

SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 2



2010 Trial Examination

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: B

Explanation:

By completing the square, the equation of $x^2 - 4y^2 - 4mx - 8my = 4$ becomes

$$(x - 2m)^2 - 4m^2 - 4(y + m)^2 + 4m^2 = 4$$

$$(x - 2m)^2 - 4(y + m)^2 = 4$$

$$\frac{(x - 2m)^2}{4} - (y + m)^2 = 1$$

This is a hyperbola with centre $(2m, -m)$ and the semi-axes $a = 2$ and $b = 1$

Question 2

Answer: A

Explanation:

For $f(x)$ to have non-stationary point of inflection at $x = 1$, second derivative at this point must be zero. Because $f(x)$ is an increasing function, the first derivative must be positive.

$$f'(x) = 2Ax - 1 + \frac{B}{x}, \quad f''(x) = 2A - \frac{B}{x^2}$$

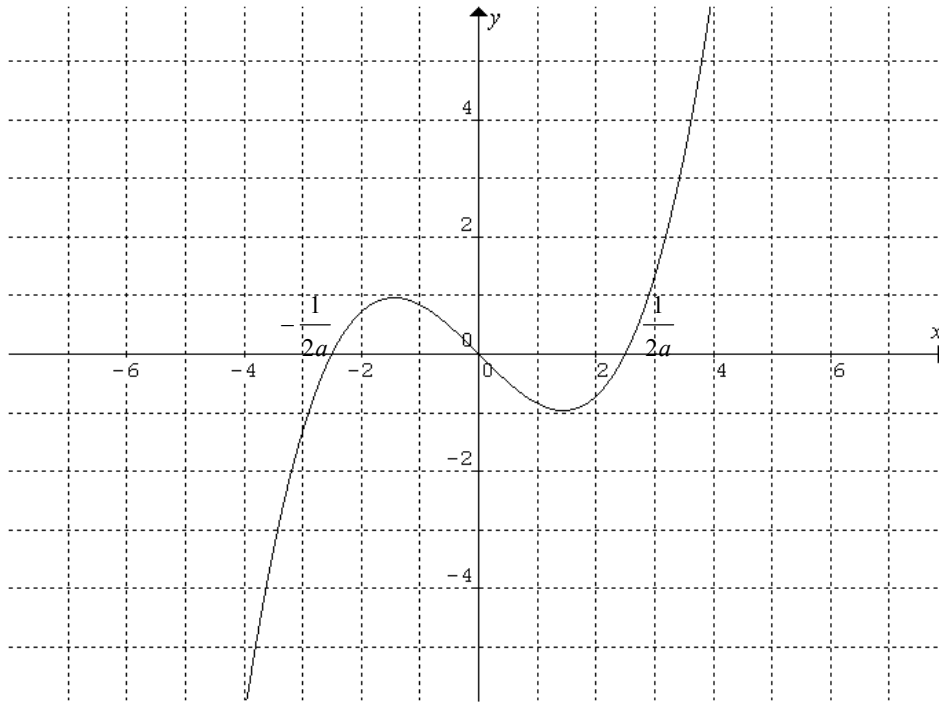
When $x = 1$, $2A - 1 + B > 0$, $2A - B = 0$

After substituting $B = 2A$ into $2A - 1 + B > 0$, we have $A > \frac{1}{4}$ and $B > \frac{1}{2}$

Question 3*Answer:* D*Explanation:*

$f(x) = \sqrt{4a^2x^3 - x}$, $a \in \mathbb{R}^+$, is defined when $4a^2x^3 - x \geq 0$.

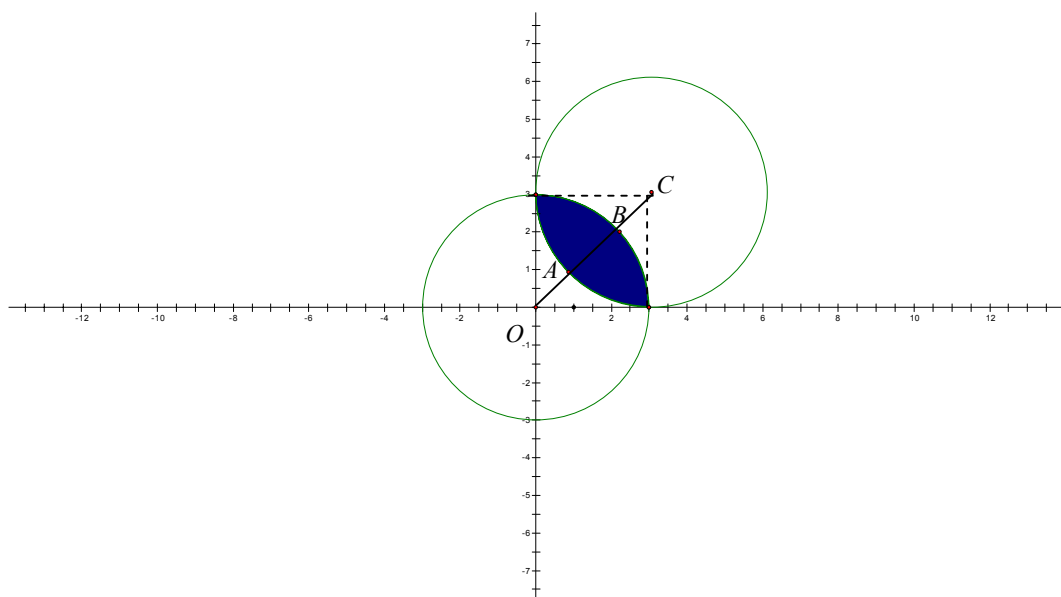
Solving the inequality graphically, $x \in \left[-\frac{1}{2a}, 0\right] \cup \left[\frac{1}{2a}, +\infty\right)$.

**Question 4***Answer:* D*Explanation:*

For the stationary point of inflection, both $f'(x)$ and $f''(x)$ must equal zero which is true at $(0, 0)$.

Question 5*Answer:* B*Explanation:*

Alternative A is obviously incorrect as for $S = \{z : |z| \leq 3\} \cap \{z : |z - 3 - 3i| \leq 3\}$, z belongs to the shaded region, and therefore $0 \leq \text{Arg}(z) \leq \frac{\pi}{2}$.



Further, for every z from the shaded region, $|z| \leq 3$. The minimum value of $|z| = OA$.

As $OA = BC = 3\sqrt{2} - 3$, it follows that $3(\sqrt{2} - 1) \leq |z| \leq 3$.

Question 6*Answer:* E*Explanation:*

Substituting $z = x + iy$ into $|z| - z = 1 + 2i$ and rearranging

$$\sqrt{x^2 + y^2} - x - iy = 1 + 2i$$

$$\sqrt{x^2 + y^2} = (x + 1) + (2 + y)i$$

Equating real and imaginary parts, $2 + y = 0$, $\sqrt{x^2 + y^2} = x + 1$ and solving for x and y gives

$x = \frac{3}{2}$, $y = -2$. Therefore, $z = \frac{3}{2} - 2i$. The only correct alternative is $|z| = \frac{5}{2}$.

Question 7*Answer:* C*Explanation:*

The polynomial has real coefficients. By the conjugate root theorem, $z = 1 + ai$ is also a solution. The quadratic factor is

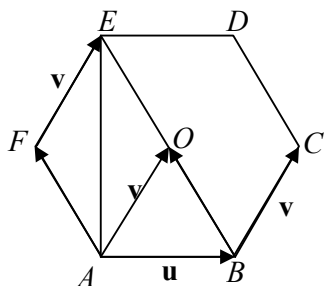
$$z^2 - (1 + ai + 1 - ai)z + (1 + ai)(1 - ai) = z^2 - 2z + 1 + a^2$$

Question 8*Answer:* E*Explanation:*

$$\mathbf{a} + n\mathbf{b} = 6\mathbf{i} + (1 + 3n)\mathbf{j} + (1 - n)\mathbf{k}, \quad \mathbf{c} = -2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$$

The scalar product of these two vectors must be 0.

$$(\mathbf{a} + n\mathbf{b}) \cdot \mathbf{c} = -12 + 3(1 + 3n) + 5(1 - n) = 0 \Rightarrow n = 1$$

Question 9*Answer:* C*Explanation:*

$$\vec{AF} = \vec{BO} = \vec{BA} + \vec{AO} = -\mathbf{u} + \mathbf{v}$$

$$\vec{AE} = \vec{AF} + \vec{FE} = -\mathbf{u} + \mathbf{v} + \mathbf{v} = -\mathbf{u} + 2\mathbf{v}$$

Question 10*Answer:* A*Explanation*

$$\text{Let } \alpha = \cos^{-1} \frac{1}{\sqrt{1+a^2}} \text{ and } \beta = \cos^{-1} \frac{a}{\sqrt{1+a^2}}.$$

$$\text{Then } \cos \alpha = \frac{1}{\sqrt{1+a^2}} \text{ and } \cos \beta = \frac{a}{\sqrt{1+a^2}} \Rightarrow \tan \alpha = a \text{ and } \tan \beta = \frac{1}{a}.$$

$$\text{Using the addition formula for tangent, } \tan(\alpha - \beta) = \frac{a - \frac{1}{a}}{1 + 1} = \frac{a^2 - 1}{2a}$$

Question 11*Answer:* D*Explanation:*

$$y = \ln \frac{1}{1+x} = -\ln(1+x) \quad x \frac{dy}{dx} + p = e^y$$

Substituting $\frac{dy}{dx} = -\frac{1}{1+x}$ into $x \frac{dy}{dx} + p = e^y$ yields

$$\frac{-x}{1+x} + p = e^{\ln \frac{1}{1+x}}$$

$$\frac{-x}{1+x} + p = \frac{1}{1+x} \Rightarrow p = \frac{x}{1+x} + \frac{1}{1+x} = 1$$

Question 12*Answer:* A*Explanation:*

Using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}$$

$$x = a \cos t, \quad \frac{dx}{dt} = -a \sin t$$

$$y = b \sin t, \quad \frac{dy}{dt} = b \cos t$$

The gradient of the tangent is $\frac{dy}{dx} = -\frac{b \cos t}{a \sin t}$.

For $t = \frac{3\pi}{4}$, the gradient of the normal is $\frac{a}{b} \tan \frac{3\pi}{4} = -\frac{a}{b}$

Question 13*Answer:* B*Explanation:*Let $u = x + 2$, then $x = u - 2$ and $dx = du$

$$\begin{aligned} \int_0^3 \frac{2x-1}{\sqrt{x+2}} dx &= \int_2^5 \frac{2u-5}{\sqrt{u}} du \\ &= \int_2^5 \left(2\sqrt{u} - \frac{5}{\sqrt{u}} \right) du \end{aligned}$$

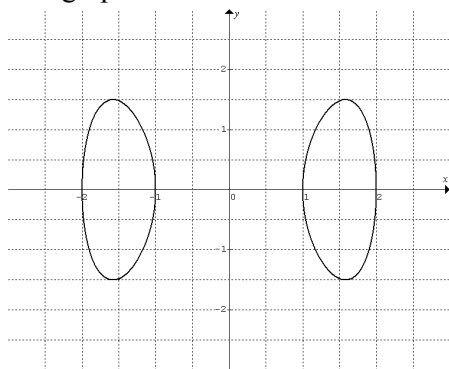
Question 14*Answer:* E*Explanation:*

The slope field given in this question appears only in the first and third quadrant. The corresponding differential equation is defined only when x and y are both positive or both negative.

$$\frac{dy}{dx} = \frac{y}{x} \ln\left(\frac{y}{x}\right) \text{ is defined for } \frac{y}{x} > 0.$$

Question 15*Answer:* D*Explanation:*

The graph below shows the relation $y^2 = -x^4 + 5x^2 - 4$.



$$\text{enclosed area} = 4 \int_1^2 \sqrt{-x^4 + 5x^2 - 4} dx = 4.646$$

Question 16*Answer:* C*Explanation:*

$$\begin{aligned} \int \frac{\sin^2 x \cos^5 x}{1 + \cos 2x} dx &= \int \frac{\sin^2 x \cos^5 x}{2 \cos^2 x} dx \\ &= \frac{1}{2} \int \sin^2 x \cos^3 x dx \\ &= \frac{1}{2} \int \sin^2 x (1 - \sin^2 x) \cos x dx \end{aligned}$$

The substitution $\sin x = u, \cos x dx = du$ gives $\frac{1}{2} \int (u^2 - u^4) du$.

Question 17*Answer:* C*Explanation:*

$$t = 0, x = 50 \times 10 = 500 \text{ grams}$$

$$\frac{dx}{dt} = \text{Rate}_{in} \times \text{Concentration}_{in} - \text{Rate}_{out} \frac{x}{V_0 + (\text{Rate}_{in} - \text{Rate}_{out})t}$$

$$\frac{dx}{dt} = 3 \times 8 - \frac{2x}{50 + t}$$

$$\frac{dx}{dt} = 24 - \frac{2x}{50 + t}, \quad t = 0, x = 500$$

Question 18*Answer:* A*Explanation:*

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + hf(x_n, y_n), \quad x_0 = 1, y_0 = 0, h = 0.1, \quad f(x, y) = x - \log_e(x + y)$$

$$x_0 = 1 \quad y_1 = 0 + 0.1(1 - \log_e 1) = 0.1$$

$$\begin{aligned} x_1 = 1.1, \quad y_2 &= 0.1 + 0.1(1.1 - \log_e 1.2) \\ &= 0.1 + 0.11 - 0.1 \log_e 1.2 \\ &= 0.21 - 0.1 \log_e 1.2 \end{aligned}$$

Question 19*Answer:* A*Explanation:*

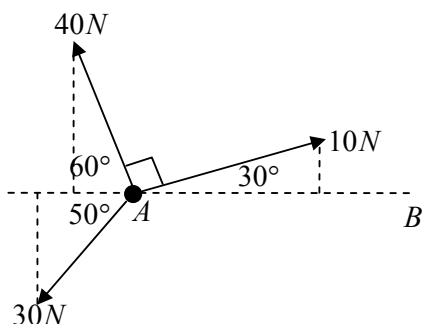
$$\text{When } t = 0, v_1 = 0 \text{ and } v_2 = -10.$$

$$v_1 = 3t^2 \Rightarrow s_1 = \int 3t^2 dt = t^3$$

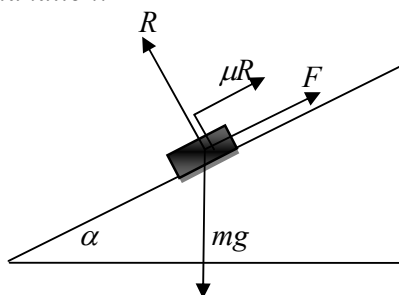
$$v_2 = (6t^2 - 10) \Rightarrow s_2 = \int (6t^2 - 10) dt = 2t^3 - 10t$$

$$t = 10, s_1 = 1000, s_2 = 1900$$

$$s_2 - s_1 = 900m$$

Question 20*Answer:* D*Explanation:*

Sum of perpendicular components = $40 \sin 60^\circ - 30 \sin 50^\circ + 10 \sin 30^\circ = 16.66$ (2 dec.pl).

Question 21*Answer:* B*Explanation:*

Resolving horizontally : $F + \mu R - mg \sin \alpha = 0$

Resolving vertically : $R - mg \cos \alpha = 0 \Rightarrow R = mg \cos \alpha$

Therefore, $F = mg \sin \alpha - \mu mg \cos \alpha = mg(\sin \alpha - \mu \cos \alpha)$.

Question 22*Answer:* B*Explanation:*

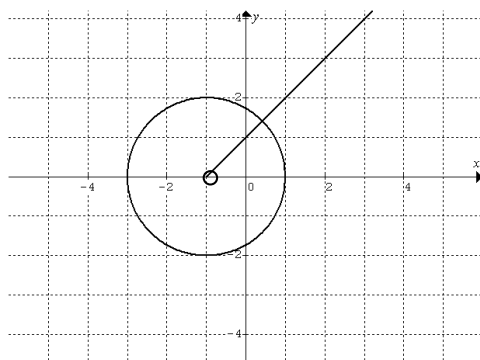
$$v = \sqrt[3]{2-x^2} = (2-x^2)^{\frac{1}{3}} \Rightarrow \frac{dv}{dx} = \frac{-2x}{3 \sqrt[3]{(2-x^2)^2}}$$

$$\begin{aligned} a &= v \frac{dv}{dx} = \sqrt[3]{2-x^2} \times \frac{-2x}{3 \sqrt[3]{(2-x^2)^2}} \\ &= \frac{-2x}{3 \sqrt[3]{2-x^2}} = \frac{-2x}{3v} \end{aligned}$$

SECTION 2

Question 1

- a. Region $A = \left\{ z : \text{Arg}(1+z) = \frac{\pi}{4} \right\}$ represents a ray with gradient of 1 with the domain $(-1, +\infty)$, while region $B = \{ z : |1+z| = 2 \}$ is a circle with centre at $(-1, 0)$ and radius 2.



Ray with correct domain (excluding -1) and gradient A2

Circle with correct centre and radius A2

- b. $y = x + 1, x > -1$ A1
 $(x+1)^2 + y^2 = 4$ A1

c. Method 1

If $|1+z_0| = 2$ and $\text{Arg}(1+z_0) = \frac{\pi}{4}$, then the polar form of $1+z_0$ is $2\text{cis}\frac{\pi}{4} = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$. M1

$z = -1 + \sqrt{2} + i\sqrt{2}$, as required. M1

Method 2

Solving the Cartesian equations simultaneously yields

$$(x+1)^2 + (x+1)^2 = 4 \Rightarrow x = -1 \pm \sqrt{2} \quad \text{M1}$$

As $x > -1$, $x = -1 + \sqrt{2}$, $y = \sqrt{2}$ and therefore, $z_0 = -1 + \sqrt{2} + i\sqrt{2}$. M1

d.

$$\text{i } (1+z_0)^6 = 2^6 \text{cis} \frac{6\pi}{4} \quad \text{M1}$$

$$= 64 \left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right) \quad \text{A1}$$

$$= -64i$$

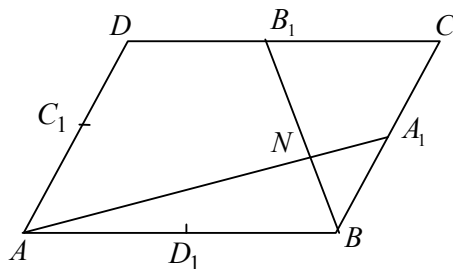
$$\text{ii } \sqrt{1+z_0} = \left(2\text{cis} \frac{\pi}{4} \right)^{\frac{1}{2}} = \sqrt{2}\text{cis} \frac{\pi}{8} \quad \text{and} \quad \sqrt{2}\text{cis} \frac{9\pi}{8} = \sqrt{2}\text{cis} \left(-\frac{7\pi}{8} \right) \quad \text{A2}$$

Question 2

a. $\vec{AA_1} = \mathbf{a} + \frac{1}{2}\mathbf{b}$, $\vec{BB_1} = \mathbf{b} - \frac{1}{2}\mathbf{a}$

A2

b.



$$\vec{AN} = n\vec{AA_1} = n\left(\mathbf{a} + \frac{1}{2}\mathbf{b}\right) \quad (1), \quad \vec{BN} = m\vec{BB_1} = m\left(\mathbf{b} - \frac{1}{2}\mathbf{a}\right)$$

A1

Also, $\vec{AN} = \vec{AB} + \vec{BN}$

$$\vec{AN} = \mathbf{a} + m\mathbf{b} - \frac{m}{2}\mathbf{a} \quad (2)$$

M1

Equating (1) and (2) yields $\mathbf{a} + m\mathbf{b} - \frac{m}{2}\mathbf{a} = n\mathbf{a} + \frac{1}{2}n\mathbf{b}$

$$\left(1 - \frac{m}{2}\right)\mathbf{a} + m\mathbf{b} = n\mathbf{a} + \frac{n}{2}\mathbf{b}$$

$$1 - \frac{m}{2} = n, \quad m = \frac{n}{2}$$

M1

Solving the last two equations simultaneously gives $n = \frac{4}{5}$, $m = \frac{2}{5}$.

Therefore, $\vec{AN} = \frac{4}{5}\vec{AA_1}$, $\vec{NA_1} = \frac{1}{5}\vec{AA_1} \Rightarrow AN : NA_1 = 4 : 1$

A1

c. $\vec{AA_1} \cdot \vec{BB_1} = \left(\mathbf{a} + \frac{1}{2}\mathbf{b}\right) \cdot \left(\mathbf{b} - \frac{1}{2}\mathbf{a}\right)$

M1

$$= \mathbf{a} \cdot \mathbf{b} - \frac{1}{2}|\mathbf{a}|^2 + \frac{1}{2}|\mathbf{b}|^2 - \frac{1}{4}\mathbf{a} \cdot \mathbf{b}$$

$$= \frac{3}{4}\mathbf{a} \cdot \mathbf{b} - \frac{1}{2}|\mathbf{a}|^2 + \frac{1}{2}|\mathbf{b}|^2 \quad (1)$$

M1

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 60^\circ$$

$$= k \times \frac{3}{2}k \times \frac{1}{2} = \frac{3}{4}k^2 \quad (2)$$

A1

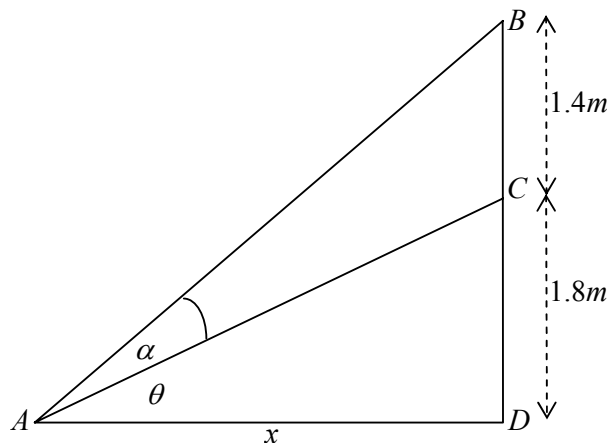
Substituting (2) and $|\mathbf{a}| = k$, $|\mathbf{b}| = \frac{2}{3}|\mathbf{a}|$ into equation (1), we have

$$\vec{AA_1} \cdot \vec{BB_1} = \frac{9}{16}k^2 - \frac{1}{2}k^2 + \frac{9}{8}k^2 = \frac{19}{16}k^2$$

A1

Question 3

a. Let $\angle CAD = \theta$.



From the triangle ACD , $x = \frac{1.8}{\tan \theta} \Rightarrow \tan \theta = \frac{1.8}{x}$, A1

From the triangle ABD , $x = \frac{3.2}{\tan(\alpha + \theta)} \Rightarrow \tan(\alpha + \theta) = \frac{3.2}{x}$. A1

Further, $\frac{\tan \alpha + \frac{1.8}{x}}{1 - \tan \alpha \times \frac{1.8}{x}} = \frac{3.2}{x}$ M1

$$\frac{1.8 + x \tan \alpha}{x - 1.8 \tan \alpha} = \frac{3.2}{x}$$

$$1.8x + x^2 \tan \alpha = 3.2x - 5.76 \tan \alpha$$
 M1

$$\tan \alpha (x^2 + 5.76) = 1.4x \Rightarrow \tan \alpha = \frac{1.4x}{x^2 + 5.76}$$

b.

i. $f(x) = \frac{1.4x}{x^2 + 5.76}$

$$f'(x) = \frac{1.4(x^2 + 5.76) - 1.4x \times 2x}{(x^2 + 5.76)^2}$$
 M1

$$= \frac{1.4(x^2 + 5.76 - 2x^2)}{(x^2 + 5.76)^2}$$
 M1

$$= \frac{1.4(5.76 - x^2)}{(x^2 + 5.76)^2}, \text{ as required.}$$

- ii. The angle α will have a maximum value when $\tan \alpha$ has a maximum and therefore when $f(x)$ has a maximum.

M1

$$f'(x) = 0 \text{ when } x = \pm\sqrt{5.76} = \pm 2.4. \text{ As } x > 0, x = 2.4$$

M1

$$\alpha = \tan^{-1} \frac{1.4 \times 2.4}{2.4^2 + 5.76} = 16.26^\circ$$

A1

c. $\frac{d\alpha}{dt} = \frac{d\alpha}{dx} \frac{dx}{dt}$

$$\frac{dx}{dt} = -1.2 \text{ ms}^{-1}$$

$$\alpha = \tan^{-1} \frac{1.4x}{x^2 + 5.76}, \text{ when } x = 4, \frac{d\alpha}{dx} = -1.626978 \dots \text{degrees per second (calculator)} \quad \text{M1}$$

$$\frac{d\alpha}{dt} = -1.626978 \dots \times -1.2 = 1.952374 \dots \approx 2 \text{ degrees per second.} \quad \text{pos. rate} \quad \text{A1}$$

correct value A1

Question 4

a. $y = \frac{e^{mx} + e^{-mx} - 2}{2m}$

$$\frac{dy}{dx} = \frac{me^{mx} - me^{-mx}}{2m}$$

$$= \frac{e^{mx} - e^{-mx}}{2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(e^{mx} - e^{-mx})^2}{4}$$

$$= \frac{e^{2mx} - 2e^{-mx}e^{mx} + e^{-2mx}}{4}$$

$$= \frac{e^{2mx} - 2 + e^{-2mx}}{4}, \text{ as required.}$$

M1

M1

b.

$$\text{arc length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$l = 2 \int_0^d \sqrt{1 + \frac{e^{2mx} - 2 + e^{-2mx}}{4}} dx$$

$$= 2 \int_0^d \sqrt{\frac{e^{2mx} + 2 + e^{-2mx}}{4}} dx$$

$$= 2 \int_0^d \sqrt{\frac{(e^{mx} + e^{-mx})^2}{4}} dx$$

$$= 2 \int_0^d \frac{e^{mx} + e^{-mx}}{2} dx$$

$$= \int_0^d (e^{mx} + e^{-mx}) dx$$

$$= \frac{1}{m} [e^{mx} - e^{-mx}]_0^d = \frac{e^{md} - e^{-md}}{m}, \text{ as required.}$$

A1

M1

A1

M1

c.

i. When $h = 20$, $x = d$ and $m = \frac{1}{32}$, we have $20 = \frac{e^{\frac{d}{32}} + e^{-\frac{d}{32}} - 2}{\frac{1}{16}}$

The solution can be found by using a graphics calculator $d = 34.135438\dots$
 The distance between the poles is 68.27 metres (2 dp).

A1

ii The length of the cable is 81.975607... $\approx$ 81.98 metres.

A1

d. $100 = \frac{e^{40m} - e^{-40m}}{m}$
 $m = 0.02956814\dots \approx 0.03$

M1

A1

Question 5

a. i. $\mathbf{r}_A = (1 + \cos(kt))\mathbf{i} + (2 + 2\sin(kt))\mathbf{j}$

$$x = 1 + \cos(kt) \Rightarrow \cos(kt) = x - 1$$

$$y = 2 + 2\sin(kt) \Rightarrow \sin(kt) = \frac{y-2}{2}$$

A1

Squaring both equations and then adding them results in the given equation.

$$\cos^2(kt) = (x-1)^2, \quad \sin^2(kt) = \left(\frac{y-2}{2}\right)^2$$

$$\cos^2(kt) + \sin^2(kt) = 1, \quad (x-1)^2 + \frac{(y-2)^2}{4} = 1$$

M1

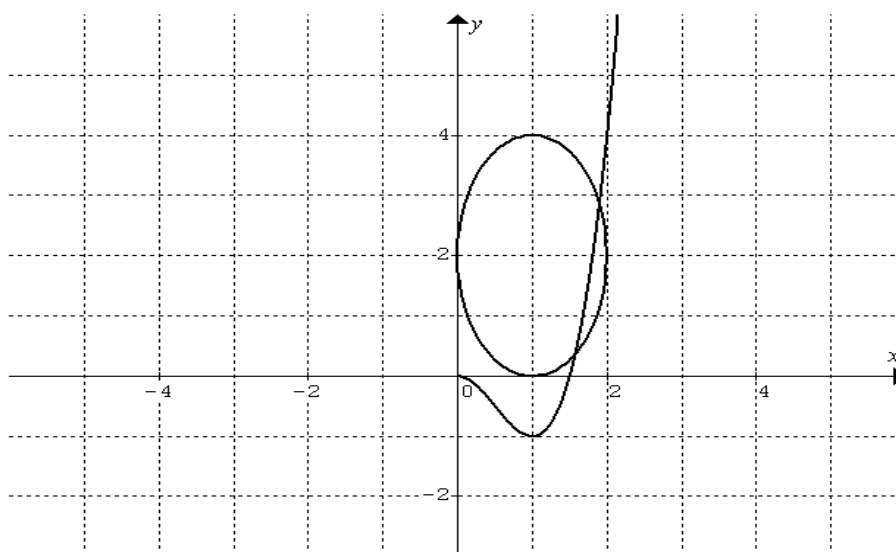
ii. $\mathbf{r}_B = t\mathbf{i} + (2t^3 - 3t^2)\mathbf{j}$

$$x = t, y = 2t^3 - 3t^2 \Rightarrow y = 2x^3 - 3x^2, \quad x \geq 0$$

A1

(domain must be given)

b.



Both graphs correct, including the domain for particle B A2

c. The velocity of particle A is $\dot{\mathbf{r}}_A = (-k \sin(kt))\mathbf{i} + (2k \cos(kt))\mathbf{j}$ M1

and the speed is

$$|\dot{\mathbf{r}}_A| = \sqrt{k^2 \sin^2(kt) + 4k^2 \cos^2(kt)}$$

$$= k\sqrt{1 + 3\cos^2(kt)}$$

M2

The maximum of $1 + 3\cos^2(kt)$ is 4. Therefore, the maximum speed of particle A is $2k$ A1

d. The points of intersection are $(1.573, 0.361)$ and $(1.899, 2.876)$ A2

e. The particles will collide when they are at the same position at the same time.

For the first point, $t = x = 1.573$, so $1 + \cos(1.573k) = 1.573 \Rightarrow k = 0.610808...$ M1

Now, for $k = 0.610808...$, $y = 2 + 2 \sin(0.610808 \times 1.573) = 3.6393...$, so this is not a point of collision.

For the second point, $t = x = 1.899$ M1

$1 + \cos(1.899k) = 1.899 \Rightarrow k = 0.23871277...$ and

$y = 2 + 2 \sin(0.23871277 \times 1.899) = 2.8758...$, so this is the first point of collision.

Therefore, the minimum value of k is $0.239(3dp)$. A1