ is where the maximum value occurs.

- b. To find the area between curves we integrate [upper curve minus lower curve] between the limits where the two graphs intersect.

$$= \int_0^1 (g(x) - f(x))dx + \int_1^2 (f(x) - g(x))dx + \int_2^3 (g(x) - f(x))dx + \int_3^4 (f(x) - g(x))dx$$

or, using symmetry

$$2 \times \left[\int_0^1 (g(x) - f(x))dx + \int_1^2 (f(x) - g(x))dx \right]$$

$$\text{where } g(x) = 2x(x-2)(x-4) \text{ and } f(x) = 6 \sin\left(\frac{\pi x}{2}\right)$$

- c. There are several pairs of transformations :

- Translation of 1 unit left and then reflection in the y axis or
- Reflection in the x axis and a translation of 3 units left or
- Reflection in the y axis and then a translation of 1 unit right

- d. Using the first method above: translation of 1 left means that x becomes $x + 1$ so the function's equation is now transformed to $m(x) = 2(x+1)(x+1-2)(x+1-4)$ or $2(x+1)(x-1)(x-3)$ and then because of the reflection in the y axis, each x becomes $-x$. Now the equation is:

$$m(x) = 2(-x+1)(-x-1)(-x-3)$$

$$\text{Therefore } m(x) = -2(x-1)(x+1)(x+3)$$

Note: The solution $m(x) = 2(x-1)(x+1)(x+3)$ was also acceptable by the examiners, the minimum turning point of the original function becoming the maximum of the transformed function and vice versa thereby satisfying the wording of the question.

Question 4

a. $f(0) = 2 - e^0 = 1$ and $f(2) = 2 - e^{-2}$ and so the range is $[1, 2 - e^{-2}]$

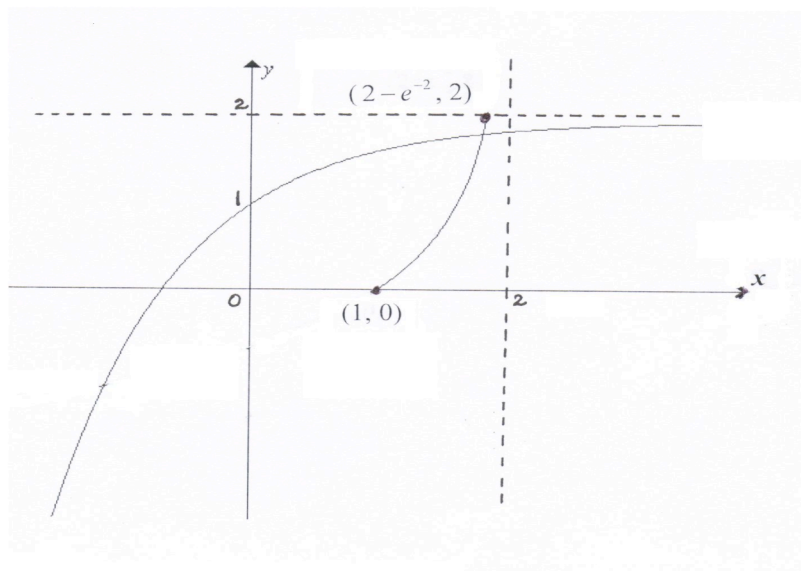
b. i. Interchanging x and $f(x)$ in the rule $f(x) = 2 - e^{-x}$ gives:

$$x = 2 - e^{-y}$$

$$e^{-y} = 2 - x \text{ and so } -y = \log_e(2 - x)$$

$$h^{-1}(x) = -\log_e(2 - x) \text{ for the domain } [1, 2 - e^{-2}].$$

ii.



c. Using the graphing calculator and solving for the point of intersection of $y_1 = 2 - e^{-x}$ and $y_2 = -\log_e(2 - x)$ over the restricted domain given in **b. i** gives, correct to two decimal places, (1.84, 1.84).

d. Area enclosed by h and $h^{-1} = \int_0^{1.84} (2 - e^{-x}) dx - \int_1^{1.84} -\log_e(2 - x) dx$

This method presents a problem because the answer for the integral of a logarithmic function is not part of the course. To overcome this difficulty, use the fact that this function and its inverse meet on the line $y = x$.

$$\text{Using symmetry: } A = 2 \left[\int_0^{1.84} (2 - e^{-x} - x) dx \right]$$

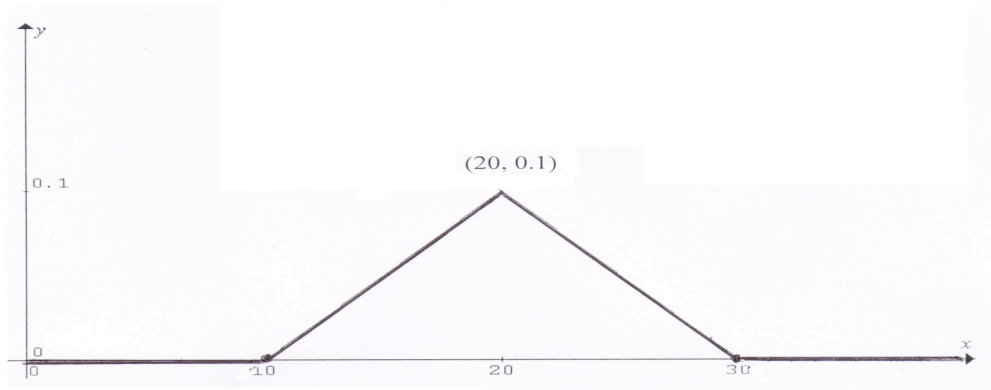
$$= 2 \left[2x + e^{-x} - \frac{x^2}{2} \right]_0^{1.84}$$

$$= 2 \left[(2 \times 1.84 + e^{-1.84} - \frac{1.84^2}{2}) - (0 + 1 - 0) \right]$$

$$= 2.29 \text{ to two decimal places.}$$

Question 5

a.



$$\begin{aligned} \text{b. } \Pr(T < 25) &= \int_0^{25} f(t) dt = \int_0^{10} 0 dt + \int_{10}^{20} 0.01(t-10) dt + \int_{20}^{25} 0.01(30-t) dt \\ &= \frac{1}{2} + \frac{3}{8} \\ &= \frac{7}{8} \end{aligned}$$

$$\text{c. } \Pr(T \leq 15) = \frac{1}{4} \times \frac{1}{2} \text{ or } \frac{1}{8} \text{ using area of part of the triangle.}$$

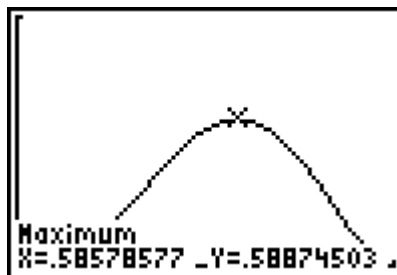
$$\Pr(T \leq 15) \setminus \Pr(T \leq 25) = \frac{1}{8} \div \frac{7}{8} \text{ which is } \frac{1}{7}$$

$$\text{d. Binomial distribution with } p = \frac{7}{8} \text{ and } n = 6$$

$$\begin{aligned} \Pr(4) + \Pr(5) + \Pr(6) &= {}^6C_4 \left(\frac{7}{8}\right)^4 \left(\frac{1}{8}\right)^2 + {}^6C_5 \left(\frac{7}{8}\right)^5 \left(\frac{1}{8}\right)^1 + \left(\frac{7}{8}\right)^6 \\ &= 0.9709 \text{ to four decimal places.} \end{aligned}$$

$$\begin{aligned} \text{e. } \Pr(3) + \Pr(4) &= {}^6C_3 p^3 (1-p)^3 + {}^6C_4 p^4 (1-p)^2 \\ &= 20p^3 (1-p)^3 + 15p^4 (1-p)^2 \\ &= 5p^3 (1-p)^2 [4(1-p) + 3p] \\ &= 5p^3 (1-p)^2 [4 - 4p + 3p] \\ &= 5p^3 (1-p)^2 (4-p) \text{ which was required.} \end{aligned}$$

- f. i. Sketching $y_1 = 5x^3(1-x)^2(4-x)$ into the calculator and finding the maximum value gives an image like that shown here.
The maximum value of Q is 0.5887 at a p value of 0.5858, both correct to four decimal places.



- ii. $\Pr(T < b) = 0.5858$

Hence $b > 20$ as the area is greater than 0.5.

$$\int_{20}^b 0.01(30-t)dt = 0.0858$$

Divide both sides by 0.01: $\int_{20}^b (30-t)dt = 8.58$

$$[30t - 0.5t^2]_{20}^b = 8.58$$

$$(30b - 0.5b^2) - (600 - 0.5 \times 400) = 8.58$$

$$30b - 0.5b^2 - 408.58 = 0$$

Using a quadratic program gives $b = 20.9$ (or 39.1 which is not in the domain).

Answer: $b = 20.9$ correct to one decimal place.