

YEAR 12 Trial Exam Paper

2016

MATHEMATICAL METHODS

Written examination 1

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the exam

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Question 1a.**Worked solution**

$$\frac{dy}{dx} = 3x^2 \cos(x) - x^3 \sin(x)$$

Mark allocation: 2 marks

- 1 method mark for recognising that the product rule is to be used
- 1 answer mark for the correct answer

**Tip**

- *Look out for the product rule. There is always either a product rule or quotient rule question of this standard on Exam 1.*

Question 1b.**Worked solution**

$$f(x) = (2 - x^2)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(2 - x^2)^{-\frac{1}{2}} \times -2x$$

$$= \frac{-x}{\sqrt{2 - x^2}}$$

$$f'(1) = -\frac{1}{1} = -1$$

Alternatively:

$$\text{Let } y = f(x) = (2 - x^2)^{\frac{1}{2}}$$

$$\text{and let } u = 2 - x^2 \Rightarrow \frac{du}{dx} = -2x$$

$$y = u^{\frac{1}{2}} \Rightarrow \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$$

$$\therefore f'(x) = \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{u}} \times -2x$$

$$= \frac{-x}{\sqrt{2 - x^2}}$$

$$f'(1) = \frac{-1}{1} = -1$$

Mark allocation: 2 marks

- 1 answer mark for finding the correct derivative $\frac{-x}{\sqrt{2 - x^2}}$
- 1 answer mark for the correct answer $f'(1) = -1$

**Tip**

- Don't forget to finish the question—the question asks for $f'(1)$. Many students forget to evaluate.

Question 2**Worked solution**

$$\begin{aligned}
 & \int_4^7 \frac{3}{3x-2} dx \\
 &= 3 \int_4^7 \frac{1}{3x-2} dx \\
 &= \log_e(3x-2) \Big|_4^7 \\
 &= \log_e(19) - \log_e(10) \\
 &= \log_e\left(\frac{19}{10}\right)
 \end{aligned}$$

$$\text{So, } k = \frac{19}{10}.$$

Mark allocation: 2 marks

- 1 method mark for getting $\log_e(3x-2)$ as the integral
- 1 answer mark for $\frac{19}{10}$ or its equivalent

Question 3**Worked solution**

$$\sin\left(2\pi x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Reference angle is $\frac{\pi}{4}$, so:

$$2\pi x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}$$

$$2\pi x = 0, \frac{\pi}{2}, 2\pi$$

$$x = 0, \frac{1}{4}, 1$$

Mark allocation: 2 marks

- 1 method mark for stating reference angle is $\frac{\pi}{4}$
- 1 answer mark for $x = 0, \frac{1}{4}, 1$

**Tip**

- *You can check your answers by substituting the values back into the original equation.*

Question 4**Worked solution**

$$9^x - 9 = 8(3^x)$$

$$3^{2x} - 8(3^x) - 9 = 0$$

Let $k = 3^x$, giving:

$$k^2 - 8k - 9 = 0$$

$$(k - 9)(k + 1) = 0$$

$$3^x = 9, \quad 3^x = -1$$

$$\therefore x = 2$$

Substituting this into either equation gives $y = 9^2 - 9 = 72$.

So $A = (2, 72)$.

Mark allocation: 3 marks

- 1 method mark for substituting $k = 3^x$
- 1 method mark for factorising the quadratic
- 1 answer mark for $A = (2, 72)$

**Tip**

- *Again, check your answer by substituting back into the equations.*

Question 5**Worked solution**

$$\log_e(x) - \log_e(x+6) = -4$$

$$\log_e\left(\frac{x}{x+6}\right) = -4$$

$$e^{-4} = \frac{x}{x+6}$$

$$e^{-4}(x+6) = x$$

$$x(e^{-4} - 1) = -6e^{-4}$$

$$x = \frac{-6e^{-4}}{e^{-4} - 1} = \frac{6e^{-4}}{1 - e^{-4}}$$

Mark allocation: 2 marks

- 1 method mark for using a logarithmic law correctly
- 1 answer mark for $x = \frac{6e^{-4}}{1 - e^{-4}}$ or equivalent

Question 6a.**Worked solution**

$$f(x) = \frac{1}{5}(x^4 - 4x^3)$$

$$f'(x) = \frac{1}{5}(4x^3 - 12x^2)$$

Let $f'(x) = 0$:

$$\Rightarrow 4x^3 - 12x^2 = 0$$

$$4x^2(x - 3) = 0$$

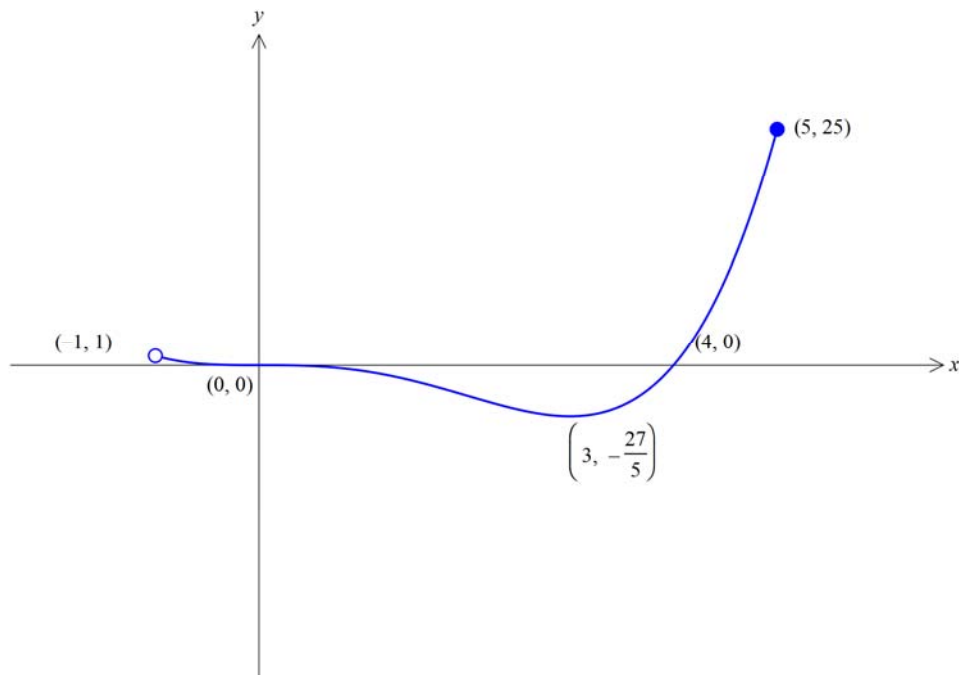
$$x = 0, x = 3$$

$$y = 0, y = \frac{-27}{5}$$

Hence, the coordinates of the stationary points are $(0, 0), \left(3, \frac{-27}{5}\right)$.

Mark allocation: 2 marks

- 1 method mark for finding the derivative and setting it to zero
- 1 answer mark for giving both coordinates

Question 6b.**Worked solution****Mark allocation: 3 marks**

- 1 method mark for the correct shape (i.e. a positive quartic)
- 1 answer mark for the correct end points
- 1 answer mark for correctly labelled intercepts and stationary points

Question 6c.**Worked solution**

$$\text{At } x = 1, f'(1) = \frac{1}{5}(4 - 12) = -\frac{8}{5}$$

$$f(1) = -\frac{3}{5}$$

$$y - y_1 = m(x - x_1)$$

$$y + \frac{3}{5} = -\frac{8}{5}(x - 1)$$

$$y + \frac{3}{5} = -\frac{8x}{5} + \frac{8}{5}$$

$$y = -\frac{8x}{5} + 1$$

Mark allocation: 2 marks

- 1 method mark for finding the derivative at $x = 1$
- 1 answer mark for the correct equation of the tangent line

Question 7**Worked solution**

$$\begin{aligned}
 f(x) &= \int \left(3\sin(x) - \cos\left(\frac{x}{2}\right) \right) dx \\
 &= \left[-3\cos(x) - 2\sin\left(\frac{x}{2}\right) + c \right] \\
 f\left(\frac{\pi}{3}\right) &= -3\cos\left(\frac{\pi}{3}\right) - 2\sin\left(\frac{\pi}{6}\right) + c = 1 \\
 \Rightarrow \frac{-3}{2} - 1 + c &= 1 \\
 c &= \frac{7}{2} \\
 f(x) &= -3\cos(x) - 2\sin\left(\frac{x}{2}\right) + \frac{7}{2}
 \end{aligned}$$

Mark allocation: 3 marks

- 1 method mark for attempting to antidifferentiate $f(x)$
- 1 method mark for attempting to find the value of c
- 1 answer mark for $f(x) = -3\cos(x) - 2\sin\left(\frac{x}{2}\right) + \frac{7}{2}$

Question 8a.**Worked solution**

Let X be the number of red fish in the sample. Hence:

$$X = 0, 1, 2, 3$$

$$\hat{p} = 0, \frac{1}{3}, \frac{2}{3}, 1$$

Mark allocation: 1 mark

- 1 answer mark for $\hat{p} = 0, \frac{1}{3}, \frac{2}{3}, 1$

**Tip**

- Remember that the distribution of the sample proportions is linked directly to the distribution of the number of red fish in the sample.

Question 8b.**Worked solution**

$$\begin{aligned} \Pr(\hat{p} > 0.25) &= \Pr(X \geq 1) \\ &= 1 - \Pr(X = 0) \\ &= 1 - \frac{8}{16} \times \frac{7}{15} \times \frac{6}{14} \\ &= 1 - \frac{1}{10} = \frac{9}{10} \end{aligned}$$

Mark allocation: 2 marks

- 1 method mark for finding $\Pr(X = 0)$
- 1 answer mark for $\frac{9}{10}$

**Tip**

- It is always easier to find $\Pr(X \geq 1)$ by finding $1 - \Pr(X = 0)$.

Question 8c.**Worked solution**

$$\Pr\left(\hat{p} = \frac{1}{3} \mid \hat{p} > \frac{1}{4}\right) = \frac{\Pr(\hat{p} = \frac{1}{3})}{\Pr(\hat{p} > \frac{1}{4})}$$

$$= \frac{\Pr(\hat{p} = \frac{1}{3})}{0.9}$$

$$\Pr(\hat{p} = \frac{1}{3}) = \Pr(X = 1)$$

$$= \frac{\binom{8}{1}\binom{8}{2}}{\binom{16}{3}} = \frac{8 \times \frac{8}{2} \times \frac{7}{1}}{\frac{16}{3} \times \frac{15}{2} \times \frac{14}{1}}$$

$$= \frac{2}{5} = 0.4$$

$$\Pr\left(\hat{p} = \frac{1}{3} \mid \hat{p} > \frac{1}{4}\right) = \frac{\Pr(\hat{p} = \frac{1}{3})}{0.9}$$

$$= \frac{0.4}{0.9} = \frac{4}{9}$$

Mark allocation: 3 marks

- 1 method mark for determining $\Pr\left(\hat{p} = \frac{1}{3} \mid \hat{p} > \frac{1}{4}\right) = \frac{\Pr(\hat{p} = \frac{1}{3})}{\Pr(\hat{p} > \frac{1}{4})}$
- 1 method mark for finding $\Pr\left(\hat{p} = \frac{1}{3}\right) = \Pr(X = 1) = 0.4$
- 1 answer mark for $\frac{4}{9}$

Question 9a.**Worked solution**

$a < 2$ as $f(x) > 0$ in order to be a probability density function.

$\int_0^a \frac{\pi}{2} \sin\left(\frac{\pi x}{2}\right) dx = 1$ since X represents a probability density function.

$$\begin{aligned} \int_0^a \frac{\pi}{2} \sin\left(\frac{\pi x}{2}\right) dx &= \left[-\cos\left(\frac{\pi x}{2}\right) \right]_0^a \\ &= -\cos\left(\frac{\pi a}{2}\right) - -\cos(0) \\ &= 1 - \cos\left(\frac{\pi a}{2}\right) \end{aligned}$$

So, $1 - \cos\left(\frac{\pi a}{2}\right) = 1$

$$\cos\left(\frac{\pi a}{2}\right) = 0$$

$$a = 1$$

Mark allocation: 2 marks

- 1 method mark for setting integral equal to one
- 1 answer mark for $a = 1$

Question 9b.**Worked solution**

$$E(X) = \int_0^1 \frac{\pi x}{2} \sin\left(\frac{\pi x}{2}\right) dx$$

$$\frac{d}{dx} \left(x \cos\left(\frac{\pi x}{2}\right) \right) = -\frac{\pi x}{2} \sin\left(\frac{\pi x}{2}\right) + \cos\left(\frac{\pi x}{2}\right)$$

$$\int \frac{d}{dx} \left(x \cos\left(\frac{\pi x}{2}\right) \right) dx = \int \left(-\frac{\pi x}{2} \sin\left(\frac{\pi x}{2}\right) + \cos\left(\frac{\pi x}{2}\right) \right) dx$$

$$\begin{aligned} \int \left(\frac{\pi x}{2} \sin\left(\frac{\pi x}{2}\right) \right) dx &= \int \cos\left(\frac{\pi x}{2}\right) dx - \left(x \cos\left(\frac{\pi x}{2}\right) \right) \\ &= \frac{2}{\pi} \sin\left(\frac{\pi x}{2}\right) - \left(x \cos\left(\frac{\pi x}{2}\right) \right) \end{aligned}$$

$$\begin{aligned} \text{So, } \int_0^1 \left(\frac{\pi x}{2} \sin\left(\frac{\pi x}{2}\right) \right) dx &= \left[\frac{2}{\pi} \sin\left(\frac{\pi x}{2}\right) - \left(x \cos\left(\frac{\pi x}{2}\right) \right) \right]_0^1 \\ &= \frac{2}{\pi} \sin \frac{\pi}{2} - \frac{2}{\pi} \sin(0) - \cos\left(\frac{\pi}{2}\right) - 0 \\ &= \frac{2}{\pi} \end{aligned}$$

Mark allocation: 3 marks

- 1 method mark for setting up $E(X) = \int_0^1 \frac{\pi x}{2} \sin\left(\frac{\pi x}{2}\right) dx$
- 1 method mark for getting $\int \left(\frac{\pi x}{2} \sin\left(\frac{\pi x}{2}\right) \right) dx = \frac{2}{\pi} \sin\left(\frac{\pi x}{2}\right) - \left(x \cos\left(\frac{\pi x}{2}\right) \right)$
- 1 answer mark for $\frac{2}{\pi}$

Question 10a.**Worked solution**

$$\frac{dy}{dx} = \frac{1}{\sqrt{a+2x}}$$

$$\text{At } x=4, \quad \frac{dy}{dx} = \frac{1}{\sqrt{a+8}}$$

$$\text{At } x=4, \quad y = \sqrt{a+8} \quad \text{and at } x=0, \quad y=0.$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\sqrt{a+8}}{4}$$

$$\text{So, } \frac{\sqrt{a+8}}{4} = \frac{1}{\sqrt{a+8}}$$

$$\Rightarrow a+8=4$$

$$a = -4$$

Mark allocation: 3 marks

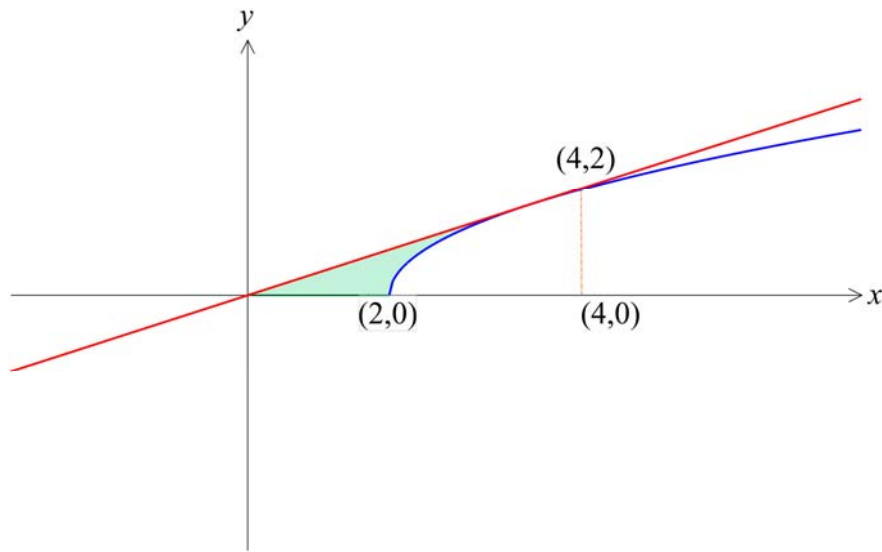
- 1 method mark for finding $\frac{dy}{dx} = \frac{1}{\sqrt{a+8}}$
- 1 method mark for setting gradient equal to the derivative at $x = a$
- 1 answer mark for $a = -4$

**Tip**

- The solution can be checked by substituting $a = -4$ into the equation $\frac{dy}{dx} = \frac{1}{\sqrt{a+8}}$. This gives $\frac{dy}{dx} = \frac{1}{2}$, for $x = 4$, so the equation of the tangent is $y = \frac{x}{2}$. The point on the curve $(4, 2)$ lies on this tangent.

Question 10b.**Worked solution**

A quick sketch produces the following.



So the shaded area can be calculated by finding the area of triangle $-\int_2^4 \sqrt{2x-4} \, dx$.

Shaded area:

$$\begin{aligned}
 &= \frac{1}{2} \times 4 \times 2 - \left[\frac{1}{3} (2x-4)^{\frac{3}{2}} \right]_2^4 \\
 &= 4 - \frac{1}{3} (4)^{\frac{3}{2}} - \frac{1}{3} (0) \\
 &= 4 - \left(\frac{8}{3} \right) = \frac{4}{3} \text{ square units}
 \end{aligned}$$

Mark allocation: 3 marks

- 1 method mark for determining the correct region
- 1 answer mark for finding $\int \sqrt{2x-4} \, dx = \frac{(2x-4)^{\frac{3}{2}}}{3}$
- 1 answer mark for $\frac{4}{3}$

**Tip**

- Always draw a sketch to determine the region required. Use areas of simple shapes to help with calculating the area.

END OF WORKED SOLUTIONS