

Year 2002

VCE

Specialist Mathematics Written Examination 2

Detailed Suggested Solutions

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Kilbaha Pty Ltd Publishers ABN 47 065 111 373
PO Box 2227
Kew Vic 3101
Australia
Tel: 03 9817 5374
Fax: 03 9817 4334
chemas@chemas.com
www.chemas.com

Question 1

a. $\overrightarrow{OS} = 8\hat{i} - 4\hat{j}$ $\overrightarrow{OC} = 2\hat{i} - 6\hat{j}$ $\overrightarrow{CS} = \overrightarrow{CO} + \overrightarrow{OS} = \overrightarrow{OS} - \overrightarrow{OC}$ $\overrightarrow{CS} = 8\hat{i} - 4\hat{j} - 2\hat{i} + 6\hat{j}$ $\overrightarrow{CS} = 6\hat{i} + 2\hat{j}$ $ \overrightarrow{CS} = \sqrt{36 + 4} = 6.3 \text{ km}$ to the nearest tenth of a kilometre	b.(i) $\overrightarrow{OP} = 6m\hat{i} - 2m\hat{j}$ $\overrightarrow{PS} = \overrightarrow{PO} + \overrightarrow{OS}$ $\overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP}$ $\overrightarrow{PS} = 8\hat{i} - 4\hat{j} - 6m\hat{i} + 2m\hat{j}$ $\overrightarrow{PS} = (8 - 6m)\hat{i} + (-4 + 2m)\hat{j}$ $\overrightarrow{OP} \bullet \overrightarrow{PS} = 6m(8 - 6m) - 2m(-4 + 2m)$ $\overrightarrow{OP} \bullet \overrightarrow{PS} = 56m - 40m^2$
b.(ii) Shortest distance is when PS is perpendicular to the shoreline. This means $\overrightarrow{OP} \bullet \overrightarrow{PS} = 0$ $\overrightarrow{OP} \bullet \overrightarrow{PS} = 56m - 40m^2 = 0$ $8m(7 - 5m) = 0$ $m = 0 \text{ or } m = \frac{7}{5}$ But $m > 0$ $\therefore m = \frac{7}{5}$ $\therefore P \text{ is point } (6 \times \frac{7}{5}, -2 \times \frac{7}{5})$ $\therefore P \text{ is point } (8.4, -2.8)$	b.(iii) When $m = \frac{7}{5}$ $\overrightarrow{PS} = (0.8 - 6 \times \frac{7}{5})\hat{i} + (-4 + 2 \times \frac{7}{5})\hat{j}$ $\overrightarrow{PS} = 0.4\hat{i} - 1.2\hat{j}$ $ \overrightarrow{PS} = \sqrt{0.4^2 + 1.2^2} = 1.265$ $ \overrightarrow{PS} = 1.3 \text{ km to the nearest tenth of a kilometre}$

Question 1 (continued)

c.(i)

$$\dot{r}_c(t) = 15\hat{i} - 5\hat{j}$$

$$r_c(t) = \int 15\hat{i} - 5\hat{j} dt$$

$$r_c(t) = 15t\hat{i} - 5t\hat{j} + \vec{c} \text{ where } \vec{c} \text{ is a vector constant}$$

$$\text{When } t = 0, r(0) = 2\hat{i} - 6\hat{j}$$

$$\therefore r_c(t) = 15t\hat{i} - 5t\hat{j} + 2\hat{i} - 6\hat{j}$$

$$\therefore r_c(t) = (2 + 15t)\hat{i} - (6 + 5t)\hat{j}$$

c.(ii)

$$\dot{r}_s(t) = 12\hat{i} + (3\sin t - 8)\hat{j}$$

$$r_s(t) = \int (12\hat{i} + (3\sin t - 8)\hat{j}) dt$$

$$r_s(t) = 12t\hat{i} + (-3\cos t - 8t)\hat{j} + \vec{c} \text{ where } \vec{c} \text{ is a vector constant}$$

$$r_s(0) = 0\hat{i} - 3\hat{j} + \vec{c} = 8\hat{i} - 4\hat{j}$$

$$\therefore \vec{c} = 8\hat{i} - \hat{j}$$

$$r_s(t) = 12t\hat{i} + (-3\cos t - 8t)\hat{j} + 8\hat{i} - \hat{j}$$

$$r_s(t) = (12t + 8)\hat{i} - (3\cos t + 8t + 1)\hat{j}$$

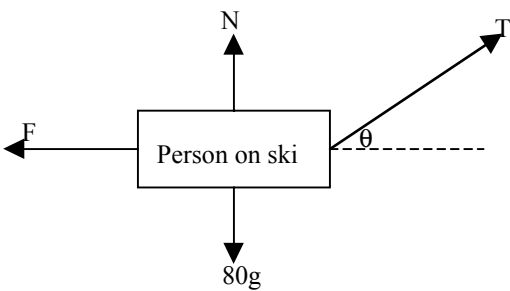
d.

$$r_c(2) = 32\hat{i} - 16\hat{j}$$

$$r_s(2) = 32\hat{i} - 15.8\hat{j}$$

Hence, both ships are the same distance east of O but the cargo ship is 0.2 km south of the sailing ship.

Question 2

a. 	b. $ma = T \cos 60^\circ - \mu N$ $\Rightarrow T \cos 60^\circ = 0.3N + 80 \times 2 \quad (1)$ $N + T \sin 60^\circ - 80g = 0$ $\Rightarrow N = 80g - T \sin 60^\circ \quad (2)$ Substituting (2) in (1) $T \cos 60^\circ = 0.3(80g - T \sin 60^\circ) + 160$ $T \cos 60^\circ + 0.3T \sin 60^\circ = 0.3 \times 80g + 160$ $T(\cos 60^\circ + 0.3 \sin 60^\circ) = 0.3 \times 80g + 160$ $T = \frac{0.3 \times 80 \times 9.8 + 160}{\cos 60^\circ + 0.3 \sin 60^\circ}$ $T = 520.1316N$ $T = 520N \text{ to the nearest integer.}$
c.(i) $T(\theta) = \frac{0.3 \times 80 \times 9.8 + 160}{\cos \theta + 0.3 \sin \theta}$ $T(\theta) = \frac{395.2}{\cos \theta + 0.3 \sin \theta}$ $T(\theta) = 395.2(\cos \theta + 0.3 \sin \theta)^{-1}$ $\frac{dT}{d\theta} = 395.2(-1)(\cos \theta + 0.3 \sin \theta)^{-2} \times$ $(-\sin \theta + 0.3 \cos \theta)$ $\Rightarrow \frac{-395.2(0.3 \cos \theta - \sin \theta)}{(\cos \theta + 0.3 \sin \theta)^2} = 0 \text{ for TP}$ $\Rightarrow 0.3 \cos \theta - \sin \theta = 0$ $\Rightarrow \sin \theta = 0.3 \cos \theta$ $\Rightarrow \tan \theta = 0.3$ $\Rightarrow \theta = 16.699^\circ = 16.7^\circ \text{ to the nearest tenth of a degree}$	c.(ii) $T_{\min} = T(16.699^\circ) = \frac{395.2}{\cos 16.699^\circ + 0.3 \sin 16.699^\circ}$ $T(16.699^\circ) = 378.5$ $T(16.699^\circ) = 379N \text{ to the nearest newton}$

Question 3 a.(i) $u = 0$ $s = 400$ $t = 8$ $s = ut + \frac{1}{2}at^2$ $400 = 0 + \frac{1}{2} \times a \times 64$ $400 = 32a$ $a = \frac{400}{32} = 12.5 \text{ m/sec}^2$	a.(ii) $v = u + at$ $v = 0 + 12.5 \times 8$ $v = 100 \text{ m/sec. Shown}$ b.(i) $ma = -T - R$ $400a = -5000 - 0.5v^2$ $a = \frac{-(0.5v^2 + 5000)}{400}$ $a = \frac{-(v^2 + 10^4)}{800}$
b.(ii) $a = v \frac{dv}{dx} = \frac{-(10^4 + v^2)}{800}$ $\frac{dv}{dx} = \frac{-(10^4 + v^2)}{800v} \text{ Shown}$	b.(iii) $\frac{dv}{dx} = \frac{-(10^4 + v^2)}{800v}$ Invert both sides $\frac{dx}{dv} = \frac{-800v}{10^4 + v^2}$ $x = -800 \int_{100}^0 \frac{v}{10^4 + v^2} dv$ $x = 400 \int_0^{100} \frac{2v}{10^4 + v^2} dv$ $x = 400 \log_e(10^4 + v^2) \Big _0^{100}$ $x = 400[\log_e(10^4 + 10^4) - \log_e 10^4]$ $x = 277.3$ $x = 277 \text{ m to the nearest metre.}$

c.

$$a = \frac{dv}{dt} = \frac{-(v^2 + 10^4)}{800}$$

Invert both sides

$$\frac{dt}{dv} = \frac{-800}{v^2 + 10^4}$$

$$t = -800 \int_{100}^0 \frac{dv}{v^2 + 100^2}$$

$$t = 800 \int_0^{100} \frac{dv}{v^2 + 100^2}$$

$$t = 8 \int_0^{100} \frac{100}{v^2 + 100^2}$$

$$t = 8 \left[\tan^{-1} \frac{v}{100} \right]_0^{100}$$

$$t = 8 \left[\tan^{-1} 1 - \tan^{-1} 0 \right]$$

$$t = 8 \left[\frac{\pi}{4} - 0 \right]$$

$$t = 2\pi \text{ secs.}$$

Question 4

a.

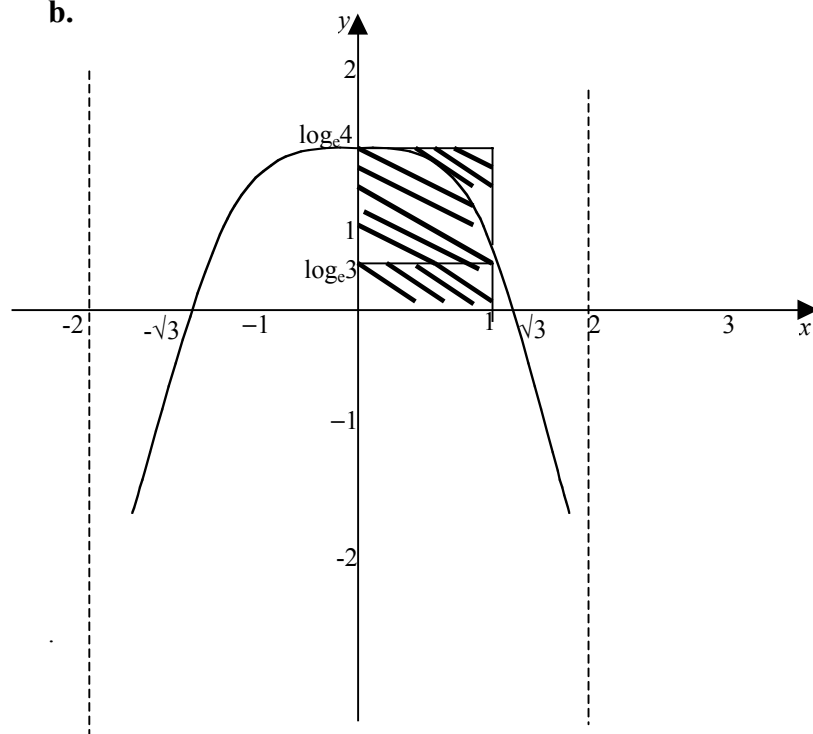
$$4 - x^2 > 0$$

$$4 > x^2$$

$$x < |2|$$

Domain, D is $(-2, 2)$

b.



x intercept when $y = 0$

$$\Rightarrow \log_e(4 - x^2) = 0$$

$$\Rightarrow 4 - x^2 = 1$$

$$\Rightarrow x^2 = 3$$

$$\Rightarrow x = \pm\sqrt{3}$$

y intercept when $x = 0$

$$y = \log_e 4$$

asymptotes at $x = \pm 2$

As $x \rightarrow \pm 2$, $f(x) \rightarrow -\infty$

$$f'(x) = \frac{-2x}{4 - x^2} = 0 \text{ for turning point}$$

When $x = 0$, local maximum

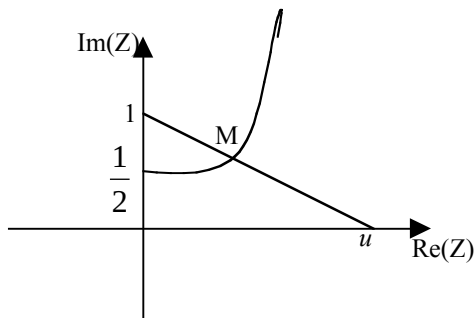
Question 4

c. When $x = 1$, $y = \log_e 3$ Area of lower rectangle $= 1 \times \log_e 3 = \log_e 3$ When $x = 0$, $y = \log_e 4$ Area of upper rectangle $= 1 \times \log_e 4 = \log_e 4$ Area of lower rectangle $< A <$ Area of upper rectangle $\therefore \log_e 3 < A < \log_e 4$	d.(i) $y = x \log_e (4 - x^2)$ $\frac{dy}{dx} = x \times \frac{1}{4 - x^2} \times -2x + \log_e (4 - x^2) \times 1$ $\frac{dy}{dx} = \log_e (4 - x^2) - \frac{2x^2}{4 - x^2}$
d.(ii) $-x^2 + 4 \Bigg) \frac{-1}{x^2}$ $\frac{x^2 - 4}{4}$ $\frac{x^2}{4 - x^2} = -1 + \frac{4}{4 - x^2}$ $\frac{4}{4 - x^2} = \frac{A}{2 - x} + \frac{B}{2 + x}$ $\frac{4}{4 - x^2} = \frac{A(2 + x) + B(2 - x)}{4 - x^2}$ $4 = A(2 + x) + B(2 - x)$ When $x = -2$ $4 = 4B$ $\Rightarrow B = 1$ When $x = 2$ $4 = 4A$ $\Rightarrow A = 1$ $\int \frac{x}{4 - x^2} dx = \int -1 + \frac{1}{2 - x} + \frac{1}{2 + x} dx$ $\int \frac{x}{4 - x^2} dx = -x + \log_e (2 - x) \times (-1) + \log_e (2 + x)$ $\int \frac{x}{4 - x^2} dx = -x + \log_e \frac{2 + x}{2 - x}$	d.(iii) $A = \int_0^1 \log_e (4 - x^2) dx$ From d(i) $\frac{d}{dx} x \log_e (4 - x^2) = \log_e (4 - x^2) - \frac{2x^2}{4 - x^2}$ $\therefore x \log_e (4 - x^2) = \int \log_e (4 - x^2) - \int \frac{2x^2}{4 - x^2}$ $\therefore \int \log_e (4 - x^2) dx = x \log_e (4 - x^2) + \int \frac{2x^2}{4 - x^2} dx$ $A = x \log_e (4 - x^2) \Big _0^1 + 2 \int_0^1 \frac{x^2}{4 - x^2} dx$ From d(ii) $A = x \log_e (4 - x^2) \Big _0^1 + 2 \left[\log_e \frac{2 + x}{2 - x} - x \right]_0^1$ $A = (\log_e 3 - 0) + 2(\log_e 3 - 1 - 0)$ $A = \log_e 3 + 2 \log_e 3 - 2$ $A = 3 \log_e 3 - 2$

Question 4

e.(i) $y_{20} = y_{19} + h \frac{dy}{dx}$ $y_{20} = y_{19} + 0.05 \log_e(4 - x_{19}^2)$ $x_{19} = x_0 + 19 \times h$ $x_{19} = 0 + 19 \times 0.05$ $x_{19} = 0.95$ $y_{20} = y_{19} + 0.05 \log_e(4 - 0.95^2)$ $y_{20} = y_{19} + 0.05653$	e.(ii) $y_{20} = 1.2464 + 0.0565$ $y_{20} = 1.3029$
	e.(iii) Euler's method approximates a solution, $y = A$, to the given differential equation by using a point (x_1, y_1) on the tangent to the curve at (x_0, y_0) rather than the actual point on the curve when $x = x_1$ The gradient of the tangent at the point (x_0, y_0) is $\frac{dy}{dx} = \frac{y_1 - y_0}{h}$ $\therefore y_1 = y_0 + h \frac{dy}{dx}$ Each of these 20 subintervals which represent areas under the tangent lines at the 20 points approximates the actual area under the curve from $x = x_0$ up to the particular x value.

Question 5

a.(i) $x + (y-1)i = (x-u) + yi$ So Z is equidistant from both L and N Hence, Z lies on the perpendicular bisector of LN	a.(ii) $x^2 + (y-1)^2 = (x-u)^2 + y^2$ $x^2 + y^2 - 2y + 1 = x^2 - 2xu + u^2 + y^2$ $-2y + 1 = -2xu + u^2$ $2y - 1 = 2xu - u^2$ $2y = 2ux - u^2 + 1$ Shown
b.(i) $2y = 2ux - u^2 + 1$ When $x = u$ $2y = 2u^2 - u^2 + 1$ $2y = u^2 + 1$ $y_w = \frac{1}{2}(u^2 + 1)$ $\therefore w = u + \frac{1}{2}(u^2 + 1)i$	b.(ii) $x = u \quad u > 0$ $y = \frac{1}{2}(u^2 + 1)$ $y = \frac{1}{2}(x^2 + 1)$ 
c. $y = \frac{1}{2}x^2 + \frac{1}{2}$ At w gradient = $\frac{dy}{dx} = x = u$ Gradient of LN = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{1-0}{0-u} = -\frac{1}{u}$ $\therefore m_{LN} \times m_w = u \times -\frac{1}{u} = -1$ But $m_{LN} \times$ gradient of perpendicular bisector of LN = -1 $\therefore m_w =$ gradient of perpendicular bisector of LN $\therefore$ perpendicular bisector of LN is tangent to the curve at w	

End of suggested solutions 2002 VCE Specialist Mathematics Examination 2