

Year 2013

VCE

Specialist Mathematics

Trial Examination 2

Solutions



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- Teachers and students are reminded that for the purposes of school requirements and external assessments, students must submit work that is clearly their own.

SECTION 1

ANSWERS

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E

SECTION 1

Question 1

Answer C

$$(1) \frac{(x+a)^2}{a^2} + \frac{(y+a)^2}{b^2} = 1 \quad (2) \frac{(y+a)^2}{b^2} - \frac{(x+a)^2}{a^2} = 1 \quad \text{add (1)+(2)}$$

$$\frac{2(y+a)^2}{b^2} = 2$$

$$\frac{(y+a)^2}{b^2} = 1$$

$$(y+a)^2 = b^2$$

$$y+a = \pm b \quad \text{so } y = -a \pm b$$

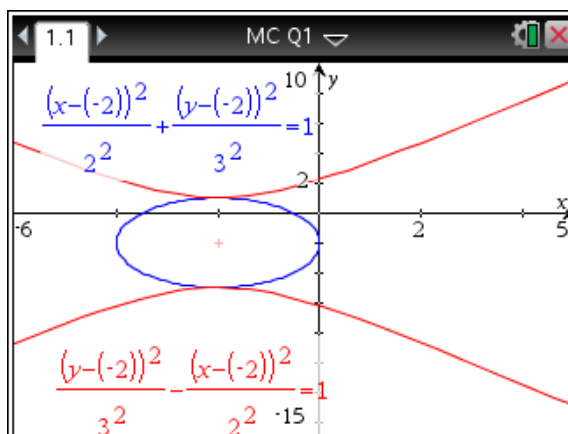
when $y = -a \pm b$ into (1)

$$\frac{(x+a)^2}{a^2} + \frac{b^2}{b^2} = 1$$

$$(x+a)^2 = 0$$

$$\Rightarrow x = -a$$

the graphs touch at two points $(-a, b-a)$ and $(-a, -(a+b))$



Question 2

Answer E

$$y = \frac{1}{ax^2 + b(1-a)x - b^2} \quad \text{the graph has vertical asymptotes when the denominator is zero}$$

$$ax^2 + b(1-a)x - b^2 = 0$$

$$(x-b)(ax+b) = 0$$

$$x = b, -\frac{b}{a}$$

also as $x \rightarrow \infty$ $y \rightarrow 0$ so that the x -axis or $y = 0$ is also a horizontal asymptote.

The asymptotes are $x = b$, $x = -\frac{b}{a}$ and $y = 0$.

Question 3

Answer C

$$y = \frac{ax^n + b}{x^2} = ax^{n-2} + \frac{b}{x^2}, \quad \text{but the line } y = ax \text{ is an oblique asymptote with a negative gradient,}$$

therefore $n = 3$ and $a < 0$. The graph crosses the x -axis when $y = 0$ at $ax^3 + b = 0$, so that the

x -intercept is $x^3 = -\frac{b}{a} \Rightarrow x = \sqrt[3]{-\frac{b}{a}}$, since $a < 0$, we must have $b < 0$, since this x -intercept is negative.

Question 4

Answer B

$y = a \sin^{-1}(bx - c) + d$ the domain and range are both equal to $[0, \pi]$

the domain, satisfies

$$|bx - c| \leq 1$$

$$-1 \leq bx - c \leq 1$$

$$c - 1 \leq bx \leq c + 1$$

$$\frac{c-1}{b} \leq x \leq \frac{c+1}{b}$$

$$\frac{c-1}{b} = 0 \quad \text{and} \quad \frac{c+1}{b} = \pi$$

$$c = 1 \quad \text{and} \quad \frac{2}{b} = \pi \quad \text{so that} \quad b = \frac{2}{\pi}$$

the range satisfies

$$\left[\frac{-a\pi}{2} + d, \frac{a\pi}{2} + d \right] = [0, \pi]$$

$$\text{so that } a = 1 \quad \text{and} \quad d = \frac{\pi}{2}$$

Question 5

Answer B

Let $z = a + bi = r \operatorname{cis}(\theta)$

The conjugate of z , is the reflection in the real axis, so that $\bar{z} = a - bi = r \operatorname{cis}(-\theta)$

Now rotate $\bar{z}$ by 90° anticlockwise $i\bar{z} = i(a - bi) = ai - bi^2 = b + ai = r \operatorname{cis}\left(\frac{\pi}{2} - \theta\right)$

Now multiply by -1 , so that $-i\bar{z} = -b - ai = -r \operatorname{cis}\left(\frac{\pi}{2} - \theta\right)$

Question 6

Answer C

$z = (a + b)i$ and $z = (a - b)i$, satisfy $P(z) = 0$. Since the polynomial has real coefficients, the conjugates are also zeros, so that $z = -(a + b)i$ and $z = -(a - b)i$.

These are distinct since a and b are non-zero real constants with $|a| \neq |b|$. The minimum

degree of the polynomial is 4, and $P(z) = (z^2 + (a + b)^2)(z^2 + (a - b)^2) = z^4 + \dots$

Question 7

Answer D

$$z^3 - ia^3 = 0$$

$$z^3 = ia^3 = -1 \times i^2 \times ia^3 = -i^3 a^3 = (-ai)^3$$

one root is $z = -ai$, at point J , but there are three roots, and they all must be equally spaced

around the circle, by $\frac{2\pi}{3} = 120^\circ$, the other two roots, must be B and F .

Question 8

Answer E

From projectile motion theory, using $V = 19.6 = 2g$ and $\alpha = 30^\circ$

the time of flight is $T = \frac{2V \sin(\alpha)}{g} = \frac{2 \times 2g \times \sin(30^\circ)}{g} = 4 \times \frac{1}{2} = 2$ s.

the maximum height reached is $H = \frac{V^2 \sin^2(\alpha)}{2g} = \frac{(2g)^2 (\sin^2(30^\circ))}{2g} = \frac{4g^2 \times \frac{1}{4}}{2g} = \frac{g}{2} = 4.9$ m.

the range is $R = \frac{V^2 \sin(2\alpha)}{g} = \frac{(2g)^2 \sin(60^\circ)}{g} = \frac{4g^2 \times \frac{\sqrt{3}}{2}}{g} = 2g\sqrt{3}$ m.

So the cricket ball reaches a maximum height of 4.9 metres and hits the ground after two seconds, $2g\sqrt{3}$ metres from the point of impact.

Question 9

Answer B

Using constant acceleration formulae, with $s = -1$, $u = 4.9$, $a = -g = -9.8$

$s = ut + \frac{1}{2}at^2 \Rightarrow -1 = 4.9t - 4.9t^2$ solving use CAS with $t \geq 0$, gives $t = 1.17$

Question 10

Answer E

$$|z - b|^2 + |z + bi|^2 = 2b^2 \quad \text{let } z = x + yi$$

$$|(x - b) + yi|^2 + |x + (y + b)i|^2 = 2b^2$$

$$(x - b)^2 + y^2 + x^2 + (y + b)^2 = 2b^2 \quad \text{expanding}$$

$$x^2 - 2bx + b^2 + y^2 + x^2 + y^2 + 2by + b^2 = 2b^2$$

$$2x^2 - 2bx + 2y^2 + 2by = 0$$

$$x^2 - bx + y^2 + by = 0 \quad \text{completing the square}$$

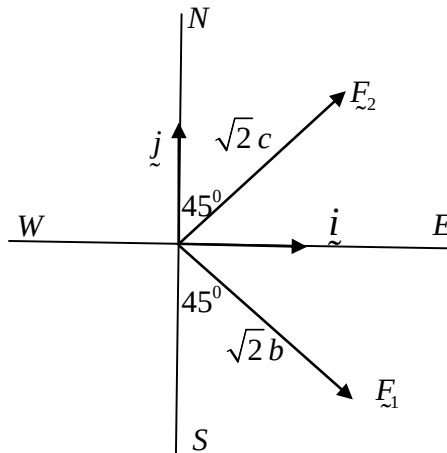
$$x^2 - bx + \frac{b^2}{4} + y^2 + by + \frac{b^2}{4} = \frac{b^2}{2}$$

$$\left(x - \frac{b}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 = \frac{b^2}{2}$$

circle, centre $\left(\frac{b}{2}, -\frac{b}{2}\right)$ and radius $r^2 = \frac{b^2}{2}$ since $r > 0$ $r = \frac{b}{\sqrt{2}} = \frac{\sqrt{2}b}{2}$

Question 11

Answer D



1.1	MC Q11	
$[b \ -b] \rightarrow f1$	$[b \ -b]$	
$[c \ c] \rightarrow f2$	$[c \ c]$	
$\text{norm}(f1)$	$ b \cdot \sqrt{2}$	
$\text{norm}(f2)$	$ c \cdot \sqrt{2}$	
$f1+f2$	$[b+c \ c-b]$	
$\text{norm}(f1+f2)$	$\sqrt{2} \cdot (b^2+c^2)$	
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$$\vec{F}_1 = \sqrt{2}b(\sin(45^\circ)\hat{i} - \cos(45^\circ)\hat{j}) = b(\hat{i} - \hat{j}) \quad |\vec{F}_1| = \sqrt{2}b$$

$$\vec{F}_2 = \sqrt{2}c(\sin(45^\circ)\hat{i} + \cos(45^\circ)\hat{j}) = c(\hat{i} + \hat{j}) \quad |\vec{F}_2| = \sqrt{2}c$$

$$\vec{F}_1 + \vec{F}_2 = (b+c)\hat{i} + (c-b)\hat{j}$$

$$|\vec{F}_1 + \vec{F}_2| = \sqrt{(b+c)^2 + (c-b)^2} = \sqrt{b^2 + 2bc + c^2 + c^2 - 2bc + b^2} \\ = \sqrt{2(b^2 + c^2)}$$

Question 12

Answer D

Resolving the forces horizontally, the equation of motion is $ma = F - kv$, although, m , V , k , F , D and T are all constants, we cannot use the constant acceleration formulae, since the acceleration is not constant, and depends on the speed. So Alan, Ben and Colin are all incorrect, all their relationships are derived from using the constant acceleration formulae.

Use $a = \frac{dv}{dt}$ then $m \frac{dv}{dt} = F - kv \Rightarrow \frac{dt}{dv} = \frac{m}{F - kv}$ integrating, between $v = 0$ and $v = V$

$$\Rightarrow \int_0^T 1 dt = T = \int_0^V \frac{m}{F - kv} dv, \text{ so that Edward is correct.}$$

Use $a = v \frac{dv}{dx}$ then $mv \frac{dv}{dx} = F - kv \Rightarrow \frac{dx}{dv} = \frac{mv}{F - kv}$ integrating, between $v = 0$ and $v = V$,

$$\Rightarrow \int_0^D 1 dx = D = \int_0^V \frac{mv}{F - kv} dv, \text{ so that David is correct.}$$

Question 13

Answer C

If the lift is moving up, the equation of motion is

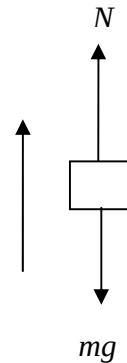
$$ma = N - mg$$

$$m = 50 \text{ kg and } N = 60 \text{ kg-wt} = 60g \text{ newtons}$$

$$50a = 60g - 50g = 10g$$

$$a = \frac{g}{5} = 1.96 \text{ ms}^{-2}$$

C. is correct, the lift moves up with an acceleration equal to 1.96 ms^{-2} .



Question 14

Answer A

$$m_N = -\frac{dx}{dy} = 2\sqrt{\frac{y}{x}} \text{ inverting both sides}$$

$$\frac{dy}{dx} = -\frac{1}{2}\sqrt{\frac{x}{y}} = -\sqrt{\frac{x}{4y}}$$

$$\frac{dy}{dx} + \sqrt{\frac{x}{4y}} = 0$$

Question 15

Answer E

$|\underline{u}| = |\underline{v}| = L$. The angle between the vectors $\underline{u}$ and $\underline{v}$ is θ . So that

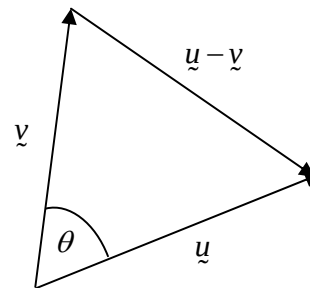
$$\cos(\theta) = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}||\underline{v}|} = \frac{\underline{u} \cdot \underline{v}}{L^2}$$

$$\underline{u} \cdot \underline{v} = L^2 \cos(\theta)$$

$$\begin{aligned} |\underline{u} - \underline{v}|^2 &= (\underline{u} - \underline{v}) \cdot (\underline{u} - \underline{v}) \\ &= \underline{u} \cdot \underline{u} - 2\underline{u} \cdot \underline{v} + \underline{v} \cdot \underline{v} = |\underline{u}|^2 - 2\underline{u} \cdot \underline{v} + |\underline{v}|^2 \\ &= 2L^2 - 2L^2 \cos(\theta) \\ &= 2L^2 (1 - \cos(\theta)) \\ &= 2L^2 \left(1 - \left(1 - 2\sin^2\left(\frac{\theta}{2}\right) \right) \right) \end{aligned}$$

$$|\underline{u} - \underline{v}|^2 = 4L^2 \sin^2\left(\frac{\theta}{2}\right), \text{ since } L > 0$$

$$|\underline{u} - \underline{v}| = 2L \sin\left(\frac{\theta}{2}\right)$$



Question 16

Answer A

$$\underline{a} = \underline{i} - \underline{j} + \underline{k} \quad , \quad \underline{b} = 2\underline{i} + 2\underline{j} - \underline{k} \quad \text{and} \quad \underline{c} = \frac{1}{3}(\underline{i} - 5\underline{j} + 4\underline{k})$$

$$\underline{c} = \frac{1}{3}(m\underline{a} + n\underline{b}) = \frac{1}{3}(\underline{i} - 5\underline{j} + 4\underline{k})$$

$$\underline{c} = \frac{1}{3}(m(\underline{i} - \underline{j} + \underline{k}) + n(2\underline{i} + 2\underline{j} - \underline{k})) = \frac{1}{3}((m + 2n)\underline{i} + (2n - m)\underline{j} + (m - n)\underline{k})$$

$$(1) \quad m + 2n = 1 \quad (2) \quad 2n - m = -5 \quad (3) \quad m - n = 4$$

$$(1) + (2) \Rightarrow 4n = -4 \Rightarrow n = -1 \quad \text{and} \quad m = 3, \text{ checks in (3) so that } \underline{c} = \frac{1}{3}(3\underline{a} - \underline{b})$$

so that the vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ are linearly dependant. **A.** is true.

$$\underline{a} \cdot \underline{b} = 2 - 2 - 1 = -1 \quad |\underline{b}| = \sqrt{4 + 4 + 1} = 3 \quad \hat{\underline{b}} = \frac{1}{3}(2\underline{i} + 2\underline{j} - \underline{k})$$

the vector resolute of the vector $\underline{a}$ parallel to the vector $\underline{b}$ is

$$(\underline{a} \cdot \hat{\underline{b}})\hat{\underline{b}} = -\frac{1}{9}(2\underline{i} + 2\underline{j} - \underline{k}) \quad \text{C. is false}$$

the vector resolute of the vector $\underline{a}$ perpendicular to the vector $\underline{b}$ is

$$\underline{a} - (\underline{a} \cdot \hat{\underline{b}})\hat{\underline{b}} = (\underline{i} - \underline{j} + \underline{k}) + \frac{1}{9}(2\underline{i} + 2\underline{j} - \underline{k}) = \frac{1}{9}(11\underline{i} - 7\underline{j} + 8\underline{k}) \quad \text{D. is false}$$

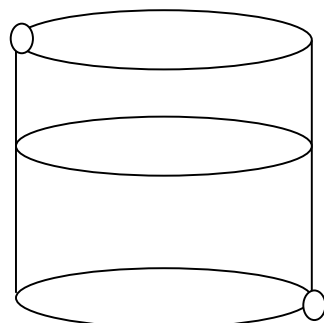
$$\underline{a} + \underline{b} = 3\underline{i} + \underline{j} \quad \text{E. is false}$$

Question 17

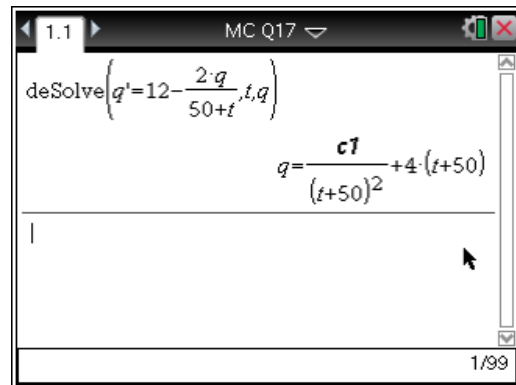
Answer C

inflow
4 kg/litre at
3 litre/min

50 litres



outflow at 2 litre/min



$$\frac{dQ}{dt} = \text{inflow} - \text{outflow}, \text{ and } V(t) = 50 + (3 - 2)t = 50 + t$$

$$\frac{dQ}{dt} = 4 \times 3 - \frac{2Q}{50+t} = 12 - \frac{2Q}{50+t} \quad \text{solving using CAS} \quad Q = 4(50+t) + \frac{c}{(50+t)^2}$$

Question 18

Answer A

from the direction field diagram, consider the points (x, y) and the gradient m at these points.

$$(1, 1) \quad m = 1$$

$$(1, -1) \quad m = 1$$

$$(-1, 1) \quad m = -1$$

$$(-1, -1) \quad m = -1, \text{ since the gradient is } \frac{dy}{dx},$$

$$\frac{dy}{dx} = y^2 x \text{ is the only differential equation}$$

which satisfies all these conditions.

Question 19

Answer D

$$\underline{v}(t) = \underline{\dot{r}}(t) = e^t \cos(t) \underline{i} + e^t \sin(t) \underline{j}$$

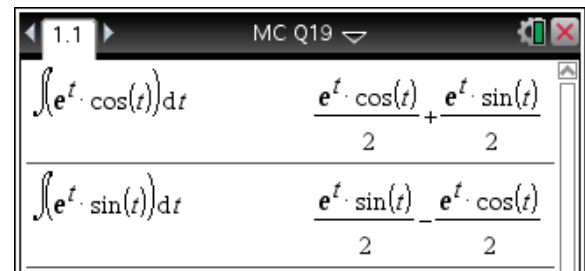
$$\underline{r}(t) = \int e^t \cos(t) dt \underline{i} + \int e^t \sin(t) dt \underline{j}$$

integrating using CAS

$$\underline{r}(t) = \frac{1}{2} e^t (\cos(t) + \sin(t)) \underline{i} + \frac{1}{2} e^t (\sin(t) - \cos(t)) \underline{j} + \underline{c}$$

$$\text{but } \underline{r}(0) = \underline{0} = \frac{1}{2} \underline{i} - \frac{1}{2} \underline{j} + \underline{c} \text{ so that } \underline{c} = -\frac{1}{2} \underline{i} + \frac{1}{2} \underline{j}$$

$$\underline{r}(t) = \frac{1}{2} [e^t (\cos(t) + \sin(t)) - 1] \underline{i} + \frac{1}{2} [e^t (\sin(t) - \cos(t)) + 1] \underline{j}$$



MC Q19

$\int e^t \cdot \cos(t) dt$	$\frac{e^t \cdot \cos(t)}{2} + \frac{e^t \cdot \sin(t)}{2}$
$\int e^t \cdot \sin(t) dt$	$\frac{e^t \cdot \sin(t)}{2} - \frac{e^t \cdot \cos(t)}{2}$

Question 20

Answer B

$$\frac{dy}{dx} = f(x) = \tan(x)$$

$$h = \frac{1}{5} = 0.2, \quad x_0 = 0 \text{ and } y_0 = 0$$

$$x_1 = 0.2, \quad x_2 = 0.4, \quad x_3 = 0.6 = \frac{3}{5}$$

using Euler's Method

$$y_1 = y_0 + hf(x_0)$$

$$y_1 = 0 + 0.2 \tan(0) = 0$$

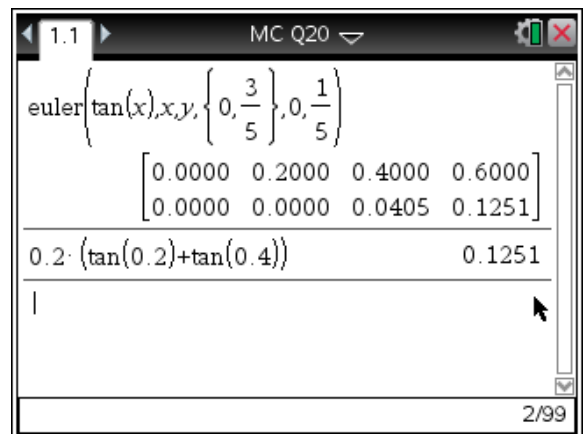
$$y_2 = y_1 + hf(x_1)$$

$$y_2 = 0 + 0.2 \tan(0.2)$$

$$y_3 = y_2 + hf(x_2)$$

$$y_3 = 0.2 \tan(0.2) + 0.2 \tan(0.4)$$

$$y_3 = 0.2(\tan(0.2) + \tan(0.4))$$



MC Q20

euler($\tan(x), x, y, \{0, \frac{3}{5}\}, 0, \frac{1}{5}$)

	0.0000	0.2000	0.4000	0.6000
	0.0000	0.0000	0.0405	0.1251
0.2 * (tan(0.2) + tan(0.4))	0.1251			

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Question 21

Answer A

If $m_2 > m_1$ the mass m_2 moves downwards, resolving downwards,

$$(1) \quad m_2 g - T = m_2 a$$

The mass m_1 moves upwards, resolving upwards,

$$(2) \quad T - m_1 g = m_1 a$$

Solving for a and T gives

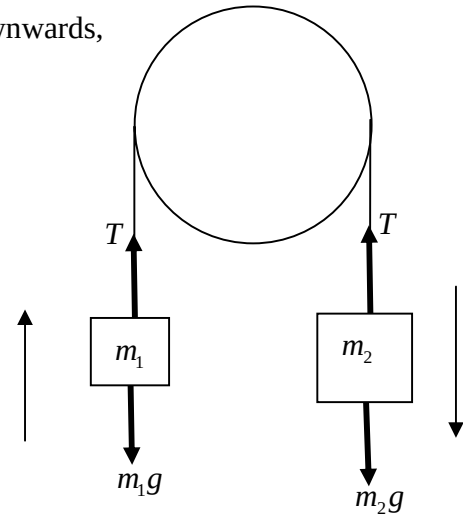
$$a = \frac{(m_2 - m_1)g}{m_1 + m_2} \quad \text{and} \quad T = \frac{2m_1 m_2 g}{m_1 + m_2}$$

So that **D.** and **E.** are both true.

B. and **C.** are also true.

$$\text{If } m_2 = 2m_1 \text{ then } a = \frac{(2m_1 - m_1)g}{m_1 + 2m_1} = \frac{g}{3}$$

A. is false.



Question 22

Answer E

$$v = e^{cx} \quad \frac{dv}{dx} = ce^{cx}$$

$$F = ma = mv \frac{dv}{dx} = mce^{2cx} \quad \text{when } x = \frac{1}{c}$$

$$F = mce^{2c \times \frac{1}{c}} = mce^2$$

END OF SECTION 1 SUGGESTED ANSWERS

SECTION 2

Question 1

a. $\overrightarrow{OA} = -2\hat{i} + 3\hat{j} + 6\hat{k}$
 $\overrightarrow{OC} = 3\hat{i} + 6\hat{j} - 2\hat{k}$
 $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OA} + \overrightarrow{OC} = (-2\hat{i} + 3\hat{j} + 6\hat{k}) + (3\hat{i} + 6\hat{j} - 2\hat{k})$ M1
 $\overrightarrow{OB} = \hat{i} + 9\hat{j} + 4\hat{k}$
 $B(1, 9, 4)$ A1

b. $\overrightarrow{OE} = \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}(\hat{i} + 9\hat{j} + 4\hat{k})$
 $E\left(\frac{1}{2}, \frac{9}{2}, 2\right)$ A1

c. $D(x, y, z)$ $\overrightarrow{OD} = x\hat{i} + y\hat{j} + z\hat{k}$
 $\overrightarrow{ED} = \overrightarrow{OD} - \overrightarrow{OE} = (x\hat{i} + y\hat{j} + z\hat{k}) - \frac{1}{2}(\hat{i} + 9\hat{j} + 4\hat{k})$ A1
 $\overrightarrow{ED} = \left(x - \frac{1}{2}\right)\hat{i} + \left(y - \frac{9}{2}\right)\hat{j} + (z - 2)\hat{k}$
 Since $\overrightarrow{ED}$ is perpendicular to $\overrightarrow{OE}$, $\overrightarrow{ED} \cdot \overrightarrow{OE} = 0$ M1
 $\frac{1}{2}\left(x - \frac{1}{2}\right) + \frac{9}{2}\left(y - \frac{9}{2}\right) + 2(z - 2) = 0$
 $\left(x - \frac{1}{2}\right) + 9\left(y - \frac{9}{2}\right) + 4(z - 2) = 0$ A1
 $x + 9y + 4z = \frac{1}{2} + \frac{81}{2} + 8$
 (1) $x + 9y + 4z = 49$

d. $|\overrightarrow{OD}| = \sqrt{x^2 + y^2 + z^2} = \frac{7}{6}\sqrt{274}$
 (2) $x^2 + y^2 + z^2 = \frac{6713}{18}$ A1
 $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (x + 2)\hat{i} + (y - 3)\hat{j} + (z - 6)\hat{k}$
 $|\overrightarrow{AD}| = \sqrt{(x + 2)^2 + (y - 3)^2 + (z - 6)^2} = \frac{7}{6}\sqrt{274}$ squaring and expanding M1
 $x^2 + 4x + 4 + y^2 - 6y + 9 + z^2 - 12z + 36 = \frac{6713}{18}$ using (2)
 (3) $4x - 6y - 12z = -49$

$$\text{similarly } \overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = (x-1)\underline{i} + (y-9)\underline{j} + (z-4)\underline{k}$$

$$|\overrightarrow{BD}| = \sqrt{(x-1)^2 + (y-9)^2 + (z-4)^2} = \frac{7}{6}\sqrt{274} \text{ squaring and expanding} \quad \text{M1}$$

$$x^2 - 2x + 1 + y^2 - 18y + 81 + z^2 - 8z + 16 = \frac{6713}{18} \quad \text{using (2)}$$

$$(4) \quad x + 9y + 4z = 49 \quad \text{same as (1)}$$

$$\text{similarly } \overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = (x-3)\underline{i} + (y-6)\underline{j} + (z+2)\underline{k}$$

$$|\overrightarrow{CD}| = \sqrt{(x-3)^2 + (y-6)^2 + (z+2)^2} = \frac{7}{6}\sqrt{274}$$

$$x^2 - 6x + 9 + y^2 - 12y + 36 + z^2 + 4z + 4 = \frac{6713}{18} \quad \text{using (2)}$$

$$(5) \quad 6x + 12y - 4z = 49 \quad \text{A1}$$

e. Solving (1) (2) (3) and (5) using CAS

$$\text{gives } x = -\frac{31}{2} \text{ and } y = \frac{59}{6} \text{ and } z = -6 \text{ or } x = \frac{33}{2} \text{ and } y = -\frac{5}{6} \text{ and } z = 10$$

$$\text{both sets of answers are acceptable, and both give } |\overrightarrow{ED}| = \frac{56}{3} \quad \text{A1}$$

eq1:=x+9·y+4·z=49	x+9·y+4·z=49
eq2:=x ² +y ² +z ² = $\frac{6713}{18}$	x ² +y ² +z ² = $\frac{6713}{18}$
eq3:=4·x-6·y-12·z=-49	4·x-6·y-12·z=-49
eq5:=6·x+12·y-4·z=49	6·x+12·y-4·z=49
solve($\begin{pmatrix} \text{eq1} \\ \text{eq2} \\ \text{eq3} \\ \text{eq5} \end{pmatrix}, \{x,y,z\}$)	x=- $\frac{31}{2}$ and y= $\frac{59}{6}$ and z=-6 or x= $\frac{33}{2}$ and y=- $\frac{5}{6}$ and z=10
$\begin{bmatrix} \frac{1}{2} & \frac{9}{2} & 2 \end{bmatrix} \rightarrow e$	$\begin{bmatrix} \frac{1}{2} & \frac{9}{2} & 2 \end{bmatrix}$
$\begin{bmatrix} -\frac{31}{2} & \frac{59}{6} & -6 \end{bmatrix} \rightarrow d$	$\begin{bmatrix} -\frac{31}{2} & \frac{59}{6} & -6 \end{bmatrix}$
norm(d-e)	$\frac{56}{3}$
$\begin{bmatrix} \frac{33}{2} & -\frac{5}{6} & 10 \end{bmatrix} \rightarrow d$	$\begin{bmatrix} \frac{33}{2} & -\frac{5}{6} & 10 \end{bmatrix}$
norm(d-e)	$\frac{56}{3}$

Question 2

- a. Given that $\sin\left(\frac{\pi}{8}\right) = \frac{1}{2}\sqrt{2-\sqrt{2}}$ using $\cos^2(A) = 1 - \sin^2(A)$

$$\cos^2\left(\frac{\pi}{8}\right) = 1 - \sin^2\left(\frac{\pi}{8}\right) = 1 - \left(\frac{1}{2}\sqrt{2-\sqrt{2}}\right)^2 \quad \text{M1}$$

$$\cos^2\left(\frac{\pi}{8}\right) = 1 - \frac{1}{4}(2 - \sqrt{2}) = \frac{1}{4}(4 - (2 - \sqrt{2}))$$

$$\cos^2\left(\frac{\pi}{8}\right) = \frac{1}{4}(2 + \sqrt{2}) \quad \text{A1}$$

$$\cos\left(\frac{\pi}{8}\right) = \pm \frac{1}{2}\sqrt{2 + \sqrt{2}} \quad \text{but } \cos\left(\frac{\pi}{8}\right) > 0$$

$$\cos\left(\frac{\pi}{8}\right) = \frac{1}{2}\sqrt{2 + \sqrt{2}}$$

- b. Using $\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$ with $A = \frac{\pi}{8}$

$$\tan\left(\frac{\pi}{4}\right) = 1 = \frac{2\tan\left(\frac{\pi}{8}\right)}{1 - \tan^2\left(\frac{\pi}{8}\right)} \quad \text{let } \tan\left(\frac{\pi}{8}\right) = t \quad \text{M1}$$

$$\frac{2t}{1 - t^2} = 1 \Rightarrow 2t = 1 - t^2$$

$$t^2 + 2t = 1 \quad \text{completing the square}$$

$$t^2 + 2t + 1 = 2$$

$$(t + 1)^2 = 2$$

$$t + 1 = \pm\sqrt{2}$$

$$t = \tan\left(\frac{\pi}{8}\right) = -1 \pm \sqrt{2} \quad \text{since } \tan\left(\frac{\pi}{8}\right) > 0 \text{ take the positive} \quad \text{A1}$$

$$\tan\left(\frac{\pi}{8}\right) = \sqrt{2} - 1$$

- c. let $w = 1 + (\sqrt{2} - 1)i$ since $\tan^{-1}(\sqrt{2} - 1) = \frac{\pi}{8}$ $\text{Arg}(w) = \frac{\pi}{8}$

$$\text{Consider } iw = i + (\sqrt{2} - 1)i^2 = 1 - \sqrt{2} + i$$

$$iw \text{ is a rotation of } 90^\circ \text{ anticlockwise from } w, \quad \text{A1}$$

$$\text{so that } \text{Arg}(iw) = \frac{\pi}{8} + \frac{\pi}{2} = \frac{5\pi}{8}, \quad k = \frac{5}{8} \quad \text{A1}$$

d. $u = \sqrt{2+\sqrt{2}} + i\sqrt{2-\sqrt{2}}$
 $u = 2\cos\left(\frac{\pi}{8}\right) + 2i\sin\left(\frac{\pi}{8}\right)$
 $u = 2\text{cis}\left(\frac{\pi}{8}\right)$

A1

e. $u^{10} = 2^{10}\text{cis}\left(10 \times \frac{\pi}{8}\right)$
 $u^{10} = 2^{10}\text{cis}\left(\frac{5\pi}{4}\right) = 2^{10}\text{cis}\left(\frac{5\pi}{4} - 2\pi\right)$
 $u^{10} = 1024\text{cis}\left(-\frac{3\pi}{4}\right)$

A1

$$u^{10} = 1024\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) = 1024\left(-\frac{\sqrt{2}}{2} + i \times -\frac{\sqrt{2}}{2}\right)$$

$$u^{10} = -512\sqrt{2} - 512\sqrt{2}i = -512\sqrt{2}(1+i)$$

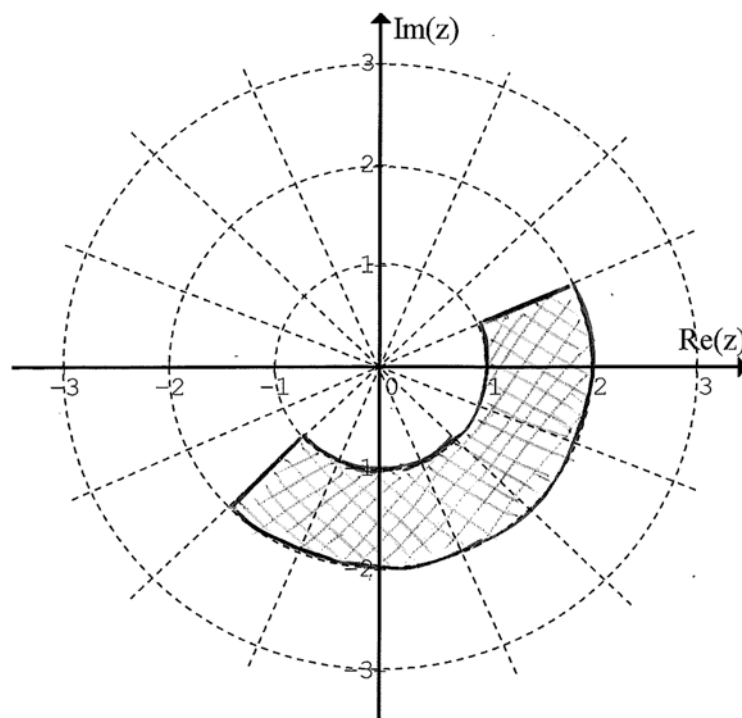
A1

f.i. $\text{Arg}(u^{10}) = -\frac{3\pi}{4}$

Inside the circles, of radius 1 and 2, with the wedge between the angles of

$\frac{\pi}{8} = 22.5^\circ$ and $-\frac{3\pi}{4} = -135^\circ$, all boundaries included.

G2



ii. $\text{Area} = \frac{1}{2}(r_2^2 - r_1^2)(\theta_2 - \theta_1)$

$$\text{Area} = \frac{1}{2}(2^2 - 1^2)\left(\frac{\pi}{8} + \frac{3\pi}{4}\right)$$

$$\text{Area} = \frac{21\pi}{16}$$

A1

g. $u = 2\text{cis}\left(\frac{\pi}{8}\right)$

$$u^n = 2^n \text{cis}\left(\frac{n\pi}{8}\right) = 2^n \left(\cos\left(\frac{n\pi}{8}\right) + i \sin\left(\frac{n\pi}{8}\right) \right)$$

$$\text{Im}(u^n) = 0 \Rightarrow \sin\left(\frac{n\pi}{8}\right) = 0$$

$$\frac{n\pi}{8} = k\pi$$

$$n = 8k \text{ where } k \in \mathbb{Z}$$

A1

$\text{angle}(1 - \sqrt{2} + i)$	$\frac{5 \cdot \pi}{8}$
$u := \sqrt{2 + \sqrt{2}} + \sqrt{2 - \sqrt{2}} \cdot i$	$\sqrt{\sqrt{2} + 2} + \sqrt{2 - \sqrt{2}} \cdot i$
$\text{angle}(u)$	$\tan^{-1}\left(\frac{\sqrt{8} - 2}{2}\right)$
u^{10}	$-512 \cdot \sqrt{2} - 512 \cdot \sqrt{2} \cdot i$
$\text{angle}(u^{10})$	$-\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) - \frac{\pi}{2}$

Question 3

a. $\int_{50}^{120} (bt + c) dt = 875$ A1

$$= \left[\frac{bt^2}{2} + ct \right]_{50}^{120}$$

$$= \frac{b}{2}(120^2 - 50^2) + c(120 - 50) = 875$$

$$5950b + 70c = 875$$
A1

b. since the function is continuous at $t = 50$

$$v(50) = a \sin\left(\frac{\pi}{2}\right) = a = 50b + c$$
M1

since the function is continuous at $t = 120$

$$v(120) = 120b + c = \frac{32}{\pi} \cos^{-1}(0) = \frac{32}{\pi} \times \frac{\pi}{2} = 16$$

$$(1) \quad 5950b + 70c = 875$$

$$(2) \quad a = 50b + c$$
A1

$$(3) \quad 120b + c = 16$$

$$70 \times (3) \quad 8400b + 70c = 1120 \quad \text{subtract from (1)}$$

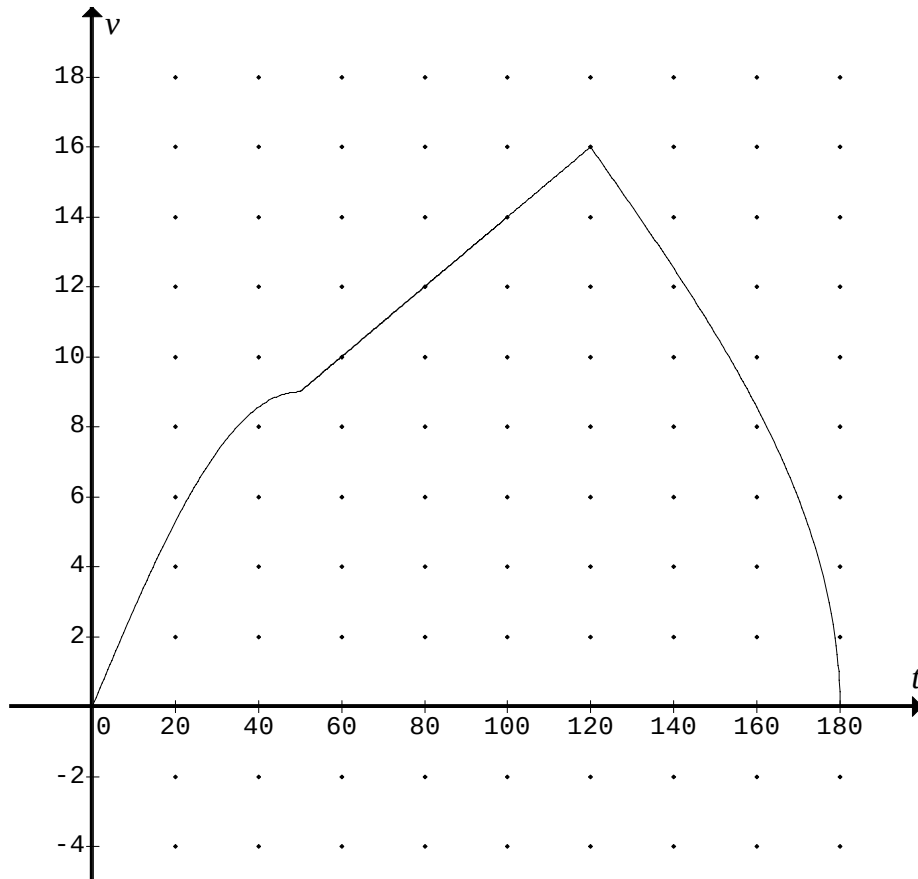
$$2450b = 245 \Rightarrow b = \frac{1}{10} \quad \text{substitute into (3) gives } c = 4$$

$$\text{substitute } b = \frac{1}{10} \text{ and } c = 4 \text{ into (2) gives } a = 9$$

$$\text{so that } a = 9, b = \frac{1}{10} \text{ and } c = 4$$

$\int_{50}^{120} (b \cdot t + c) dt$	$5950 \cdot b + 70 \cdot c$
$eq1 := 5950 \cdot b + 70 \cdot c = 875$	$5950 \cdot b + 70 \cdot c = 875$
$eq2 := a = 50 \cdot b + c$	$a = 50 \cdot b + c$
$eq3 := 120 \cdot b + c = 16$	$120 \cdot b + c = 16$
$\text{linSolve}\left(\begin{Bmatrix} eq1 \\ eq2, \{a, b, c\} \\ eq3 \end{Bmatrix}\right)$	$\left\{9, \frac{1}{10}, 4\right\}$

- c. continuous graph correct shape, over $[0, 180]$
passing through $(0, 0)$, $(50, 9)$, $(120, 16)$, $(180, 0)$, zero elsewhere. G2



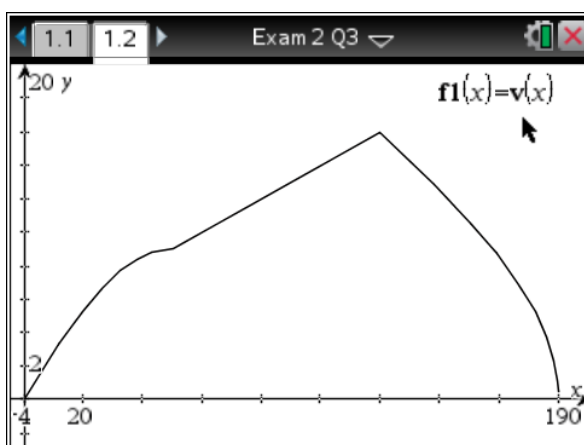
d.i. $D = \int_0^{50} 9 \sin\left(\frac{\pi t}{100}\right) dt + 875 + \int_{120}^{180} \frac{32}{\pi} \cos^{-1}\left(\frac{t-120}{60}\right) dt$ A1

ii. $D = \frac{900}{\pi} + 875 + 611.15$
 $D = 1773 \text{ m.}$ A1

e. $a = \frac{d}{dt} \left(\frac{32}{\pi} \cos^{-1}\left(\frac{t-120}{60}\right) \right) \Big|_{t=150} = -\frac{16\sqrt{3}}{45\pi}$
 $a(150) = -0.196 \text{ m/s}^2$ A1

- f. $30 \text{ km/hr} = \frac{30 \times 1000}{60 \times 60} = \frac{25}{3} = 8\frac{1}{3} \text{ m/s}$
- solving using CAS $9 \sin\left(\frac{\pi t}{100}\right) = 8\frac{1}{3}$ over $0 < t < 50 \Rightarrow t = 37.6713$
- solving $\frac{32}{\pi} \cos^{-1}\left(\frac{t-120}{60}\right) = 8\frac{1}{3}$ over $120 < t < 180 \Rightarrow t = 161.0155$ M1
- percentage of time, the speed is under 30 km/hr is
- $$\frac{37.6713 + (180 - 161.0155)}{180} \times 100$$
- $$= 31.5\%$$
- A1

$\int_0^{50} \left(9 \cdot \sin\left(\frac{\pi \cdot t}{100}\right) \right) dt$	$\frac{900}{\pi}$
$\int_{120}^{180} \left(\frac{32}{\pi} \cdot \cos^{-1}\left(\frac{t-120}{60}\right) \right) dt$	611.1550
$\frac{d}{dt} \left(\frac{32}{\pi} \cdot \cos^{-1}\left(\frac{t-120}{60}\right) \right) \Big _{t=150}$	$\frac{-16 \cdot \sqrt{3}}{45 \cdot \pi}$
$\frac{d}{dt} \left(\frac{32}{\pi} \cdot \cos^{-1}\left(\frac{t-120}{60}\right) \right) \Big _{t=150}$	-0.1960
$\text{solve}\left(9 \cdot \sin\left(\frac{\pi \cdot t}{100}\right) = \frac{25}{3}, t \mid 0 < t < 50\right)$	$t = 37.6713$
$\text{solve}\left(\frac{32}{\pi} \cdot \cos^{-1}\left(\frac{t-120}{60}\right) = \frac{25}{3}, t \mid 120 < t < 180\right)$	$t = 161.0155$
$\frac{37.671329685402 + 180 - 161.01553812137}{180} \cdot 100$	31.4754
Define $v(t) = \begin{cases} 9 \cdot \sin\left(\frac{\pi \cdot t}{100}\right), & 0 \leq t \leq 50 \\ \frac{t}{10} + 4, & 50 \leq t \leq 120 \end{cases}$	Done



Question 4

a. $x = \cos(t)(4\cos(t) - \sec(t)) = \cos(t)\left(4\cos(t) - \frac{1}{\cos(t)}\right) = 4\cos^2(t) - 1$

$$x + 1 = 4\cos^2(t)$$

$$x^2 = \cos^2(t)(4\cos(t) - \sec(t))^2$$

$$y^2 = \sin^2(t)(4\cos(t) - \sec(t))^2 \quad \text{M1}$$

$$x^2 + y^2 = (\cos^2(t) + \sin^2(t))(4\cos(t) - \sec(t))^2 = (4\cos(t) - \sec(t))^2$$

Consider the *LHS* substituting

$$4x^2 - (x+1)(x^2 + y^2) \quad \text{M1}$$

$$= 4\cos^2(t)(4\cos(t) - \sec(t))^2 - 4\cos^2(t)(4\cos(t) - \sec(t))^2$$

$$= 0 = \text{RHS}$$

b.i. $4x^2 - (x+1)(x^2 + y^2) = 0$

$$4x^2 - x^3 - xy^2 - x^2 - y^2 = 0$$

$$3x^2 - x^3 - xy^2 - y^2 = 0$$

Using implicit differentiation, product rule on second last term

$$\frac{d}{dx}(3x^2) - \frac{d}{dx}(x^3) - \frac{d}{dx}(xy^2) - \frac{d}{dx}(y^2) = 0 \quad \text{M1}$$

$$6x - 3x^2 - y^2 - 2xy\frac{dy}{dx} - 2y\frac{dy}{dx} = 0 \quad \text{A1}$$

$$6x - 3x^2 - y^2 = \frac{dy}{dx}(2xy + 2y)$$

$$\frac{dy}{dx} = \frac{6x - 3x^2 - y^2}{2y(x+1)}$$

ii. $x = x(t) = \cos(t)(4\cos(t) - \sec(t)) \quad \frac{dx}{dt} = -8\sin(t)\cos(t)$

$$y = y(t) = \sin(t)(4\cos(t) - \sec(t)) \quad \frac{dy}{dt} = \frac{-4(2\sin^2(t) - 1)\cos^2(t) - 1}{\cos^2(t)}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{4(2\sin^2(t) - 1)\cos^2(t) + 1}{8\sin(t)\cos^3(t)} \quad \text{by CAS} \quad \text{A1}$$

iii. for turning points, $\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dt} = 0$ solving $\frac{dy}{dt} = 0$ using CAS

$$4(2\sin^2(t)-1)\cos^2(t)+1=0 \text{ over}$$

$$t \in [0, \pi] \text{ gives } t = 0.598, 2.544$$

A1

$$x(0.598) = 1.732 \quad y(0.598) = 1.180$$

$$x(2.544) = 1.732 \quad y(2.544) = -1.180$$

$(1.732, 1.180)$ is a local maximum

and $(1.732, -1.180)$ is a local minimum

A1

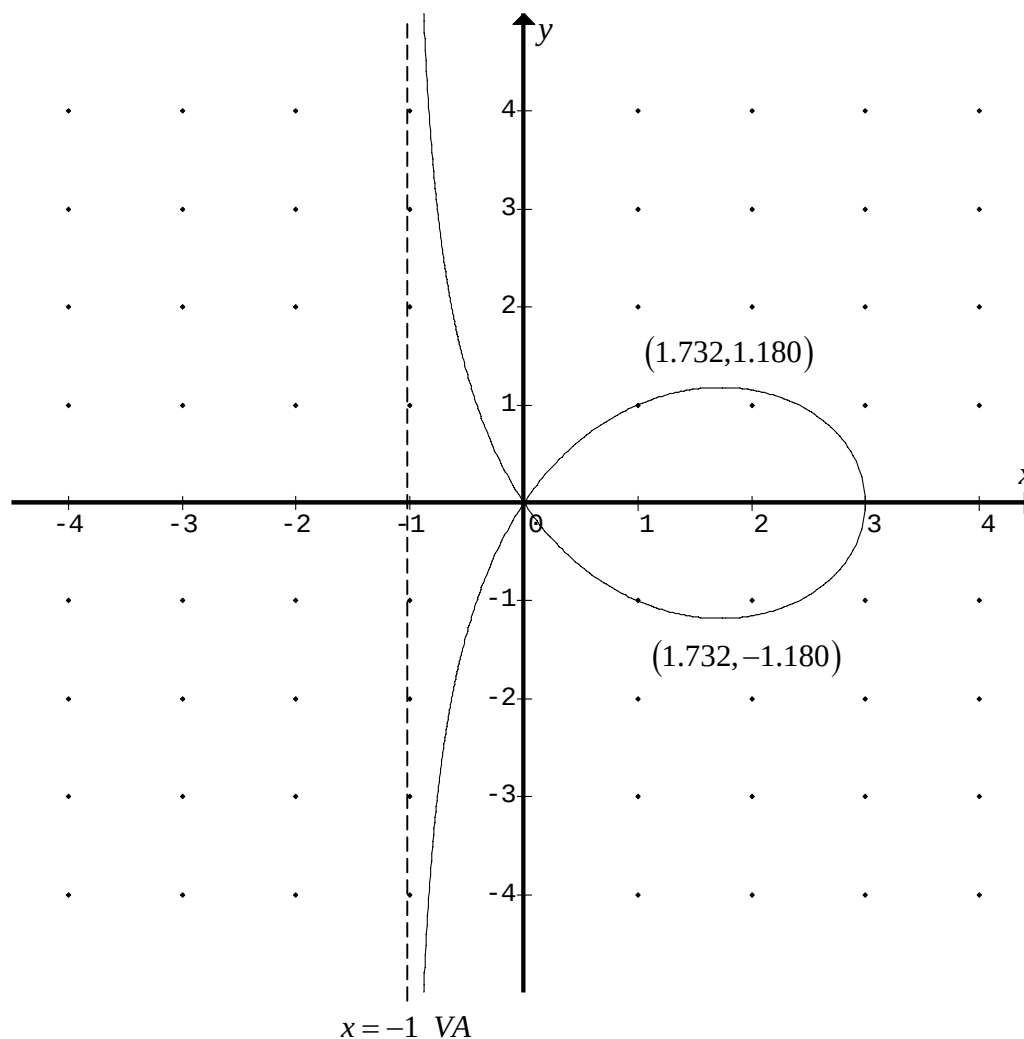
c. from d.i. $y = \pm x \sqrt{\frac{3-x}{x+1}}$ the graph has a vertical asymptote at $x = -1$

A1

Since $x = 4\cos^2(t) - 1$ the domain is $(-1, 3]$, passes through $(0, 0)$, $(3, 0)$

correct graph, shape, maximum and minimum

A1



d.i. $V = \pi \int_a^b y^2 dx$

$$4x^2 - (x+1)(x^2 + y^2) = 0$$

$$4x^2 = (x+1)(x^2 + y^2)$$

$$x^2 + y^2 = \frac{4x^2}{x+1}$$

M1

$$y^2 = \frac{4x^2}{x+1} - x^2 = \frac{4x^2 - x^2(x+1)}{x+1} = \frac{4x^2 - x^3 - x^2}{x+1}$$

$$y^2 = \frac{3x^2 - x^3}{x+1} = \frac{x^2(3-x)}{x+1}$$

$$V = \pi \int_0^3 \frac{x^2(3-x)}{x+1} dx$$

ii. let $u = x+1$ $\frac{du}{dx} = 1$

terminals when $x=3$ $u=4$ and when $x=0$ $u=1$

$$x = u-1 \quad \text{and} \quad x^2 = u^2 - 2u + 1$$

M1

$$3-x = 4-u$$

$$V = \pi \int_1^4 \frac{(u^2 - 2u + 1)(4-u)}{u} du$$

$$V = \pi \int_1^4 \left(u - 2 + \frac{1}{u}\right)(4-u) du$$

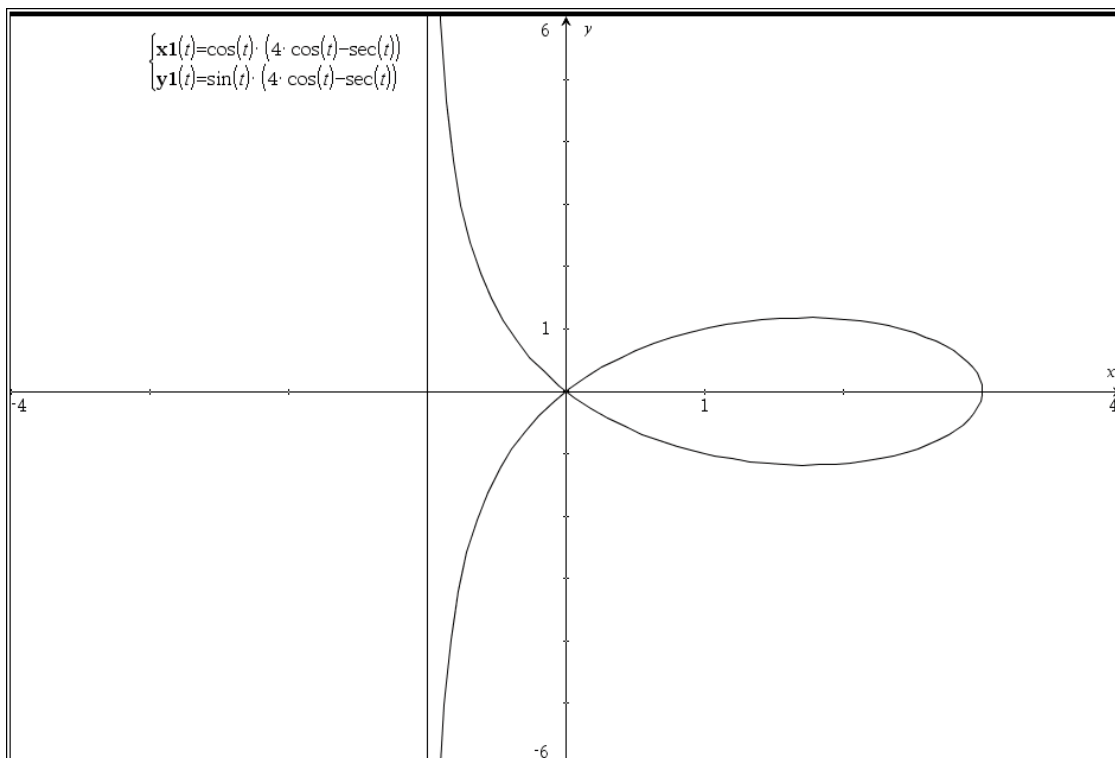
$$V = \pi \int_1^4 4\left(u - 2 + \frac{1}{u}\right) - u\left(u - 2 + \frac{1}{u}\right) du$$

$$V = \pi \int_1^4 \left(-u^2 + 6u - 9 + \frac{4}{u}\right) du$$

A1

$$a=1, b=4, p=-1, q=6, r=-9, s=4$$

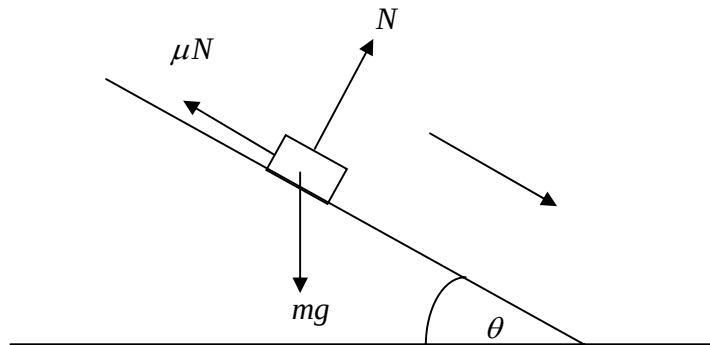
Define $x1(t)=\cos(t) \cdot (4 \cdot \cos(t)-\sec(t))$	Done
Define $y1(t)=\sin(t) \cdot (4 \cdot \cos(t)-\sec(t))$	Done
$\frac{d}{dt}(x1(t))$	$-8 \cdot \sin(t) \cdot \cos(t)$
$\frac{d}{dt}(y1(t))$	$-\frac{4 \cdot (2 \cdot (\sin(t))^2 - 1) \cdot (\cos(t))^2 + 1}{(\cos(t))^2}$
$\frac{d}{dt}(y1(t))$	$\frac{4 \cdot (2 \cdot (\sin(t))^2 - 1) \cdot (\cos(t))^2 + 1}{8 \cdot \sin(t) \cdot (\cos(t))^3}$
$\frac{d}{dt}(x1(t))$	
$4 \cdot x^2 - (x+1) \cdot (x^2 + y^2) = 0$	$-x^3 + 3 \cdot x^2 - x \cdot y^2 - y^2 = 0$
$\text{impDif}(-x^3 + 3 \cdot x^2 - x \cdot y^2 - y^2 = 0, x, y)$	$-\frac{(3 \cdot x^2 - 6 \cdot x + y^2)}{2 \cdot (x+1) \cdot y}$
$\text{solve}(4 \cdot (2 \cdot (\sin(t))^2 - 1) \cdot (\cos(t))^2 + 1 = 0, t) 0 < t < 3.14$	$t = 0.598031 \text{ or } t = 2.54356$
$x1(0.59803094704303)$	1.73205
$y1(0.59803094704303)$	1.17996
$x1(2.5435617065467)$	1.73205
$y1(2.5435617065467)$	-1.17996
	12/99



Question 5

- a.i.** N is the normal reaction, mg the weight force, and μN the frictional resistance force, all forces are in newtons
all forces correct.

A1



- ii.** resolving down and parallel to the plane.

$$(1) \quad ma = mg \sin(\theta) - \mu N$$

A1

resolving perpendicular to the plane.

$$(2) \quad N - mg \cos(\theta) = 0 \Rightarrow N = mg \cos(\theta) \text{ into (1)}$$

$$ma = mg \sin(\theta) - \mu mg \cos(\theta) = mg (\sin(\theta) - \mu \cos(\theta))$$

M1

$$a = g (\sin(\theta) - \mu \cos(\theta)) \quad \text{but } \theta = 60^\circ$$

$$a = g (\sin(60^\circ) - \mu \cos(60^\circ))$$

$$a = g \left(\frac{\sqrt{3}}{2} - \frac{\mu}{2} \right)$$

$$a = \frac{g}{2} (\sqrt{3} - \mu)$$

b.i. $\mu = \sqrt{3} - 1 \quad a = \frac{g}{2} (\sqrt{3} - (\sqrt{3} - 1)) = \frac{g}{2}$

Using constant acceleration formulae with

$$a = \frac{g}{2} \quad u = 0 \quad v = \frac{g}{4} \quad s = D = ? \quad t = T = ?$$

$$v = u + at$$

$$\frac{g}{4} = 0 + \frac{g}{2} T$$

$$T = \frac{1}{2}$$

A1

ii. $D = \left(\frac{u+v}{2} \right) T = \frac{1}{2} \left(0 + \frac{g}{4} \right) \frac{1}{2}$

$$D = \frac{g}{16} \text{ or } \frac{49}{80} \quad \text{A1}$$

c. $\mu = \sqrt{3} - 1 - \frac{v^2}{5g}$

$$a = \frac{g}{2} \left(\sqrt{3} - \left(\sqrt{3} - 1 - \frac{v^2}{5g} \right) \right) = \frac{g}{2} \left(1 + \frac{v^2}{5g} \right) = \frac{g}{2} \left(\frac{5g + v^2}{5g} \right)$$

$$a = \frac{49 + v^2}{10} \quad \text{A1}$$

i. to find the time, use $a = \frac{dv}{dt} = \frac{49 + v^2}{10}$

$$\frac{dt}{dv} = \frac{10}{49 + v^2}$$

$$T = \int_0^{\frac{g}{4}} \frac{10}{49 + v^2} dv \quad \text{A1}$$

$$T = \frac{10}{7} \left[\tan^{-1} \left(\frac{v}{7} \right) \right]_0^{\frac{g}{4}}$$

$$T = \frac{10}{7} \left[\tan^{-1} \left(\frac{g}{28} \right) - \tan^{-1}(0) \right]$$

$$T = \frac{10}{7} \tan^{-1} \left(\frac{g}{28} \right) \text{ or } \frac{10}{7} \tan^{-1} \left(\frac{7}{20} \right) \quad \text{A1}$$

ii. to find the distance travelled, use $a = v \frac{dv}{dx} = \frac{49 + v^2}{10}$ M1

$$D = \int_0^{\frac{g}{4}} \frac{10v}{49 + v^2} dv \quad \text{A1}$$

$$D = 5 \left[\log_e (49 + v^2) \right]_0^{\frac{g}{4}}$$

$$D = 5 \left[\log_e \left(49 + \frac{g^2}{16} \right) - \log_e (49) \right]$$

$$D = 5 \log_e \left(1 + \frac{g^2}{784} \right) \text{ or } 5 \log_e \left(\frac{449}{400} \right) \quad \text{A1}$$

END OF SECTION 2 SUGGESTED ANSWERS