

Year 2013

VCE

Specialist Mathematics

Trial Examination 1

Solutions



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- Teachers and students are reminded that for the purposes of school requirements and external assessments, students must submit work that is clearly their own.

Question 1

$$\int \frac{3x-5}{\sqrt{25-9x^2}} dx \quad \text{separate out into two integrals}$$

$$= 3 \int \frac{x}{\sqrt{25-9x^2}} dx - 5 \int \frac{1}{\sqrt{25-9x^2}} dx \quad \text{M1}$$

$$\text{let } u = 25-9x^2 \quad \text{let } v = 3x$$

$$\frac{du}{dx} = -18x \quad \frac{dv}{dx} = 3 \quad \text{M1}$$

$$= -\frac{3}{18} \int u^{-\frac{1}{2}} du - \frac{5}{3} \int \frac{1}{\sqrt{25-v^2}} dv$$

$$= -\frac{1}{6} \left(2u^{\frac{1}{2}} \right) + \frac{5}{3} \cos^{-1} \left(\frac{v}{5} \right) + c \quad \text{A1}$$

$$= -\frac{1}{3} \sqrt{25-9x^2} + \frac{5}{3} \cos^{-1} \left(\frac{3x}{5} \right) + c \quad \text{for } |x| < \frac{5}{3} \quad b = \frac{5}{3} \quad \text{A1}$$

$$\text{alternatively } = -\frac{1}{3} \sqrt{25-9x^2} - \frac{5}{3} \sin^{-1} \left(\frac{3x}{5} \right) + c \quad \text{for } |x| < \frac{5}{3} \quad b = \frac{5}{3}$$

Question 2

$$y = Ax^2 e^{-3x} \quad \text{differentiating using the product rule} \quad \text{M1}$$

$$\frac{dy}{dx} = A(2xe^{-3x} - 3x^2 e^{-3x}) = Ae^{-3x}(2x - 3x^2) \quad \text{differentiating using the product rule again}$$

$$\frac{d^2 y}{dx^2} = -3Ae^{-3x}(2x - 3x^2) + Ae^{-3x}(2 - 6x) = Ae^{-3x}[-3(2x - 3x^2) + (2 - 6x)]$$

$$\frac{d^2 y}{dx^2} = Ae^{-3x}(9x^2 - 12x + 2) \quad \text{A1}$$

$$\text{substituting into } \frac{d^2 y}{dx^2} + 6 \frac{dy}{dx} + 9y = 8e^{-3x} \quad \text{M1}$$

$$Ae^{-3x}[(9x^2 - 12x + 2) + 6(2x - 3x^2) + 9x^2] = 8e^{-3x}$$

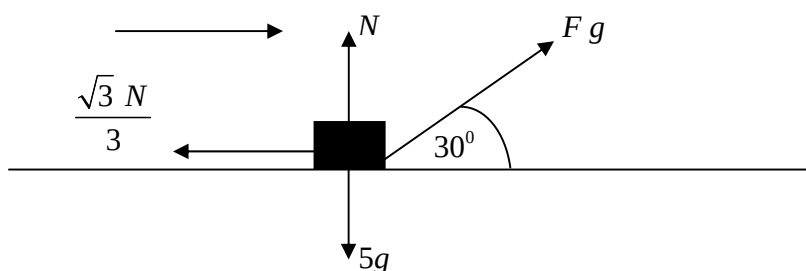
$$Ae^{-3x}[9x^2 - 12x + 2 + 12x - 18x^2 + 9x^2] = 8e^{-3x}$$

$$2Ae^{-3x} = 8e^{-3x}$$

$$2A = 8$$

$$A = 4 \quad \text{A1}$$

Question 3



all the forces are newtons

resolving parallel to the plane (1) $Fg \cos(30^\circ) - \frac{\sqrt{3}}{3} N = 5a$

resolving perpendicular to the plane (2) $Fg \sin(30^\circ) + N - 5g = 0$

(2) $\Rightarrow N = 5g - Fg \sin(30^\circ) = 5g - \frac{Fg}{2} = \frac{g}{2}(10 - F)$ A1

so if $F > 10 \Rightarrow N < 0$, so that the box leaves the table. A1

(1) $\Rightarrow Fg \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3} \left(\frac{g}{2}(10 - F) \right) = 5a$

$\frac{g\sqrt{3}}{3} \left(F - \frac{1}{2}(10 - F) \right) = 5a$ M1

$\frac{g\sqrt{3}}{6}(3F - 10) = 5a$

$\frac{10}{3} < F < 10 \Rightarrow a > 0$ the box moves with constant acceleration. A1

Question 4

$z = 2\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right) = 2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = -2 + 2i$ A1

The conjugate $\bar{z} = -2 - 2i$ is also a root, by the conjugate root theorem,

the sum of the roots $z + \bar{z} = -4$, the product of the roots $z\bar{z} = 4 - 4i^2 = 8$

since b is real $(z^2 + 4z + 8)$ is a factor M1

$P(z) = z^3 + z^2 + bz - 24 = (z^2 + 4z + 8)(z - 3)$

expanding $z: b = 8 - 12 = -4$

$b = -4$, and all the roots are $z = -2 \pm 2i$, $z = 3$ A1

Question 5

$$\cos(\theta) = \frac{2}{3} \text{ draw a triangle}$$

by Pythagoras

$$x = \sqrt{3^2 - 2^2} = \sqrt{5}$$

$$\sin(\theta) = \frac{\sqrt{5}}{3}$$

$$z = 3 \operatorname{cis}(\theta)$$

$$z^2 = 9 \operatorname{cis}(2\theta) = 9(\cos(2\theta) + i \sin(2\theta))$$

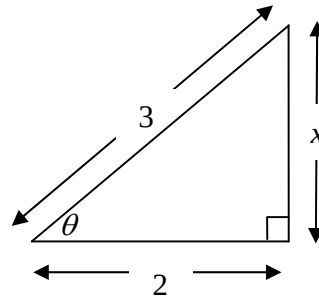
$$z^2 = 9(\cos^2(\theta) - \sin^2(\theta)) + 18 \sin(\theta) \cos(\theta) i$$

$$z^2 = 9\left(\frac{4}{9} - \frac{5}{9}\right) + 18 \times \frac{\sqrt{5}}{3} \times \frac{2}{3} i$$

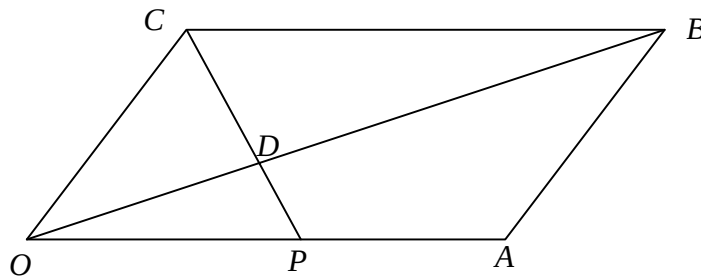
$$z^2 = -1 + 4\sqrt{5} i$$

M1

A1



Question 6



$OACB$ is a parallelogram, $\overrightarrow{OA} = \overrightarrow{CB}$ and $\overrightarrow{OC} = \overrightarrow{AB}$

since P is the midpoint of OA , $\overrightarrow{OP} = \overrightarrow{PA} = \frac{1}{2}\overrightarrow{OA}$, given that $\overrightarrow{PD} = \frac{1}{3}\overrightarrow{PC}$

A1

consider $\overrightarrow{OD} = \overrightarrow{OP} + \overrightarrow{PD}$

$$= \frac{1}{2}\overrightarrow{OA} + \frac{1}{3}\overrightarrow{PC} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{3}(\overrightarrow{PO} + \overrightarrow{OC})$$

M1

$$= \frac{1}{2}\overrightarrow{OA} + \frac{1}{3}\left(\overrightarrow{OC} - \frac{1}{2}\overrightarrow{OA}\right)$$

$$= \left(\frac{1}{2} - \frac{1}{6}\right)\overrightarrow{OA} + \frac{1}{3}\overrightarrow{OC} \quad \text{since } \overrightarrow{OC} = \overrightarrow{AB}$$

M1

$$= \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{AB})$$

So that $\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OB}$ this shows that O , D and B are collinear.

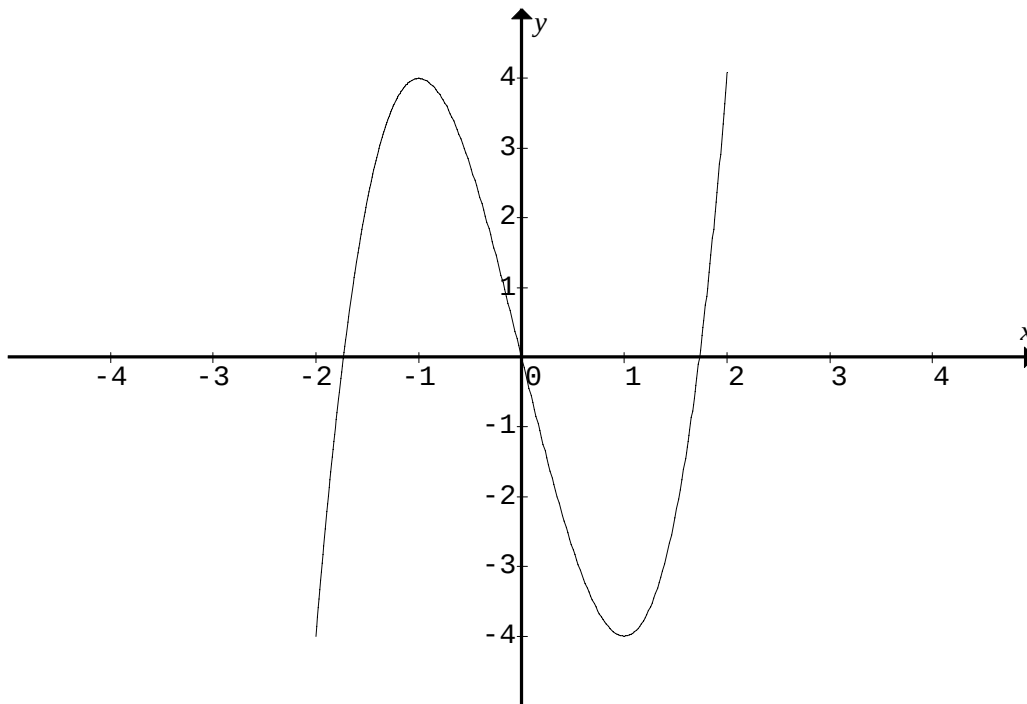
A1

Question 7

a. $\cos(3A) = \cos(2A + A)$
 $= \cos(2A)\cos(A) - \sin(2A)\sin(A)$ M1
 $= (2\cos^2(A) - 1)\cos(A) - 2\sin^2(A)\cos(A)$
 $= (2\cos^2(A) - 1)\cos(A) - 2(1 - \cos^2(A))\cos(A)$
 $= 2\cos^3(A) - \cos(A) - 2\cos(A) + 2\cos^3(A)$
 $\cos(3A) = 4\cos^3(A) - 3\cos(A)$

b.i. $\underline{r}(t) = 2\cos(t)\underline{i} + 4\cos(3t)\underline{j}$ for $t \geq 0$
 $x = 2\cos(t)$ $y = 4\cos(3t)$
 $\cos(t) = \frac{x}{2}$, $\cos(3t) = \frac{y}{4}$
 $\cos(3t) = 4\cos^3(t) - 3\cos(t)$
 $\frac{y}{4} = 4\left(\frac{x}{2}\right)^3 - 3\left(\frac{x}{2}\right)$
 $y = 2x^3 - 6x$ A1
 but since $t \geq 0$, $x = 2\cos(t) \Rightarrow x \in [-2, 2]$
 $y = 4\cos(3t) \Rightarrow y \in [-4, 4]$ A1

ii. the particle moves on part of a cubic,
 $y = 2x^3 - 6x = 2x(x^2 - 3) = 2x(x + \sqrt{3})(x - \sqrt{3})$
 crosses the x-axis when $y = 0$ at $(\sqrt{3}, 0)$, $(0, 0)$, $(-\sqrt{3}, 0)$
 $\frac{dy}{dx} = 6x^2 - 6 = 6(x^2 - 1) = 6(x + 1)(x - 1)$
 when $\frac{dy}{dx} = 0$ there are turning points at $(-1, 4)$ and $(1, -4)$ A1
 correct graph, on restricted domain, with key features. G1



Question 8

$$v = \frac{dx}{dt} = \frac{5}{100 - 9t^2}$$

distance travelled in two seconds $D = \int_0^2 \frac{5}{100 - 9t^2} dt$ A1

by partial fractions

$$\frac{5}{100 - 9t^2} = \frac{A}{10 - 3t} + \frac{B}{10 + 3t} = \frac{A(10 + 3t) + B(10 - 3t)}{(10 - 3t)(10 + 3t)} = \frac{10(A + B) + 3t(A - B)}{100 - 9t^2}$$
 M1

$$(1) A + B = \frac{1}{2} \quad (2) A - B = 0 \quad \text{solving } A = B = \frac{1}{4}$$

$$D = \frac{1}{4} \int_0^2 \left(\frac{1}{10 - 3t} + \frac{1}{10 + 3t} \right) dt$$

$$D = \frac{1}{4} \left[-\frac{1}{3} \log_e(10 - 3t) + \frac{1}{3} \log_e(10 + 3t) \right]_0^2$$
 M1

$$D = \frac{1}{12} \left[\log_e \left(\frac{10 + 3t}{10 - 3t} \right) \right]_0^2 = \frac{1}{12} \left(\log_e \left(\frac{16}{4} \right) - \log_e(1) \right)$$

$$D = \frac{1}{12} \log_e(4) \text{ metres}$$
 A1

Question 9

a. $x^2 + 4x - 4y^2 = 0$

$x^2 + 4x + 4 - 4y^2 = 4$ completing the square

$(x+2)^2 - 4y^2 = 4$

$\frac{(x+2)^2}{4} - y^2 = 1$, hyperbola centre $(-2, 0)$

A1

crosses the x-axis, when

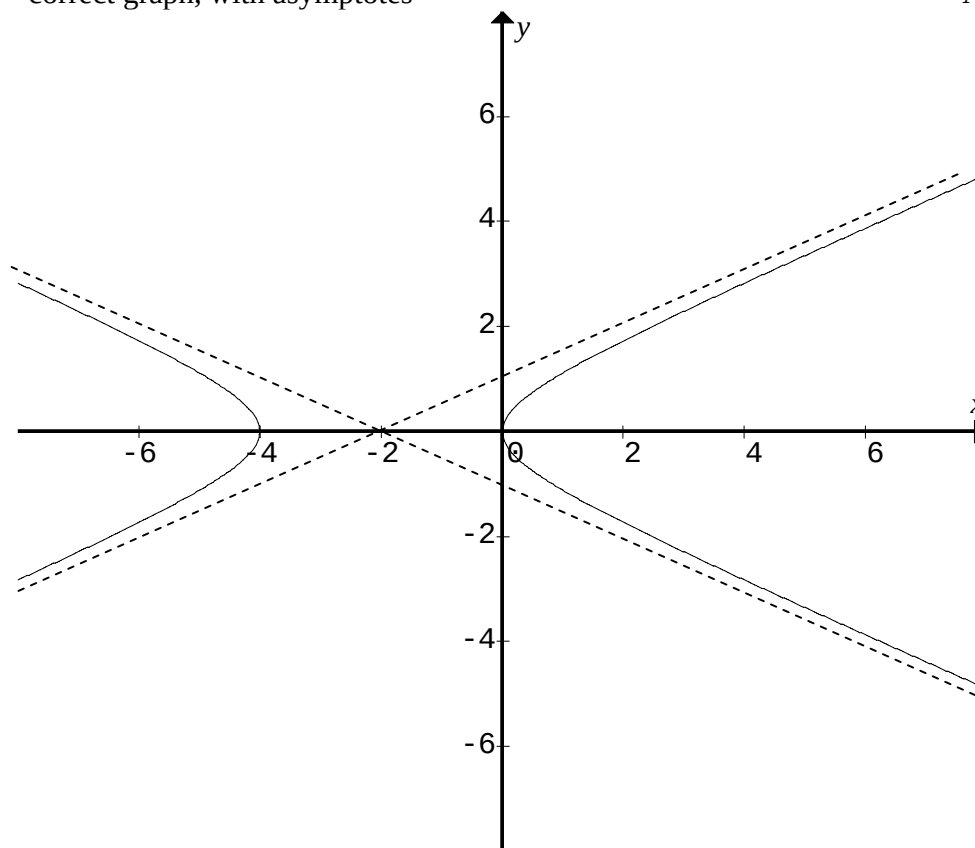
$y = 0$, $(x+2)^2 = 4$, $x+2 = \pm 2$, $x = 0, -4$ $(0, 0)$, $(-4, 0)$,

asymptotes $y = \pm \frac{1}{2}(x+2)$ $y = \frac{x}{2} + 1$ and $y = -\frac{1}{2}x - 1$

A1

correct graph, with asymptotes

A1



b. $x^2 + 4x - 4y^2 = 0$ using implicit differentiation

$2x + 4 - 8y \frac{dy}{dx} = 0$

$8y \frac{dy}{dx} = 2x + 4 = 2(x+2)$

$\frac{dy}{dx} = \frac{x+2}{4y}$

A1

Question 10

i. $y = \tan(2x)\sec(2x) = \frac{\sin(2x)}{\cos^2(2x)}$

asymptotes when denominator is zero, when $\cos(2x) = 0 \Rightarrow 2x = (2n+1)\frac{\pi}{2}$

$$x = (2n+1)\frac{\pi}{4} \text{ where } n \in \mathbb{Z} \quad \text{A1}$$

ii. area $A = \int_0^{\frac{\pi}{6}} \tan(2x)\sec(2x)dx = \int_0^{\frac{\pi}{6}} \frac{\sin(2x)}{\cos^2(2x)}dx$

let $u = \cos(2x) \quad \frac{du}{dx} = -2\sin(2x)$ M1

terminals when $x = \frac{\pi}{6} \quad u = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and when $x = 0 \quad u = \cos(0) = 1$

$$A = -\frac{1}{2} \int_1^{\frac{1}{2}} \frac{1}{u^2} du = -\frac{1}{2} \int_1^{\frac{1}{2}} u^{-2} du \quad \text{M1}$$

$$A = \frac{1}{2} \left[\frac{1}{u} \right]_1^{\frac{1}{2}} = \frac{1}{2} (2-1)$$

$$A = \frac{1}{2} \quad \text{A1}$$

iii. volume $V = \pi \int_a^b y^2 dx$

$$V = \pi \int_0^{\frac{\pi}{6}} \tan^2(2x)\sec^2(2x)dx$$

let $u = \tan(2x) \quad \frac{du}{dx} = 2\sec^2(2x)$ M1

terminals when $x = \frac{\pi}{6} \quad u = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$ and when $x = 0 \quad u = \tan(0) = 0$

$$V = \frac{\pi}{2} \int_0^{\sqrt{3}} u^2 du$$

$$= \frac{\pi}{2} \left[\frac{1}{3} u^3 \right]_0^{\sqrt{3}} = \frac{\pi}{6} \left((\sqrt{3})^3 - 0 \right)$$

$$V = \frac{\pi\sqrt{3}}{2} \quad \text{A1}$$

END OF SUGGESTED SOLUTIONS