

Year 2002

VCE

Mathematical Methods (CAS) Written Examination 2

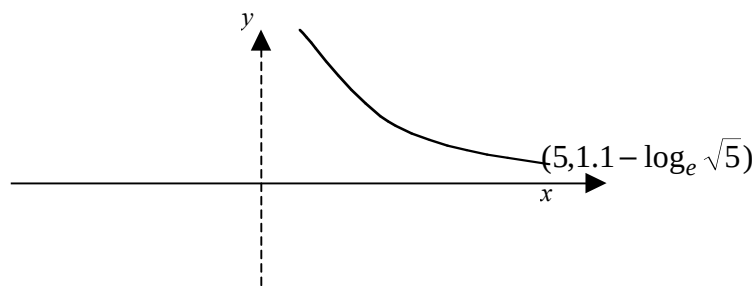
Detailed Suggested Solutions

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Question 1(a) (i)



The vertical asymptote is the y axis, i.e. the line $x = 0$

When $x = 5$,

$$y = 1.1 - 0.5 \log_e 5$$

$$y = 1.1 - \log_e 5^{0.5}$$

$$y = 1.1 - \log_e \sqrt{5}$$

Question 1(a) (ii)

An inverse function exists because there is a one-to-one correspondence for every point on this section of the graph.

Question 1(a) (iii)

$$f(x) = 1.1 - 0.5 \log_e x$$

Interchange x and y

$$x = 1.1 - 0.5 \log_e y$$

$$0.5 \log_e y = 1.1 - x$$

$$\log_e y^{\frac{1}{2}} = 1.1 - x$$

$$\sqrt{y} = e^{1.1-x}$$

Square both sides

$$y = (e^{1.1-x})^2 \Rightarrow y = e^{2.2-2x}$$

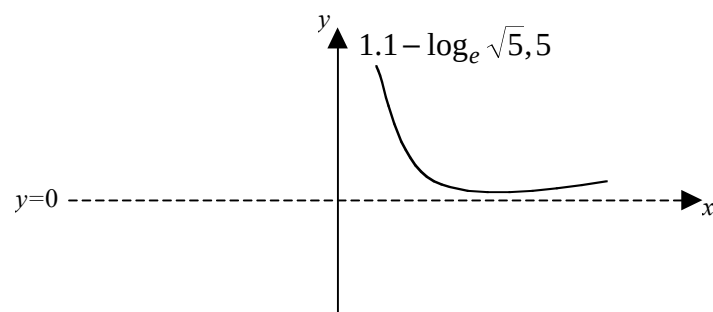
Question 1(a) (iv)

Domain $f^{-1}(x) = \text{range of } f(x)$

From graph in (i) this can be seen to be $\geq 1.1 - 0.5 \log_e 5$

$$\text{Domain } f^{-1}(x) = [1.1 - 0.5 \log_e 5, \infty)$$

Question 1(a) (v)



The x axis, i.e. the line $y = 0$ is a horizontal asymptote.

Question 1(b)

$$t = a - b \log_e x$$

When $x = 1, t = 0.5$

$$0.5 = a - b \log_e 1$$

$$0.5 = a - 0$$

$$\therefore a = 0.5$$

When $x = 1.5, t = 0.3$

$$0.3 = 0.5 - b \log_e 1.5$$

$$b \log_e 1.5 = 0.2$$

$$b = \frac{0.2}{\log_e 1.5}$$

Question 1

(c) (i)

$$\frac{T}{k} = 1.1 - 0.5 \log_e x$$

$$0.5 \log_e x = 1.1 - \frac{T}{k}$$

$$\log_e x = 2\left(1.1 - \frac{T}{k}\right)$$

$$x = e^{2\left(1.1 - \frac{T}{k}\right)}$$

(c) (ii)

$$g: (0, 5] \rightarrow R$$

$$g(x) = k(1.1 - 0.5 \log_e x)$$

$$k > 0$$

$g(x) = T$ has a solution for all x in the given domain

(see 1(a)(i))

$$T = k(1.1 - 0.5 \log_e x)$$

$$k = \frac{T}{1.1 - 0.5 \log_e x}$$

k is largest when $1.1 - 0.5 \log_e x$ is smallest

$\Rightarrow k$ is largest when x is largest i.e. when $x = 5$

When $x = 5$

$$k = \frac{T}{1.1 - 0.5 \log_e 5}$$

$$k = \frac{T}{0.29528}$$

$$k = 3.3866T$$

$$k = 3.387T \text{ to 3 dp.}$$

Question 2(a)

(i)

$$E(X) = 150 = \int_0^a x f(x) dx$$

$$\int_0^a \frac{2x^2}{a^2} dx = 150$$

$$\frac{2}{a^2} \left[\frac{x^3}{3} \right]_0^a = 150$$

$$\frac{2}{a^2} \times \frac{a^3}{3} - 0 = 150$$

$$\frac{2a}{3} = 150$$

$$a = \frac{150 \times 3}{2}$$

$$a = 225 \quad \text{Shown}$$

TI-89 calculator screen showing the definition of $e(x) = \int_0^a (x \cdot f(x)) dx$ and the solution for a when $e(x) = 150$, resulting in $a = 225$.

(ii)

$$\Pr(X \leq 200) = \int_0^{200} f(x) dx$$

$$\Pr(X \leq 200) = \frac{1}{225^2} \int_0^{200} 2x dx$$

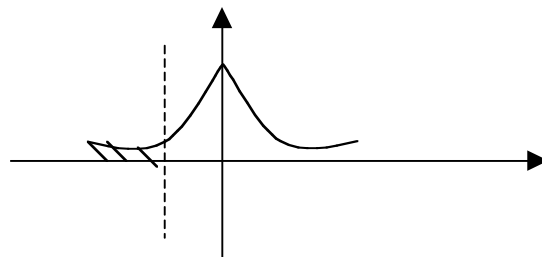
$$\Pr(X \leq 200) = \frac{1}{225^2} [x^2]_0^{200}$$

$$\Pr(X \leq 200) = \left(\frac{200}{225} \right)^2 - 0$$

$$\Pr(X \leq 200) = 0.790 \text{ to 3 dp}$$

TI-89 calculator screen showing the calculation of $\Pr(X \leq 200)$ using the formula $\frac{1}{225^2} \int_0^{200} 2x dx$, resulting in 0.790 .

Question 2(b)



$$Z = \frac{Z - \mu}{\sigma}$$

$$Z = \frac{16 - 20}{2} = -2$$

$$\Pr(X < 16) = \Pr(Z < -2)$$

$$\Pr(X < 16) = \Pr(Z > 2)$$

$$\Pr(X < 16) = 1 - \Pr(Z < 2)$$

$$\Pr(X < 16) = 1 - 0.9772$$

$$\Pr(X < 16) = 0.0228$$

$$\Pr(X < 16) = 0.023 \text{ to 3 d.p.}$$

Question 2(c)

Because of symmetry, the mean is half way between 19 and 28.

$$\text{Hence, } \mu = \frac{19 + 28}{2} = \frac{47}{2} = 23.5 \text{ mm}$$

$$\Pr(X) < 28 = 1 - 0.08 = 0.92$$

$$\text{From tables } Z = 1.41$$

$$Z = \frac{X - \mu}{\sigma}$$

$$1.41 = \frac{28 - 23.5}{\sigma}$$

$$1.41\sigma = 4.5$$

$$\sigma = \frac{4.5}{1.41}$$

$$\sigma = 3.19$$

$$\sigma = 3.2 \text{ mm to the nearest 0.1 mm.}$$

Question 2(d) (i)

Since mean of Fhaisi antennae is 20 mm, then 0.5 of Fhaisi butterflies have antennae less than 20 mm.

Pr. Jojo and antennae less than 20 mm or Fhaisi and antennae less than 20 mm

$$= \frac{1}{5} \times 0.1370 + \frac{4}{5} \times 0.5$$

$$= 0.0274 + 0.4$$

$$= 0.4274$$

Proportion of all butterflies with antennae less than 20 mm = 0.427 to 3 d.p.

Question 2(d) (ii)

Pr Fhaisi given that the antenna is less than 20mm

$$\begin{aligned} &= \frac{\text{Pr Fhaisi and antenna} < 20}{\text{Pr Fhaisi and antenna} < 20 + \text{Pr Jojo and antenna} < 20} \\ &= \frac{0.8 \times 0.5}{0.4274} = 0.9359 = 0.936 \text{ to 3 d.p.} \end{aligned}$$

Question 3(a)(i)

$$2x^4 - x^3 - 5x^2 + 3x = x(2x - 3)(ax^2 + bx + c)$$

$$x(2x^3 - x^2 - 5x + 3) = x(2ax^3 + 2bx^2 + 2cx - 3ax^2 - 3bx - 3c)$$

Equating coefficients

$$2 = 2a \Rightarrow a = 1$$

$$-1 = 2b - 3a$$

$$\Rightarrow -1 = 2b - 3$$

$$\Rightarrow 2b = 2$$

$$\Rightarrow b = 1$$

$$3 = -3c \Rightarrow c = -1$$

Question 3(a)(ii)

$$x(2x - 3)(x^2 + x - 1) = 0$$

$$\Rightarrow x = 0 \text{ or } x = \frac{3}{2} \text{ or } x = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$\Rightarrow x = 0, \frac{3}{2}, \frac{-1 \pm \sqrt{5}}{2}$$

Question 3(b)(i)

Equation of normal is of the form, $y = mx + c$

$$\text{At } x = 1, \frac{dy}{dx} = \frac{1}{2}(8 - 3 - 10 + 3) = -1$$

This is the gradient of the tangent at A

So, gradient of normal at A = 1

Hence, $y = 1x + c$

$$\text{At A when } x = 1, y = \frac{1}{2}(2 - 1 - 5 + 3) = -\frac{1}{2}$$

Substituting this point $(1, -\frac{1}{2})$ in the equation of the normal gives

$$-\frac{1}{2} = 1 + c$$

$$c = -1\frac{1}{2}$$

$$\therefore y = 1x - 1\frac{1}{2}$$

$$\therefore y = x - 1.5$$

Shown.

Question 3(b)(ii)

At points of intersection of the curve and the normal to A

$$x^4 - \frac{1}{2}x^3 - \frac{5}{2}x^2 + \frac{3}{2}x = x - 1.5$$

$$\Rightarrow 2x^4 - x^3 - 5x^2 + 3x = 2x - 3$$

$$\Rightarrow 2x^4 - x^3 - 5x^2 + x + 3 = 0$$

We know that $(x-1)$ is a factor since the x coordinate of A is 1

$$\begin{array}{r} 2x^3 + x^2 - 4x - 3 \\ \therefore x-1 \overline{) 2x^4 - x^3 - 5x^2 + x + 3} \\ \underline{2x^4 - 2x^3} \\ x^3 - 5x^2 + x + 3 \\ \underline{x^3 - x^2} \\ -4x^2 + x + 3 \\ \underline{-4x^2 + 4x} \\ -3x + 3 \\ \underline{-3x + 3} \\ 0 \end{array}$$

$$\text{Hence, } (x-1)(2x^3 + x^2 - 4x - 3) = 0$$

$$\therefore x-1=0 \text{ or } 2x^3 + x^2 - 4x - 3 = 0$$

$$\text{Let } f(x) = 2x^3 + x^2 - 4x - 3$$

$$f(-1) = -2 + 1 + 4 - 3 = 0$$

$\therefore x+1$ is a factor

$\therefore$ B is the point where $x = -1$

$$\text{When } B = -1, \text{ gradient of curve} = \frac{dy}{dx} = 4(-1)^3 - \frac{3}{2}(-1)^2 - 5(-1) + \frac{3}{2}$$

$$\frac{dy}{dx} = -4 - \frac{3}{2} + 5 + \frac{3}{2} = 1$$

$\therefore$ gradient of tangent at B = 1

But gradient of line $y = x - 1.5$ is 1

$\therefore$ the normal is a tangent to the curve at B.

$$\text{At B, when } x = -1, y = \frac{1}{2}(2 + 1 - 5 - 3) = -\frac{5}{2} = -2.5$$

$\therefore$ coordinates of B are $(-1, -2.5)$

Question 3(c)(i)

$$A = \int_{-1}^1 \left(x^4 - \frac{1}{2}x^3 - \frac{5}{2}x^2 + \frac{3}{2}x - x + 1.5 \right) dx$$

$$A = \int_{-1}^1 \left(x^4 - \frac{1}{2}x^3 - \frac{5}{2}x^2 + \frac{1}{2}x + 1.5 \right) dx$$

Question 3(c)(ii)

$$A = \left[\frac{x^5}{5} - \frac{x^4}{8} - \frac{5x^3}{6} + \frac{x^2}{4} + 1.5x \right]_{-1}^1$$

$$A = \left(\frac{1}{5} - \frac{1}{8} - \frac{5}{6} + \frac{1}{4} + 1.5 \right) - \left(-\frac{1}{5} - \frac{1}{8} + \frac{5}{6} + \frac{1}{4} - 1.5 \right)$$

$$A = 1.73 \text{ to 2 d.p.}$$

Question 4(a) (i)

$$\text{Maximum height occurs when } \sin\left(\frac{\pi t}{3}\right) = 1$$

$$x_{\max} = 15 + 6 = 21 \text{ metres.}$$

Question 4(a) (ii)

$$\text{Height} = 9 \text{ when } \sin\left(\frac{\pi t}{3}\right) = -1$$

$$\sin\left(\frac{3\pi}{2}\right) = -1$$

$$\therefore \frac{\pi t}{3} = \frac{3\pi}{2}$$

$$\Rightarrow \frac{t}{3} = \frac{3}{2}$$

$$\Rightarrow t = 4.5 \text{ sec}$$

So platform 9 m from ground for the first time when $t = 4.5$ sec

Question 4(b)

$$6 = 15 + e^{0.04t} \sin\left(\frac{\pi t}{3}\right)$$

$$e^{0.04t} \sin\left(\frac{\pi t}{3}\right) + 9 = 0$$

Using graphics calculator and $y =$ and table, the time correct to 2 d.p. is 58.03 sec.

Question 4(c)

$$15 + e^{0.04t} \sin\left(\frac{\pi t}{3}\right) = 15$$

$$\Rightarrow e^{0.04t} \sin\left(\frac{\pi t}{3}\right) = 0$$

$$\text{But } e^{0.04t} \neq 0$$

$$\therefore \sin\left(\frac{\pi t}{3}\right) = 0 \quad 40 \leq t \leq 59$$

$$\frac{\pi t}{3} = 0, \pi, 2\pi, 3\pi, \dots$$

$$\pi t = 0, 3\pi, 6\pi, 9\pi, \dots$$

$$t = 0, 3, 6, 9, \dots$$

i.e. t is a multiple of 3

$$t = 42, 45, 48, 51, 54, 57$$

$\therefore$ the platform is exactly 15 metres above the ground 6 times in the given time frame.

Question 4(d)

$$24 = 15 + e^{0.04t} \sin\left(\frac{\pi t}{3}\right)$$

$$e^{0.04t} \sin\left(\frac{\pi t}{3}\right) - 9 = 0$$

Using graphics calculator and $y =$ and table, the time correct to the nearest second is 55 sec.

Question 4(e)(i)

$$\frac{dy}{dt} = e^{0.04t} \left(\frac{\pi}{3}\right) \cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{3}\right) 0.04e^{0.04t}$$

Question 4(e)(ii)

$$e^{0.04t} \left(\frac{\pi}{3} \cos\left(\frac{\pi t}{3}\right) + 0.04 \sin\left(\frac{\pi t}{3}\right) \right) = 0$$

$$\text{But } e^{0.04t} \neq 0$$

$$\therefore \left(\frac{\pi}{3} \cos\left(\frac{\pi t}{3}\right) + 0.04 \sin\left(\frac{\pi t}{3}\right) \right) = 0$$

$$\Rightarrow 0.04 \sin \frac{\pi t}{3} = -\frac{\pi}{3} \cos \frac{\pi t}{3}$$

$$\Rightarrow 0.04 \tan \frac{\pi t}{3} = -\frac{\pi}{3}$$

$$\Rightarrow \tan \frac{\pi t}{3} = -26.1799$$

$$\Rightarrow \frac{\pi t}{3} = 1.533, \pi - 1.533, 2\pi - 1.533, \dots, 20\pi - 1.533$$

$$\Rightarrow t = 1.54, 4.54, \dots, 58.54$$

From graphics calculator the graph can be seen to be closest to the ground when $t = 58.54$ sec.

$$\text{Distance above ground (when } t = 58.54) = 15 + e^{0.04 \times 58.54} \sin \frac{\pi \times 58.54}{3} = 4.61 \text{ metres.}$$

This can also be done using the graphics calculator.

Question 4(f)

$$\frac{dy}{dt} \leq 11.5437 \text{ (when } t = 60 \text{ from graphics calculator)}$$

$$\frac{1}{11.5437} \frac{dy}{dt} \leq 1$$

$$\frac{11}{11.5437} \frac{dy}{dt} \leq 11$$

$$\therefore a = \frac{11}{11.5437} = 0.953 \text{ to 3 d.p.}$$

End of suggested solutions 2002 VCE Mathematical Methods (CAS) Examination 2

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