

Year 2013

VCE Mathematical

Methods CAS

Trial Examination 2

Suggested Solutions



KILBAHA MULTIMEDIA PUBLISHING
PO BOX 2227
KEW VIC 3101
AUSTRALIA

TEL: (03) 9018 5376
FAX: (03) 9817 4334
kilbaha@gmail.com
<http://kilbaha.com.au>

IMPORTANT COPYRIGHT NOTICE

- This material is copyright. Subject to statutory exception and to the provisions of the relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Kilbaha Multimedia Publishing.
- The contents of this work are copyrighted. Unauthorised copying of any part of this work is illegal and detrimental to the interests of the author.
- For authorised copying within Australia please check that your institution has a licence from **Copyright Agency Limited**. This permits the copying of small parts of the material, in limited quantities, within the conditions set out in the licence.

Reproduction and communication for educational purposes The Australian Copyright Act 1968 (the Act) allows a maximum of one chapter or 10% of the pages of this work, to be reproduced and/or communicated by any educational institution for its educational purposes provided that educational institution (or the body that administers it) has given a remuneration notice to Copyright Agency Limited (CAL) under the Act.

For details of the CAL licence for educational institutions contact
CAL, Level 15, 233 Castlereagh Street, Sydney, NSW, 2000

Tel: (02) 9394 7600

Fax: (02) 9394 7601

Email: info@copyright.com.au

Web: <http://www.copyright.com.au>

- While every care has been taken, no guarantee is given that these answers are free from error. Please contact us if you believe you have found an error.

SECTION 1

ANSWERS

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E

SECTION 1

Question 1

Answer B

$$f(x) = \sqrt{b^2 - x^2}$$

$$f(0) = b \quad f\left(\frac{b}{2}\right) = \sqrt{b^2 - \frac{b^2}{4}} = \frac{\sqrt{3}b}{2}$$

$$\text{average rate of change } \bar{f} = \frac{f\left(\frac{b}{2}\right) - f(0)}{\frac{b}{2} - 0} = \frac{\frac{\sqrt{3}b}{2} - b}{\frac{b}{2}} = \frac{b\left(\frac{\sqrt{3}-2}{2}\right)}{\frac{b}{2}} = \sqrt{3} - 2$$

Question 2

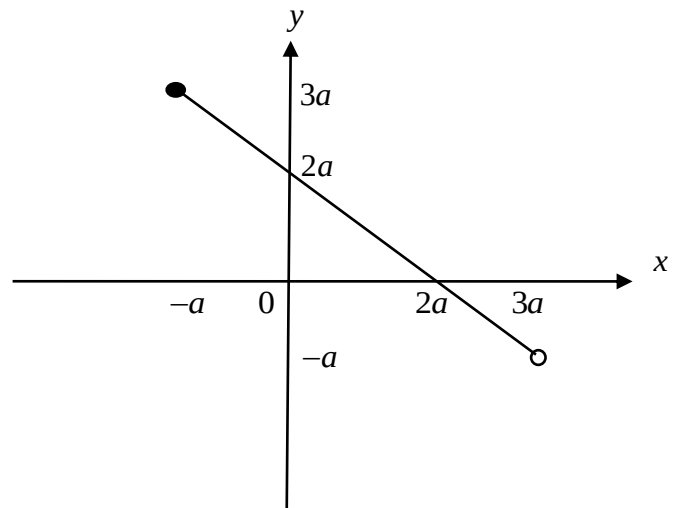
Answer B

$$f: [-a, 3a) \rightarrow \mathbb{R}, f(x) = 2a - x$$

$$f(3a) = -a$$

$$f(-a) = 3a$$

The range is $(-a, 3a]$



Question 3

Answer B

$$\text{The period } T = \frac{\pi}{n} = \frac{\pi}{\frac{b\pi}{3}} = \frac{3}{b}$$

Question 4

Answer A

$$f(x) = g(x) \log_e(h(x)) \text{ using the product rule}$$

$$f'(x) = g'(x) \log_e(h(x)) + \frac{g(x)h'(x)}{h(x)}$$

$$f'(2) = g'(2) \log_e(h(2)) + \frac{g(2)h'(2)}{h(2)}$$

$$\text{Now } g(2) = 3, g'(2) = 4, h(2) = e^2 \text{ and } h'(2) = 2$$

$$f'(2) = 4 \times \log_e(e^2) + \frac{3 \times 2}{e^2} = 8 + \frac{6}{e^2}$$

Question 5

Answer A

$$\Delta = \begin{vmatrix} 1 & -k & 1 \\ 2 & -1 & k \\ 1 & 1 & 1 \end{vmatrix} = -k^2 + k + 2 = -(k-2)(k+1)$$

When $\Delta = 0$ or $k = 2$, $k = -1$ there is no unique solution.

When $k = -1$ there is no solution.

When $k = 2$ there is infinitely many solutions.

$\det \begin{pmatrix} 1 & -k & 1 \\ 2 & -1 & k \\ 1 & 1 & 1 \end{pmatrix}$	$-(k-2) \cdot (k+1)$
$\text{solve} \left(\det \begin{pmatrix} 1 & -k & 1 \\ 2 & -1 & k \\ 1 & 1 & 1 \end{pmatrix} = 0, k \right)$	$k = -1 \text{ or } k = 2$
$\text{eq1} := x - k \cdot y + z = 14$	$x - k \cdot y + z = 14$
$\text{eq2} := 2 \cdot x - y + k \cdot z = 10$	$2 \cdot x - y + k \cdot z = 10$
$\text{eq3} := x + y + z = -2 \cdot k$	$x + y + z = -2 \cdot k$
$\text{linSolve} \left(\begin{pmatrix} \text{eq1} \\ \text{eq2} \\ \text{eq3} \end{pmatrix}, \{x, y, z\} \right)$	$\left\{ -2 \cdot (k+1), \frac{-2 \cdot (k+7)}{k+1}, \frac{4 \cdot (k+4)}{k+1} \right\}$
$\text{linSolve} \left(\begin{pmatrix} \text{eq1} \\ \text{eq2} \\ \text{eq3} \end{pmatrix}, \{x, y, z\} \right) k = -1$	"No solution found"
$\text{linSolve} \left(\begin{pmatrix} \text{eq1} \\ \text{eq2} \\ \text{eq3} \end{pmatrix}, \{x, y, z\} \right) k = 2$	$\{-(2-2), -6, 2\}$

Question 6

Answer C

$$f(x) = \tan(x) \quad f'(x) = \frac{1}{\cos^2(x)}$$

$$x = 60^\circ = \frac{\pi}{3} \quad h = 1^\circ = \frac{\pi}{180}$$

$$f\left(\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3} \quad f'\left(\frac{\pi}{3}\right) = \frac{1}{\cos^2\left(\frac{\pi}{3}\right)} = \frac{1}{\left(\frac{1}{2}\right)^2} = 4$$

$$f(x+h) \approx f(x) + hf'(x)$$

$$\tan(61^\circ) = \sqrt{3} + \frac{\pi}{180} \times 4 = 1.8019$$

Question 7

Answer A

$$f(x) = -x^3 + 4x^2 + 3x - 2$$

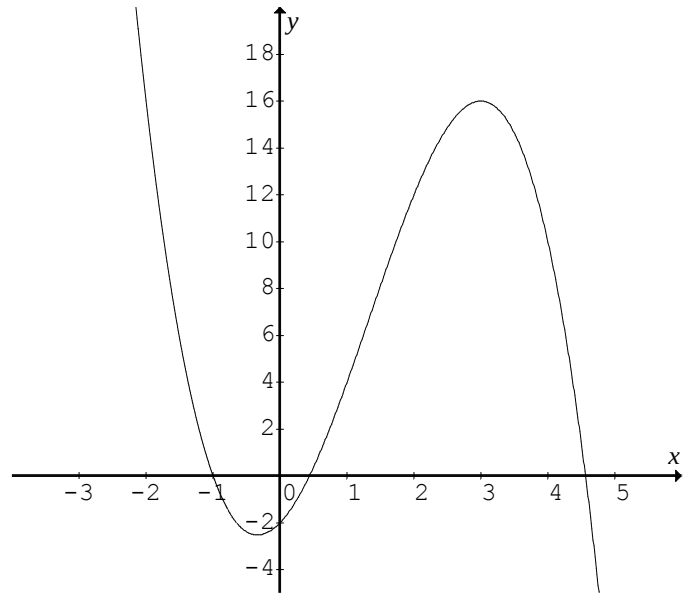
$$f'(x) = -3x^2 + 8x + 3$$

$$= -(3x^2 - 8x - 3)$$

$$= -(3x+1)(x-3)$$

Turning points at $x = -\frac{1}{3}$ and $x = 3$

To restrict the domain to make f a one-one function we require $a \leq -\frac{1}{3}$



Question 8

Answer E

$$y' = 2 - 2\log_e(2x' + 2)$$

$$\frac{y' - 2}{-2} = \log_e(2x' + 2) \quad y = \log_e(x)$$

$$\Rightarrow y = \frac{y' - 2}{-2} \text{ and } x = 2x' + 2 \Rightarrow y' = 2 - 2y \text{ and } x' = \frac{x}{2} - 1$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Question 9

Answer C

$$y = \cos(\pi bx) \quad \frac{dy}{dx} = -\pi b \sin(\pi bx) \text{ when } x = \frac{1}{6b} \quad m_T = -\pi b \sin\left(\frac{\pi}{6}\right) = -\frac{\pi b}{2}$$

$$\text{normal } m_N = \frac{2}{\pi b} = 4 \Rightarrow b = \frac{1}{2\pi}$$

Question 10

Answer E

$$f(x) = ax^2 - 2bx = x(ax - 2b), \text{ the graph}$$

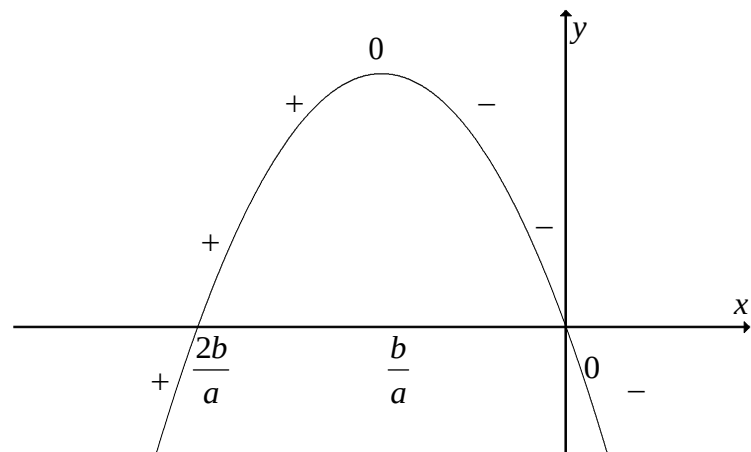
crosses the x -axis at $x = 0$ and $x = \frac{2b}{a}$.

$$f'(x) = 2ax - 2b.$$

A decreasing function has a negative gradient, so that $f'(x) < 0 \Rightarrow 2ax - 2b < 0$.

$$\Rightarrow ax < b \text{ if } a < 0 \text{ and } b > 0$$

$$\text{then } x > \frac{b}{a}.$$



Question 11

Answer C

T catches the train and D drives to work

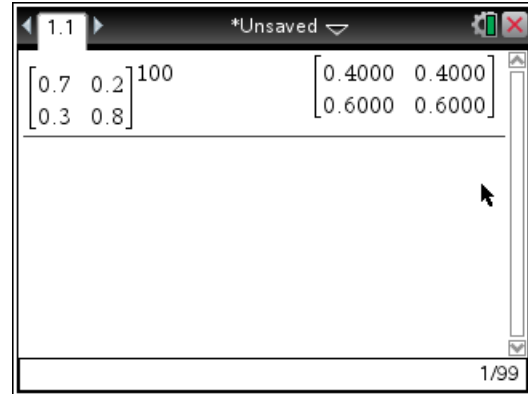
$$T \rightarrow T = 0.7 \Rightarrow T \rightarrow D = 0.3$$

$$D \rightarrow T = 0.2 \Rightarrow D \rightarrow D = 0.8$$

$T \quad D$

$$\begin{matrix} T \\ D \end{matrix} \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}^{100} = \begin{bmatrix} 0.4 & 0.4 \\ 0.6 & 0.6 \end{bmatrix}$$

long-term drives is $0.6 = \frac{3}{5}$



Question 12

Answer C

$$A = \frac{3\sqrt{3}}{2}x^2 \Rightarrow \frac{dA}{dx} = 3\sqrt{3}x \quad \text{given} \quad \frac{dA}{dt} = 36$$

$$\frac{dx}{dt} = \frac{dx}{dA} \frac{dA}{dt} = \frac{36}{3\sqrt{3}x} \quad \left. \frac{dx}{dt} \right|_{x=2} = \frac{36}{3\sqrt{3} \times 2} = \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = 2\sqrt{3} \text{ cm/min}$$

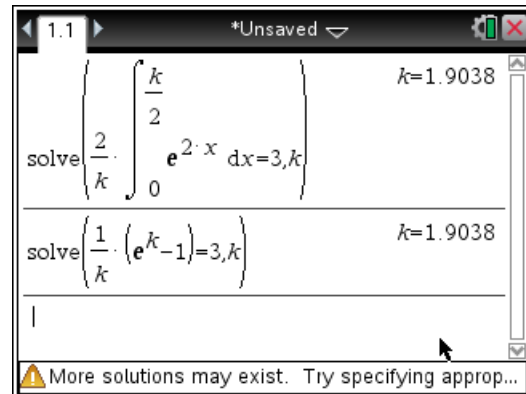
Question 13

Answer D

$$\frac{1}{\frac{k}{2} - 0} \int_0^{\frac{k}{2}} e^{2x} dx = 3$$

$$\frac{2}{k} \left[\frac{1}{2} e^{2x} \right]_0^{\frac{k}{2}} = \frac{1}{k} (e^k - 1) = 3$$

solving for k gives, $k = 1.904$



Question 14

Answer E

$$f(x) = |x| \text{ and } g(x) = x^2 \Rightarrow g^{-1}(x) = \sqrt{x}$$

$$g^{-1}(x)g(x) = \sqrt{x} \times x^2 = x^{\frac{5}{2}} \neq f(x) \quad \text{Peter is incorrect}$$

$$g^{-1}(g(x)) = g^{-1}(x^2) = \sqrt{x^2} = |x| = f(x) \quad \text{Quentin is correct}$$

$$g(g^{-1}(x)) = g(\sqrt{x}) = (\sqrt{x})^2 = x \neq f(x) \quad \text{Sara is incorrect}$$

$$f(g(x)) = f(x^2) = |x^2| = x^2 = g(x) \quad \text{Tanya is correct}$$

Question 15

Answer D

Total area under the curve is one $k \int_0^{\frac{\pi}{3}} \sin(3x) dx = 1$

$$k \left[-\frac{1}{3} \cos(3x) \right]_0^{\frac{\pi}{3}} = k \left[\left(-\frac{1}{3} \cos(\pi) + \frac{1}{3} \cos(0) \right) \right] = \frac{2k}{3} = 1 \Rightarrow k = \frac{3}{2}$$

The mean and median are both symmetrical at $x = \frac{\pi}{6}$

D. Is false $E(X^2) = \frac{\pi^2 - 4}{18}$

solve $k \int_0^{\frac{\pi}{3}} \sin(3 \cdot x) dx = 1, k$	$k = \frac{3}{2}$
$\frac{3}{2} \cdot \int_0^{\frac{\pi}{3}} (x \cdot \sin(3 \cdot x)) dx$	$\frac{\pi}{6}$
$\frac{3}{2} \cdot \int_0^{\frac{\pi}{3}} (x^2 \cdot \sin(3 \cdot x)) dx$	$\frac{\pi^2 - 4}{18}$
$\frac{3}{2} \cdot \int_0^{\frac{\pi}{4}} \sin(3 \cdot x) dx$	$\frac{\sqrt{2} + 2}{4}$

Question 16

Answer E

$$\Pr(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}$$

$$\Pr(X = 0) = e^{-\lambda}, \Pr(X = 1) = \lambda e^{-\lambda}, \Pr(X = 2) = \frac{\lambda^2 e^{-\lambda}}{2!}$$

$$\Pr(X > 2) = \Pr(X \geq 3)$$

$$\Pr(X \geq 3) = 1 - [\Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)]$$

$$\Pr(X \geq 3) = 1 - \left[e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2 e^{-\lambda}}{2!} \right]$$

$$\Pr(X \geq 3) = 1 - \frac{e^{-\lambda}}{2} (\lambda^2 + 2\lambda + 2)$$

Question 17

Answer D

$$v(t) = \frac{27}{(3t+4)^2} \text{ m/s initial velocity } v(0) = \frac{27}{16} \text{ m/s A. is true.}$$

$$\int_0^2 \frac{27}{(3t+4)^2} dt = \frac{27}{20} \text{ distance travelled in the first two seconds, B. is true.}$$

$$a = \frac{dv}{dt} = \frac{d}{dt} \left[\frac{27}{(3t+4)^2} \right] = \frac{-162}{(3t+4)^3} \text{ m/s}^2 \text{ acceleration C. is true.}$$

$$x(t) = \int \frac{27}{(3t+4)^2} dt = \frac{-27}{3(3t+4)} + c = \frac{-9}{3t+4} + c$$

$$x(0) = 0 \Rightarrow 0 = -\frac{9}{4} + c \quad c = \frac{9}{4}$$

$$x(t) = \frac{9}{4} - \frac{9}{3t+4} = \frac{9(3t+4) - 9 \times 4}{4(3t+4)} = \frac{27t}{4(3t+4)} \text{ position E. is true, D. is false}$$

Define $v(t) = \frac{27}{(3 \cdot t + 4)^2}$	Done
$v(0)$	$\frac{27}{16}$
$\int_0^2 v(t) dt$	$\frac{27}{20}$
$\frac{d}{dt}(v(t))$	$\frac{-162}{(3 \cdot t + 4)^3}$
$\int_0^t v(t) dt$	$\frac{9}{4} - \frac{9}{3 \cdot t + 4}$
comDenom $\left(\frac{9}{4} - \frac{9}{3 \cdot t + 4} \right)$	$\frac{27 \cdot t}{12 \cdot t + 16}$

Question 18

Answer B

A cubic with no turning points can be expressed in the form $f(x) = a(x+h)^3 + k$

$$f(x) = a(x+h)^3 + k = a(x^3 + 3x^2h + 3xh^2 + h^3) + k = ax^3 + 3ahx^2 + 3ah^2x + ah^3 + k$$

$$f(x) = ax^3 + bx^2 + cx + d \Rightarrow b = 3ah, \quad c = 3ah^2 \text{ and } d = ah^3 + k$$

The point of inflexion is at $(-h, k)$ and $f'(x) = 3ax^2 + 2bx + c = 3a(x+h)^2$

$f'(x) = 0$ when $x = -h$ and this equation has only one root, or for no turning points, no real factors, when $\Delta \leq 0$, thus

$$\Delta = (2b)^2 - 4 \times 3a \times c = 4b^2 - 12ac = 4(b^2 - 3ac) \leq 0 \Rightarrow b^2 \leq 3ac$$

Question 19

Answer A

The area bounded by the graph of $y = f(x)$ between the x -values of $x = a$ and $x = b$, is the same if the area is translated b units to the right, that is the area bounded by the graph of $y = f(x - b)$ between the x -values of $x = a + b$ and $x = 2b$. This is also the same area if the area is translated b units to the left, that is the area bounded by the graph of $y = f(x + b)$ between the x -values of $x = a - b$ and $x = 0$, since $f(x) > 0$ for $x \in [a - b, 2b]$ where $a, b \in \mathbb{R}$ and $b > a > 0$.

Question 20

Answer D

$$X \stackrel{d}{=} N(\mu = 15, \sigma^2 = 16) \quad \sigma = 4 \quad Z \stackrel{d}{=} N(\mu = 0, \sigma^2 = 1) \quad Z = \frac{x - \mu}{\sigma}$$

X	7	11	15	19	23
Z	-2	-1	0	1	2

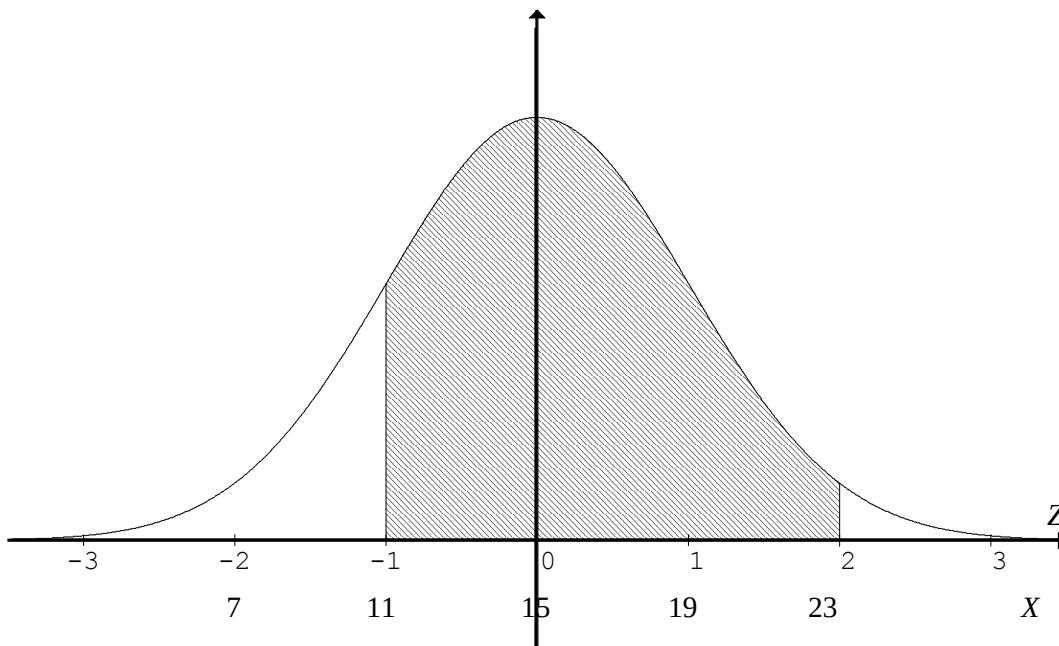
$$\Pr(-1 < Z < 2) = 1 - [\Pr(Z > 2) + \Pr(Z < -1)] \quad \text{A. is true}$$

$$= \Pr(-1 < Z < 0) + \Pr(0 < Z < 2) = \Pr(0 < Z < 1) + \Pr(0 < Z < 2) \quad \text{B. is true}$$

$$= \Pr(11 < X < 23) = 1 - [\Pr(X > 23) + \Pr(X < 11)] \quad \text{C. is true}$$

$$= 1 - [\Pr(X < 7) + \Pr(X > 19)] \quad \text{by symmetry, E. is true}$$

D. is false, all of A. B. C. E. are true



Question 21

Answer E

Since it is a discrete random variable, the probabilities add to one, so that

$$a + \frac{a}{2} + \frac{a}{3} = \frac{11a}{6} = 1 \Rightarrow a = \frac{6}{11}$$

$$E(X) = \sum x \Pr(X = x) = 1 \times a + 2 \times \frac{a}{2} + 3 \times \frac{a}{3} = 3a = \frac{18}{11}$$

$$E(X^2) = \sum x^2 \Pr(X = x) = 1 \times a + 4 \times \frac{a}{2} + 9 \times \frac{a}{3} = 6a = \frac{36}{11}$$

$$\text{var}(X) = E(X^2) - (E(X))^2 = \frac{36}{11} - \left(\frac{18}{11}\right)^2 = \frac{72}{121}$$

All of **A. B. C.** and **D.** are true, **E.** is false

$$E\left(\frac{1}{X}\right) = \sum \frac{1}{x} \Pr(X = x) = 1 \times a + \frac{1}{2} \times \frac{a}{2} + \frac{1}{3} \times \frac{a}{3} = \frac{49}{36}a = \frac{49}{36} \times \frac{6}{11} = \frac{49}{66}$$

Question 22

Answer B

$$f(x) = \log_e(3x + 4)$$

x	0	1	2	3	4
$f(x)$	$\log_e(4)$	$\log_e(7)$	$\log_e(10)$	$\log_e(13)$	$\log_e(16)$

consider four right rectangles, each of width one unit.

$$A_L = 1 \times \log_e(7) + 1 \times \log_e(10) + 1 \times \log_e(13) + 1 \times \log_e(16)$$

$$A_L = \log_e(7 \times 10 \times 13 \times 16) = \log_e(14560)$$

END OF SECTION 1 SUGGESTED ANSWERS

SECTION 2

Question 1

a.i. Let R pays by credit card, and C pays using cash
 $C \rightarrow C = 0.35 \Rightarrow C \rightarrow R = 0.65$ and $R \rightarrow R = 0.45 \Rightarrow R \rightarrow C = 0.55$
 $\Pr(\text{uses credit card three times and the first is credit card})$
 $= \Pr(RRRC) + \Pr(RRCR) + \Pr(RCRR)$ M1
 $= 0.45^2 \times 0.55 + 0.45 \times 0.55 \times 0.65 + 0.55 \times 0.65 \times 0.45$
 $= 0.433$ A1

ii. $\Pr(\text{first and fourth are credit cards})$
 $= \Pr(RCCR) + \Pr(RCRR) + \Pr(RRCR) + \Pr(RRRR)$
 $= 0.55 \times 0.35 \times 0.65 + 0.55 \times 0.65 \times 0.45 + 0.45 \times 0.55 \times 0.65 + 0.45^3$
 $= 0.538$ A1

$C \quad R$

alternatively $\begin{matrix} C \\ R \end{matrix} \begin{bmatrix} 0.35 & 0.55 \\ 0.65 & 0.45 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.462 \\ 0.538 \end{bmatrix}$ fourth is credit card 0.538

b.i. $X \stackrel{d}{=} Bi(n=16, p=?)$
 $\Pr(X=8) + \Pr(X=9) = 0.25$
 $\binom{16}{8} p^8 (1-p)^8 + \binom{16}{9} p^9 (1-p)^7 = 0.25$ by CAS M1
 $1430 p^8 (p-9)(p-1)^7 = 0.25$
solving with $0 < p < 1$ gives $p = 0.4132$ or 0.6473
but since $E(X) > 8 \Rightarrow p > 0.5$ so $p = 0.6473$ A1

ii. $X \stackrel{d}{=} Bi(n=16, p=0.65)$
 $\Pr(X > 8) = \Pr(X \geq 9) = 0.8406$ A1

$nCr(16,8) \cdot p^8 \cdot (1-p)^8 + nCr(16,9) \cdot p^9 \cdot (1-p)^7$	$1430 \cdot p^8 \cdot (p-9) \cdot (p-1)^7$
$\text{solve}(1430 \cdot p^8 \cdot (p-9) \cdot (p-1)^7 = 0.25, p) 0 < p < 1$	$p = 0.4132 \text{ or } p = 0.6473$
$\text{binomcdf}(16, 0.65, 9, 16)$	0.8406

c. X is the time in hours spent shopping, $X \stackrel{d}{=} N(\mu = ?, \sigma^2 = ?)$

$$(1) \quad \Pr(X > 3700) = 0.14$$

$$(2) \quad \Pr(X < 2990) = 0.26$$

$$(1) \Rightarrow \frac{3700 - \mu}{\sigma} = 1.0803 \quad \text{M1}$$

$$(2) \Rightarrow \frac{2990 - \mu}{\sigma} = -0.6433 \quad \text{A1}$$

$$(1) \quad 3700 - \mu = 1.0803\sigma$$

$$(2) \quad 2990 - \mu = -0.6433\sigma$$

now subtract equations (1) – (2) solving gives

$$\sigma = \$412 \text{ substituting gives } \mu = \$3,255 \quad \text{A1}$$

$eq1: \frac{3700-m}{s} = \text{invNorm}(0.86, 0, 1)$	$\frac{3700-m}{s} = 1.0803$
$eq2: \frac{2990-m}{s} = \text{invNorm}(0.26, 0, 1)$	$\frac{2990-m}{s} = -0.6433$
$\text{solve}\left(\begin{Bmatrix} eq1 \\ eq2 \end{Bmatrix}, \{m, s\}\right)$	$s=411.9130 \text{ and } m=3255.002$

d.i. Since the function is continuous at $t = 3$ $ae^1 = b \sin\left(\frac{3\pi}{6}\right) = b \Rightarrow b = ae \quad \text{A1}$

$$\int_0^3 ae^{\frac{t}{3}} dt + \int_3^6 b \sin\left(\frac{\pi t}{6}\right) dt = 1$$

Since the total area under the curve is equal to one, substituting $b = ae$

$$a \left[\int_0^3 e^{\frac{t}{3}} dt + \int_3^6 e \sin\left(\frac{\pi t}{6}\right) dt \right] = 1$$

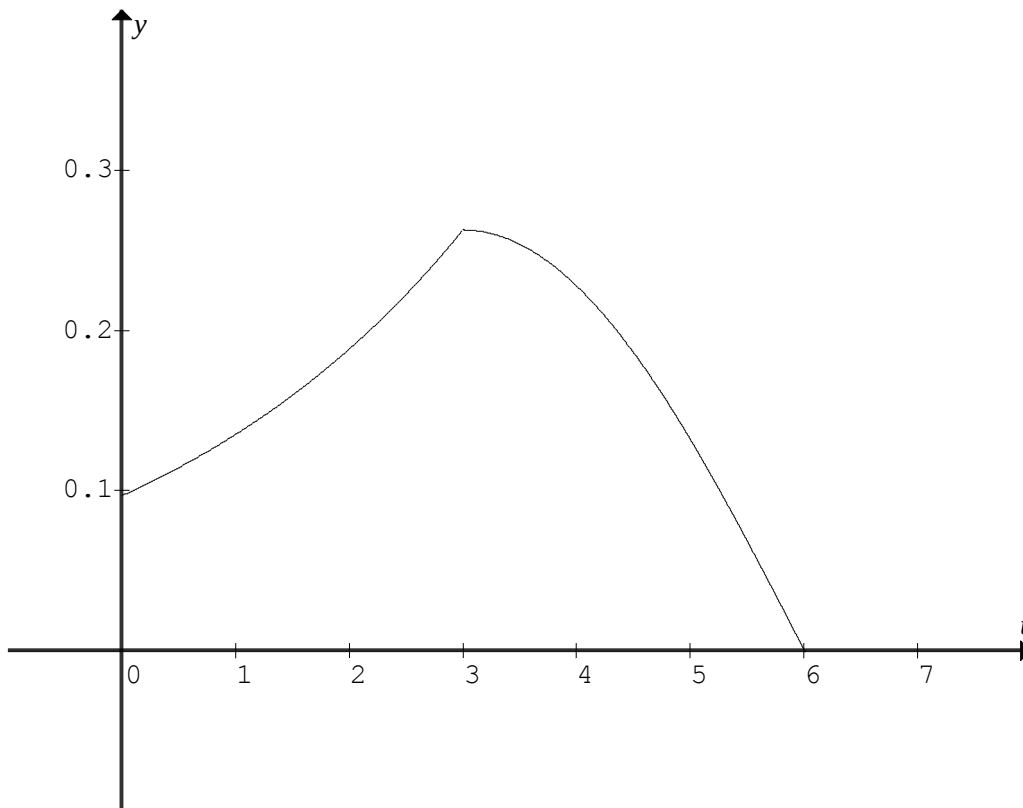
$$a \left(\left[3e^{\frac{t}{3}} \right]_0^3 - \left[\frac{6e}{\pi} \cos\left(\frac{\pi t}{6}\right) \right]_3^6 \right) = 1 \quad \text{A1}$$

$$a \left(3(e-1) - \frac{6e}{\pi} \left(\cos(\pi) - \cos\left(\frac{\pi}{2}\right) \right) \right) = a \left(3(e-1) + \frac{6e}{\pi} \right) = 1$$

$$a \left(\frac{3\pi(e-1) + 6e}{\pi} \right) = 1 \Rightarrow a = \frac{\pi}{3\pi e + 6e - 3\pi}$$

$$a = \frac{\pi}{3(e(\pi+2) - \pi)}$$

- ii. Graph passes through $(0, 0.097)$ $(3, 0.26)$ $(6, 0)$ and is continuous
and zero elsewhere G2



iii.
$$\Pr(X > 4) = b \int_4^6 \sin\left(\frac{\pi t}{6}\right) dt$$

$$= \left[-\frac{6b}{\pi} \cos\left(\frac{\pi t}{6}\right) \right]_4^6 = \left(-\frac{6b}{\pi} \left(\cos(\pi) - \cos\left(\frac{2\pi}{3}\right) \right) \right) \quad \text{with } b = \frac{\pi e}{3(e(\pi+2) - \pi)}$$

$$= \frac{e}{e(\pi+2) - \pi} \quad \text{A1}$$

iv.
$$\Pr(T < 2 | T < 3) = \frac{\Pr(T < 2)}{\Pr(T < 3)} \quad \text{M1}$$

$$= \frac{\int_0^2 a e^{\frac{t}{3}} dt}{\int_0^3 a e^{\frac{t}{3}} dt} = \frac{\left[3e^{\frac{t}{3}} \right]_0^2}{\left[3e^{\frac{t}{3}} \right]_0^3}$$

$$= \frac{e^{\frac{2}{3}} - 1}{e - 1} \quad \text{A1}$$

v. $E(T) = a \int_0^3 t e^{\frac{t}{3}} dt + \int_3^6 et \sin\left(\frac{\pi t}{6}\right) dt$ with $a = \frac{\pi}{3(e(\pi+2) - \pi)}$ M1

mean time $E(T) = 2.92$ hours A1

Define $f(t) = \begin{cases} a \cdot e^{\frac{t}{3}}, & 0 \leq t \leq 3 \\ b \cdot \sin\left(\frac{\pi \cdot t}{6}\right), & 3 \leq t \leq 6 \end{cases}$	Done
$f(3)$	$a \cdot e$
$\text{solve} \left(\int_0^6 f(t) dt = 1, a \right) b = a \cdot e$	$a = \frac{\pi}{3 \cdot (e \cdot (\pi+2) - \pi)}$
$a := \frac{\pi}{3 \cdot (e \cdot (\pi+2) - \pi)}$	$\frac{\pi}{3 \cdot (e \cdot (\pi+2) - \pi)}$
$b := a \cdot e$	$\frac{e \cdot \pi}{3 \cdot (e \cdot (\pi+2) - \pi)}$
Define $f(t) = \begin{cases} a \cdot e^{\frac{t}{3}}, & 0 \leq t \leq 3 \\ b \cdot \sin\left(\frac{\pi \cdot t}{6}\right), & 3 \leq t \leq 6 \end{cases}$	Done

$\int_4^6 f(t) dt$	$\frac{e}{e \cdot (\pi+2) - \pi}$
$\frac{\int_0^2 f(t) dt}{\int_0^3 f(t) dt}$	$\frac{2}{e^3 - 1}$
$\int_0^6 (t \cdot f(t)) dt$	2.9222

Question 2

i. $f(x) = x^3 + bx^2 + cx + 6$
 $f'(x) = 3x^2 + 2bx + c$ A1

ii. $f(2) = (2)^3 + b(2)^2 + 2c + 6 = 8 + 4b + 2c + 6 = 4b + 2c + 14$
 $f'(2) = 3(2)^2 + 4b + c = 4b + c + 12$ M1

Equation of the tangent at P

$$y - (4b + 2c + 14) = (4b + c + 12)(x - 2)$$

$$tp(x) = y = (4b + c + 12)x - 2(4b + c + 12) + (4b + 2c + 14)$$

$$tp(x) = y = (4b + c + 12)x - 4b - 10$$
 A1

iii. Solving $tp(x) = f(x)$ when $x = -1$ $tp(-1) = 6 = f(-1)$

$$tp(-1) = -(4b + c + 12) - 4b - 10 = -8b - c - 22 = 6$$

$$\Rightarrow (1) \quad -8b - c = 28$$
 M1

$$f(-1) = 6 \Rightarrow -1 + b - c + 6 = 6$$

$$\Rightarrow (2) \quad b - c = 1$$
 A1

solving these simultaneous equations,

$$(2) - (1) \Rightarrow 9b = -27 \Rightarrow b = -3 \text{ and } c = -4 \quad P(2, -6)$$

iv. so substitute $b = -3$ and $c = -4$

$$f(x) = x^3 - 3x^2 - 4x + 6 \text{ and } tp(x) = -4x + 2$$
 A1

$$A_1 = \int_{-1}^2 (f(x) - tp(x)) dx$$

$$A_1 = \int_{-1}^2 (x^3 - 3x^2 + 4) dx$$
 A1

v. Equation of the tangent at Q

$$f(-1) = 6 \text{ and } f'(x) = 3x^2 - 6x - 4 \Rightarrow f'(-1) = 3 + 6 - 4 = 5$$
 M1

$$y - 6 = 5(x + 1)$$

$$tq(x) = y = 5x + 11$$
 A1

vi. $tq(x) = f(x)$

$$5x + 11 = x^3 - 3x^2 - 4x + 6 \text{ solving gives } \Rightarrow x = 5 \text{ or } x = -1$$

$$f(5) = (5)^3 - 3(5)^2 - 20 + 6 = 36 \text{ or } tq(5) = 25 + 11 = 36$$

$$R(5, 36)$$
 A1

vii. $A_2 = \int_{-1}^5 (tq(x) - f(x)) dx$

$$A_2 = \int_{-1}^5 (-x^3 + 3x^2 + 9x + 5) dx$$

A1

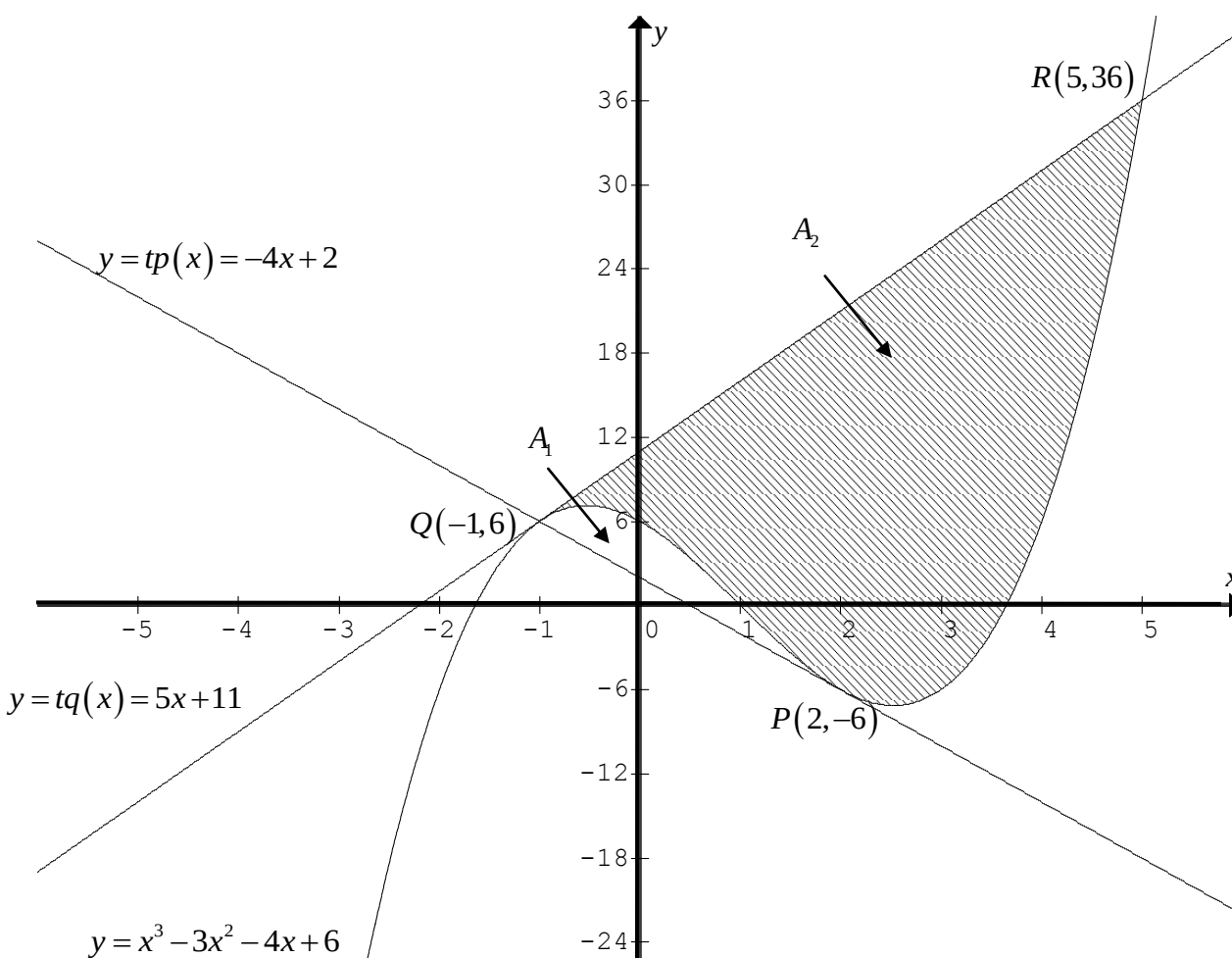
viii. $A_1 = \frac{27}{4}$ $A_2 = 108$

$$\frac{A_2}{A_1} = 16$$

A1

ix. correct graphs, tangents at points P , and Q

G2

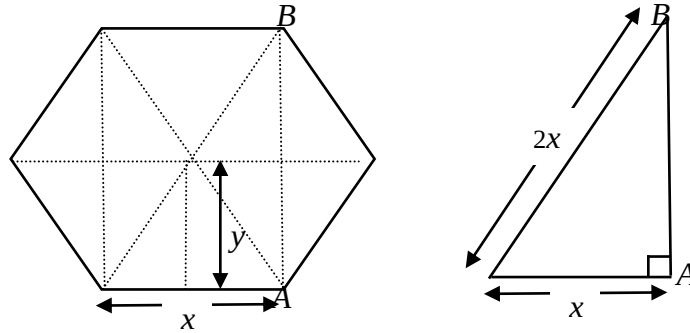


Define $f(x)=x^3+b \cdot x^2+c \cdot x+6$	Done
Define $df(x)=\frac{d}{dx}(f(x))$	Done
$f(2)$	$4 \cdot b+2 \cdot c+14$
$df(2)$	$4 \cdot b+c+12$
$\text{solve}(y-f(2)=df(2) \cdot (x-2),y)$	$y=(4 \cdot b+c+12) \cdot x-4 \cdot b-10$
Define $tp(x)=(4 \cdot b+c+12) \cdot x-4 \cdot b-10$	Done
$\text{solve}(tp(x)=f(x),x)$	$x=-(b+4)$ or $x=2$
$\text{solve}(-(b+4)=-1 \text{ and } f(-1)=6, \{b,c\})$	$b=-3$ and $c=-4$
Define $f(x)=x^3+b \cdot x^2+c \cdot x+6 b=-3 \text{ and } c=-4$	Done
Define $df(x)=\frac{d}{dx}(f(x))$	Done
Define $tp(x)=(4 \cdot b+c+12) \cdot x-4 \cdot b-10 b=-3 \text{ and } c=-4$	Done
$f(x)-tp(x)$	$x^3-3 \cdot x^2+4$
$\int_{-1}^2 (f(x)-tp(x))dx$	$\frac{27}{4}$

$\text{solve}(y-f(-1)=df(-1) \cdot (x+1),y)$	$y=5 \cdot x+11$
Define $tq(x)=5 \cdot x+11$	Done
$\text{solve}(tq(x)=f(x),x)$	$x=-1$ or $x=5$
$tq(x)-f(x)$	$-x^3+3 \cdot x^2+9 \cdot x+5$
$tq(5)$	36
$f(5)$	36
$tq(x)-f(x)$	$-x^3+3 \cdot x^2+9 \cdot x+5$
$\int_{-1}^5 (tq(x)-f(x))dx$	108
$\frac{108}{\frac{27}{4}}$	16

Question 3

- a. Consider the hexagonal base of the vase



M1

$$d(AB) = \sqrt{3}x \quad y = \frac{1}{2}d(AB) = \frac{\sqrt{3}x}{2}$$

$$\text{One triangle has area } \frac{1}{2}xy = \frac{1}{2}x \frac{\sqrt{3}x}{2} = \frac{\sqrt{3}}{4}x^2$$

$$\text{or area of one equilateral triangle } \frac{1}{2}x^2 \sin(60^\circ) = \frac{1}{2}x^2 \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}x^2$$

$$\text{Area of the base is six triangles } 6 \times \frac{\sqrt{3}}{4}x^2 = \frac{3\sqrt{3}}{2}x^2 \quad \text{A1}$$

Total surface area is the area of the base and six rectangles of length x and height h .

$$S = \frac{3\sqrt{3}}{2}x^2 + 6xh$$

- b. Volume is area of the base multiplied by the height

$$V = \frac{3\sqrt{3}}{2}x^2h \quad \text{A1}$$

$$\text{c.i. } S = \frac{3\sqrt{3}}{2}x^2 + 6xh \Rightarrow 6xh = S - \frac{3\sqrt{3}}{2}x^2 = \frac{2S - 3\sqrt{3}x^2}{2}$$

$$h = \frac{2S - 3\sqrt{3}x^2}{12x} \quad \text{M1}$$

$$V(x) = \frac{3\sqrt{3}}{2}x^2h = \frac{3\sqrt{3}}{2}x^2 \left(\frac{2S - 3\sqrt{3}x^2}{12x} \right)$$

$$V(x) = \frac{\sqrt{3}}{8}(2Sx - 3\sqrt{3}x^3) \quad \text{M1}$$

$$V(x) = \frac{\sqrt{3}}{8}(2Sx - 3\sqrt{3}x^3) > 0 \Rightarrow 2S - 3\sqrt{3}x^2 > 0 \quad 2S > 3\sqrt{3}x^2 \text{ and } x > 0$$

$$\text{so that } 0 < x < \sqrt{\frac{2S}{3\sqrt{3}}} \text{ or } 0 < x < \frac{\sqrt{2\sqrt{3}S}}{3} \quad \text{A1}$$

$$\text{ii.} \quad \frac{dV}{dx} = \frac{\sqrt{3}}{8} (2S - 9\sqrt{3}x^2) \quad \text{A1}$$

$$\text{for a maximum volume} \quad \frac{dV}{dx} = 0 \Rightarrow 2S = 9\sqrt{3}x^2$$

$$x = \sqrt{\frac{2S}{9\sqrt{3}}} = \frac{1}{3}\sqrt{\frac{2S}{\sqrt{3}}} \quad \text{A1}$$

$$\text{d.i.} \quad V = \frac{3\sqrt{3}}{2} x^2 h \Rightarrow h = \frac{2V}{3\sqrt{3} x^2} \quad \text{A1}$$

$$S = \frac{3\sqrt{3}}{2} x^2 + 6xh$$

$$S = \frac{3\sqrt{3}}{2} x^2 + 6x \left(\frac{2V}{3\sqrt{3} x^2} \right)$$

$$S(x) = \frac{3\sqrt{3}}{2} x^2 + \frac{4V}{\sqrt{3} x} = \frac{3\sqrt{3}}{2} x^2 + \frac{4V}{\sqrt{3}} x^{-1} = \sqrt{3} \left(\frac{3}{2} x^2 + \frac{4V}{3} x^{-1} \right) \quad \text{M1}$$

$$\text{ii.} \quad \frac{dS}{dx} = \sqrt{3} \left(3x - \frac{4V}{3} x^{-2} \right) = \sqrt{3} \left(3x - \frac{4V}{3x^2} \right) \quad \text{A1}$$

for a minimum surface area

$$\frac{dS}{dx} = 0 \Rightarrow 3x - \frac{4V}{3x^2} = 0 \Rightarrow x^3 = \frac{4V}{9}$$

$$x = \sqrt[3]{\frac{4V}{9}} \quad \text{A1}$$

Note that from CAS, there are many equivalent forms, for all these answers and equations.

e. solving the five non-linear equations using CAS, for four unknowns, S, V, h, x

$$(1) \quad S = \frac{3\sqrt{3}}{2} x^2 + 6xh \quad (2) \quad V = \frac{3\sqrt{3}}{2} x^2 h \quad \text{M1}$$

$$(3) \quad x = \sqrt{\frac{2S}{9\sqrt{3}}} \quad (4) \quad x = \sqrt[3]{\frac{4V}{9}} \quad \text{and} \quad (5) \quad V = 9S$$

$$\text{gives} \quad x = 18\sqrt{3} \quad \text{and} \quad h = 27 \text{ cm} \quad \text{A1}$$

$s = \frac{3 \cdot \sqrt{3}}{2} \cdot x^2 + 6 \cdot x \cdot h$	$s = \frac{3 \cdot \sqrt{3} \cdot x^2}{2} + 6 \cdot h \cdot x$
$v = \frac{3 \cdot \sqrt{3}}{2} \cdot x^2 \cdot h$	$v = \frac{3 \cdot h \cdot \sqrt{3} \cdot x^2}{2}$
$\text{solve}\left(s = \frac{3 \cdot \sqrt{3}}{2} \cdot x^2 + 6 \cdot x \cdot h, h\right)$	$h = \frac{2 \cdot s - 3 \cdot \sqrt{3} \cdot x^2}{12 \cdot x}$
$v = \frac{3 \cdot \sqrt{3}}{2} \cdot x^2 \cdot h \mid h = \frac{2 \cdot s - 3 \cdot \sqrt{3} \cdot x^2}{12 \cdot x}$	$v = \frac{\sqrt{3} \cdot s \cdot x}{4} - \frac{9 \cdot x^3}{8}$
$\frac{d}{dx}\left(\frac{\sqrt{3} \cdot s \cdot x}{4} - \frac{9 \cdot x^3}{8}\right)$	$\frac{\sqrt{3} \cdot s}{4} - \frac{27 \cdot x^2}{8}$
$\text{solve}\left(\frac{\sqrt{3} \cdot s}{4} - \frac{27 \cdot x^2}{8} = 0, x\right)$	$x = \frac{\frac{3}{4} \cdot \sqrt{2 \cdot s}}{9} \text{ and } s \geq 0 \text{ or } x = \frac{-\frac{3}{4} \cdot \sqrt{2 \cdot s}}{9} \text{ and } s \geq 0$
$\text{solve}\left(v = \frac{3 \cdot \sqrt{3}}{2} \cdot x^2 \cdot h, h\right)$	$h = \frac{2 \cdot \sqrt{3} \cdot v}{9 \cdot x^2}$
$s = \frac{3 \cdot \sqrt{3}}{2} \cdot x^2 + 6 \cdot x \cdot h \mid h = \frac{2 \cdot \sqrt{3} \cdot v}{9 \cdot x^2}$	$s = \frac{4 \cdot \sqrt{3} \cdot v}{3 \cdot x} + \frac{3 \cdot \sqrt{3} \cdot x^2}{2}$
$\frac{d}{dx}\left(\frac{4 \cdot \sqrt{3} \cdot v}{3 \cdot x} + \frac{3 \cdot \sqrt{3} \cdot x^2}{2}\right)$	$3 \cdot \sqrt{3} \cdot x - \frac{4 \cdot \sqrt{3} \cdot v}{x^2}$

16/99

$\frac{d}{dx}\left(\frac{4 \cdot \sqrt{3} \cdot v}{3 \cdot x} + \frac{3 \cdot \sqrt{3} \cdot x^2}{2}\right)$	$3 \cdot \sqrt{3} \cdot x - \frac{4 \cdot \sqrt{3} \cdot v}{x^2}$
$\text{solve}\left(3 \cdot \sqrt{3} \cdot x - \frac{4 \cdot \sqrt{3} \cdot v}{x^2} = 0, x\right)$	$x = \frac{\frac{2}{3} \cdot (3 \cdot v)^{\frac{1}{3}}}{3}$
$eq1: s = \frac{3 \cdot \sqrt{3}}{2} \cdot x^2 + 6 \cdot x \cdot h$	$s = \frac{3 \cdot \sqrt{3} \cdot x^2}{2} + 6 \cdot h \cdot x$
$eq2: v = \frac{3 \cdot \sqrt{3}}{2} \cdot x^2 \cdot h$	$v = \frac{3 \cdot h \cdot \sqrt{3} \cdot x^2}{2}$
$eq3: x = \frac{1}{3} \cdot \sqrt{\frac{2 \cdot s}{\sqrt{3}}}$	$x = \frac{\frac{3}{4} \cdot \sqrt{2 \cdot s}}{9}$
$eq4: x = \sqrt[3]{\frac{4 \cdot v}{9}}$	$x = \frac{\frac{2}{3} \cdot (3 \cdot v)^{\frac{1}{3}}}{3}$
$eq5: v = 9 \cdot s$	$v = 9 \cdot s$
$\text{solve}\left(\begin{matrix} eq1 \\ eq2 \\ eq3, \{x, h, s, v\} \\ eq4 \\ eq5 \end{matrix} \mid x > 0 \text{ and } h > 0 \text{ and } s > 0 \text{ and } v > 0\right)$	$x = 18 \cdot \sqrt{3} \text{ and } h = 27 \cdot \sqrt{3} \text{ and } s = 4374 \cdot \sqrt{3} \text{ and } v = 39366 \cdot \sqrt{3}$

1/16

Question 4

a.i. $g(x) = \int_0^x f(t) dt - 2x$

$$g(-2) = \int_0^{-2} f(t) dt + 4$$

$$= 4 - \int_{-2}^0 f(t) dt$$

Now $\int_{-2}^0 f(t) dt$ is the area of a triangle of side lengths 2 and 4

$$= 4 - \frac{1}{2} \times 2 \times 4$$

$$= 0$$

A1

ii. $g(4) = \int_0^4 f(t) dt - 8$

Now $\int_0^4 f(t) dt$ represents one-quarter of the area of a circle of radius 4

$$= \frac{1}{4} \pi \times 4^2 - 8$$

$$= 4\pi - 8$$

A1

iii. $g(-3) = \int_0^{-3} f(t) dt + 6$

$$= 6 - \int_{-3}^0 f(t) dt = 6 - \int_{-3}^0 (2t + 4) dt$$

$$= 6 - \left[t^2 + 4t \right]_{-3}^0 = 6 - [0 - (9 - 12)]$$

$$= 3$$

A1

b. $g'(x) = f(x) - 2$

where $f(x) = \begin{cases} 2x + 4 & \text{for } -3 \leq x \leq 0 \\ \sqrt{16 - x^2} & \text{for } 0 \leq x \leq 4 \end{cases}$

M1

$$g'(x) = \begin{cases} 2x + 2 & \text{for } -3 \leq x \leq 0 \\ \sqrt{16 - x^2} - 2 & \text{for } 0 \leq x \leq 4 \end{cases}$$

A1

- c. The function $g(x)$ has a maximum or minimum turning point when

$$g'(x) = 0 \text{ or } f(x) = 2$$

$$(1) \text{ for } -3 \leq x \leq 0 \Rightarrow 2x + 2 = 0 \Rightarrow x = -1 \quad \text{M1}$$

$$\begin{aligned} g(-1) &= \int_0^{-1} f(t) dt + 2 = 2 - \int_{-1}^0 (2t + 4) dx = 2 - [t^2 + 4t]_{-1}^0 \\ &= 2 - [(0) - (1 - 4)] \\ &= -1 \end{aligned}$$

$$(2) \text{ for } 0 \leq x \leq 4 \Rightarrow \sqrt{16 - x^2} - 2 = 0$$

$$16 - x^2 = 4 \Rightarrow x^2 = 12 \Rightarrow x = 2\sqrt{3} \text{ since } 0 \leq x \leq 4 \quad \text{M1}$$

$$\begin{aligned} g(2\sqrt{3}) &= \int_0^{2\sqrt{3}} f(t) dt - 4\sqrt{3} = \int_0^{2\sqrt{3}} \sqrt{16 - t^2} dt - 4\sqrt{3} \\ &= \frac{8\pi}{3} + 2\sqrt{3} - 4\sqrt{3} = \frac{8\pi}{3} - 2\sqrt{3} \approx 4.9135 \end{aligned}$$

$$\left(2\sqrt{3}, \frac{8\pi}{3} - 2\sqrt{3}\right) \text{ is the maximum turning point} \quad \text{A1}$$

$$(-1, -1) \text{ is the minimum turning point} \quad \text{A1}$$

- d. The point $(0, 0)$ is the inflexion point A1

- e. Now g has a domain $[-3, 4]$

$$g(4) = 4\pi - 8 \approx 4.5664 \text{ and } g(-3) = 3 \text{ are endpoints.}$$

$$\text{The range is } \left[-1, \frac{8\pi}{3} - 2\sqrt{3}\right], \text{ crosses x-axis at } x = -2 \text{ and } x = 0, (-2, 0) (0, 0)$$

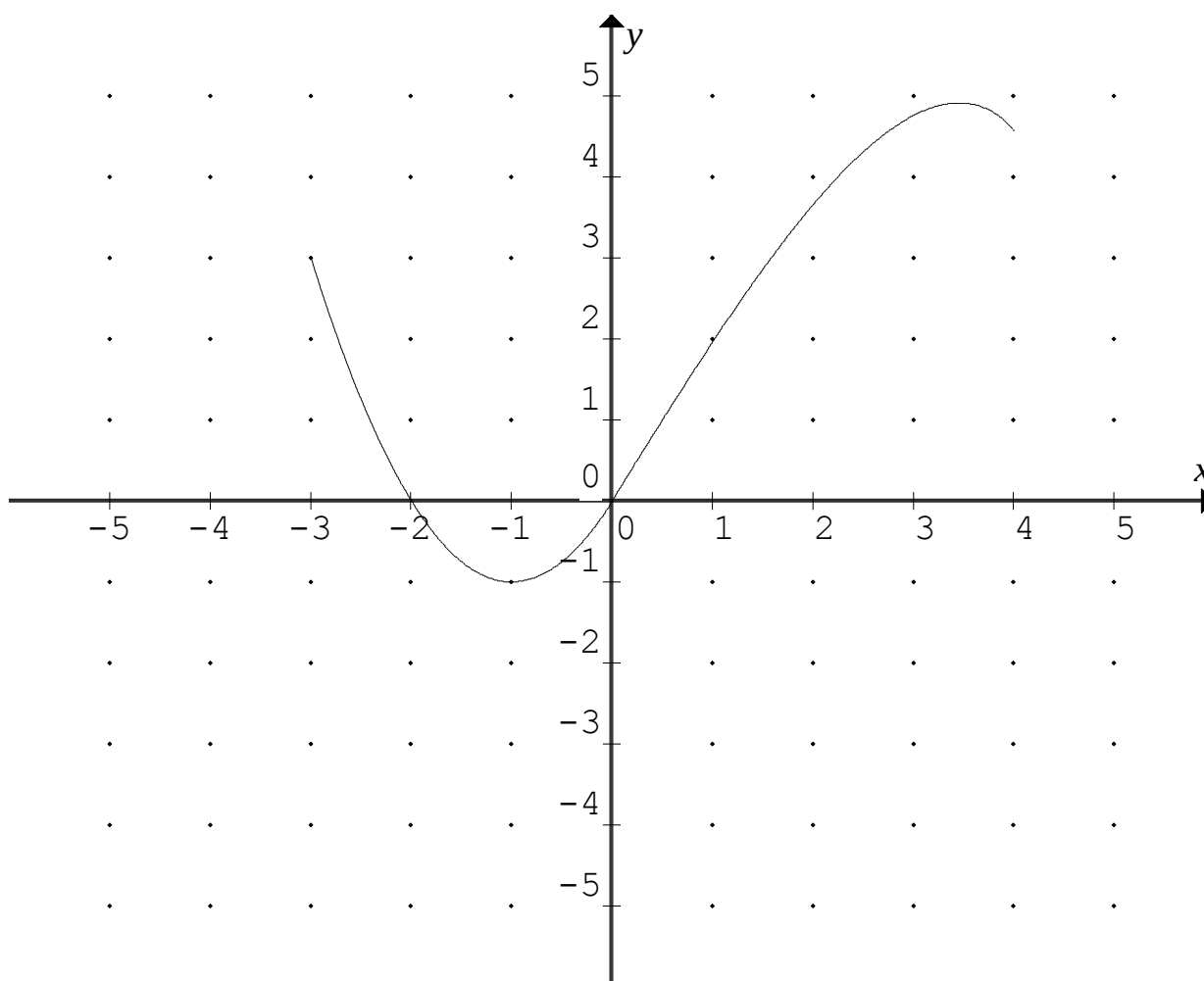
the point $(0, 0)$ is the inflexion point

correct graph over correct domain and range,

G1

all stationary points and intercepts

G1



Define $f(x) = \begin{cases} 2 \cdot x + 4, & -3 \leq x \leq 0 \\ \sqrt{16 - x^2}, & 0 \leq x \leq 4 \end{cases}$	Done
Define $g(x) = \int_0^x f(t) \, dt - 2 \cdot x$	Done
$g(-2)$	0
$g(4)$	$4 \cdot \pi - 8$
$g(-3)$	3
$g(-1)$	-1
$g(2 \cdot \sqrt{3})$	$\frac{8 \cdot \pi}{3} - 2 \cdot \sqrt{3}$
$g(0)$	0

END OF SECTION 2 SUGGESTED ANSWERS