

ICE-EM MATHEMATICS

Australian Curriculum Edition

Year **9**
Book 1

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Preface

ICE-EM Mathematics is a series of textbooks for students in years 5 to 10 throughout Australia who study the Australian Mathematics Curriculum.

Background

The International Centre of Excellence for Education in Mathematics (ICE-EM) was established in 2004 with the assistance of the Australian Government and is managed by the Australian Mathematical Sciences Institute (AMSI). The Centre originally published the series as part of a program to improve mathematics teaching and learning in Australia. AMSI is now collaborating with Cambridge University Press to publish Australian Curriculum editions of the series.

ICE-EM developed the program and textbooks in recognition of the importance of mathematics in modern society and the need to enhance the mathematical capabilities of Australian students. Students who use the series will have a strong foundation for work or further study.

You can read more about AMSI at www.amsi.org.au, and also see the mathematics involved in a variety of careers at www.mathscareers.org.au.

Features

ICE-EM Mathematics provides a progressive development from upper primary to middle secondary school. The year 10 textbooks incorporate all material for the 10A course, and selected topics in earlier books carefully prepare students for this. *ICE-EM Mathematics* is an excellent preparation for all of the Australian Curriculum's year 11 and 12 mathematics courses.

The writers of the series are some of Australia's most outstanding mathematics teachers and subject experts. The textbooks are clearly and carefully written, and contain background information, examples and worked problems.

Each chapter addresses a specific Australian Curriculum content strand and set of sub-strands. The exercises within chapters take an integrated approach to the concept of proficiency strands, rather than separating them out. Students are encouraged to develop and apply understanding, fluency, problem-solving and reasoning skills in every exercise.

The series places a strong emphasis on understanding basic ideas, along with mastering essential technical skills. Mental arithmetic and other mental processes are major focuses, as is the development of spatial intuition, logical reasoning and understanding of the concepts.

Problem-solving lies at the heart of mathematics, so *ICE-EM Mathematics* gives students a variety of different types of problems to work on, which help them develop their reasoning skills. Challenge exercises at the end of each chapter contain problems and investigations of varying difficulty that should catch the imagination and interest of students. The final chapter in each 7–10 textbook contains additional problems that cover new concepts for students who wish to explore the subject even further.

The problems and examples in the *ICE-EM Mathematics* series are written in a way that deliberately does not require the use of a calculator, except in appropriate contexts, until year 9. During primary and early secondary years, students need to become confident with mental and written calculations, using a variety of techniques. These skills are essential to students' mathematical development, and lead to a feeling of confidence and mathematical self-reliance. Furthermore, as different states have varying requirements and expectations of calculator use, the series is designed to be as calculator neutral as possible.

Additional resources

HOTmaths

Cambridge HOTmaths provides an integrated program for users of the ICE-EM Mathematics series, combining the best of textbook and interactive online resources. The presence of a HOTmaths icon (shown at the right) in the header of a chapter topic shows that resources are available for that topic. Materials are accessible from a dropdown menu in HOTmaths, organised by textbook chapter and topic/lesson structure. For more information, see www.hotmaths.com.au.



TIMES and SAM modules

The TIMES and SAM web resources were developed by the ICE-EM Mathematics author team at AMSI and are written around the structure of the Australian Curriculum. These resources have been mapped against your ICE-EM Mathematics book and are available to teachers and students by clicking on the icons in the margins in the PDF or app version of the book.

The **TIMES** (The Improving Mathematics Education in Schools) modules are written for teachers. Each module contains an in-depth discussion of a component of the curriculum along with exercises, answers, screencasts of worked examples, and notes on historical context and links forward. The **SAM** (Supporting Australian Mathematics) modules are a collection of succinct, easy-to-navigate lessons written for students in Years 7 to 9 and their teachers. Each lesson is based on an Australian Curriculum topic and includes examples and short quizzes.

Guide to the icons

The blue and green icons in the margins of this book are based on the original AMSI logo:



The blue AMSI icon indicates a resource best suited to teachers. Clicking on this icon in the PDF or app version of the book will take the teacher to the TIMES or SAM resource that matches the topic in the book.



The green AMSI icon indicates a resource best suited to students. Clicking on this icon in the PDF or app version of the book will take the student to the SAM resource that matches the topic in the book.



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David Hunt graduated from the University of Sydney in 1967 with an honours degree in Mathematics and Physics, then obtained a master's degree and a doctorate from the University of Warwick. He was appointed to a lectureship in Pure Mathematics at the University of NSW in early 1971, where he is currently an Associate Professor. David has taught courses in Pure Mathematics from first year to master's level and was Director of First Year Studies in Mathematics for five years. Many of his activities outside UNSW have centred on the Australian Mathematics Trust. He is currently Deputy Chairman of the Australian Mathematics Olympiad Committee.

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Robert McLaren graduated from the University of Melbourne in 1978 with a Bachelor of Science (Hons) and a Diploma of Education. He commenced his teaching career in 1979 at The Geelong College and has taught at a number of Victorian Independent Schools throughout his career. He has been involved in textbook writing, curriculum development and VCE examination setting and marking during his teaching life. He has taught mathematics at all secondary levels and has a particular interest in problem solving. Robert is currently Head of Upper School at Scotch College.

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Brian Woolacott graduated from the University of Melbourne in 1978 with a Bachelor of Science and a Diploma of Education. In 1979 he started his teaching career at Scotch College, Melbourne, and during his career he has taught at all secondary levels. For 13 years Brian was the Co-ordinator of Mathematics for Years 9 and 10 and during this time he was involved in co-authoring a number of textbooks for the Year 9 and 10 levels. Brian is currently the Dean of Studies at Scotch College.

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- ACMNA 212
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CHAPTER

1

Number and Algebra

Algebra

Algebra is the language of mathematics. The development of algebra revolutionised mathematics. It is used by engineers, architects, applied scientists and economists to solve practical problems. Every time you see a formula, algebra is being used.

We are now going to review and consolidate our knowledge of algebra so that we will be able to use it in more advanced work.

1A Substitution



Substitution occurs when a pronumeral is replaced by a numerical value. This allows the term or expression to be evaluated.

When evaluating an expression, it is important to remember the order in which operations are to be applied.



Order of operations

- Evaluate expressions inside brackets first.
- In the absence of brackets, carry out operations in the following order:
 - powers
 - multiplication and division from left to right
 - addition and subtraction from left to right.

Example 1

- a** Evaluate $2x$ when $x = 3$.
b Evaluate $5a + 2b$ when $a = 2$ and $b = -3$.
c Evaluate $2p(3p + q)$ when $p = 1$ and $q = -2$.

Solution

$$\begin{aligned}\mathbf{a} \quad 2x &= 2 \times 3 \\ &= 6\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad 5a + 2b &= 5 \times 2 + 2 \times (-3) \\ &= 10 - 6 \\ &= 4\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad 2p(3p + q) &= 2 \times 1 \times [3 \times 1 + (-2)] \\ &= 2 \times 1 \\ &= 2\end{aligned}$$

Exercise 1A

Example 1a

- 1** Evaluate $3a$ when $a = 7$.

- 2** Evaluate $12x$ when $x = -5$.

Example 1b

- 3** Evaluate $7m - 4n$ when:

a $m = 3$ and $n = 5$

b $m = 2$ and $n = -5$

c $m = -1$ and $n = 2$

d $m = -3$ and $n = -2$

4 Evaluate $4x + 9y$ when:

a $x = 6$ and $y = 5$

b $x = -6$ and $y = 5$

c $x = 6$ and $y = -5$

d $x = -6$ and $y = -5$

5 Evaluate $p - 2q$ when:

a $p = 7$ and $q = 2$

b $p = 7$ and $q = -2$

c $p = -7$ and $q = 2$

d $p = -7$ and $q = -2$

6 Evaluate $a + 2b - 3c$ when:

a $a = 3$, $b = 5$ and $c = 2$

b $a = 3$, $b = -5$ and $c = 2$

c $a = -3$, $b = 5$ and $c = 2$

d $a = 3$, $b = 5$ and $c = -2$

e $a = -3$, $b = -5$ and $c = 2$

f $a = 3$, $b = -5$ and $c = -2$

7 Evaluate $2x - 3y$ when:

a $x = \frac{1}{2}$ and $y = \frac{3}{4}$

b $x = -\frac{1}{3}$ and $y = \frac{1}{6}$

c $x = \frac{2}{5}$ and $y = -\frac{1}{4}$

d $x = -\frac{2}{3}$ and $y = -\frac{3}{4}$

8 Evaluate $a + 2b - 3c$ when:

a $a = 0.3$, $b = 0.5$ and $c = 0.2$

b $a = 1.3$, $b = -0.5$ and $c = 1.2$

c $a = -2.3$, $b = 1.5$ and $c = 0.2$

d $a = 2.3$, $b = 2.5$ and $c = -2.3$

9 Evaluate $p^2 - 2q$ when:

a $p = 3$ and $q = 2$

b $p = 7$ and $q = -2$

c $p = -7$ and $q = 2$

d $p = -7$ and $q = -2$

e $p = -\frac{1}{3}$ and $q = \frac{5}{6}$

f $p = -\frac{2}{5}$ and $q = \frac{1}{4}$

Example 1c

10 Evaluate $2m(m - 3n)$ when:

a $m = 3$ and $n = 5$

b $m = 2$ and $n = -5$

c $m = -1$ and $n = 2$

d $m = -3$ and $n = -2$

e $m = \frac{1}{2}$ and $n = \frac{1}{3}$

f $m = \frac{1}{3}$ and $n = \frac{1}{2}$

11 Evaluate $a(2b - c)$ when:

a $a = 3, b = 2$ and $c = 1$

c $a = 5, b = -3$ and $c = 4$

e $a = \frac{1}{2}, b = \frac{1}{4}$ and $c = \frac{1}{3}$

b $a = -2, b = 3$ and $c = -1$

d $a = 2, b = 7$ and $c = -3$

f $a = \frac{1}{3}, b = -\frac{1}{2}$ and $c = -\frac{1}{4}$

12 Evaluate $\frac{x+y}{3}$ when:

a $x = 6$ and $y = 5$

c $x = 6$ and $y = -5$

b $x = -6$ and $y = 5$

d $x = -6$ and $y = -5$

13 Evaluate $\frac{p+2q}{3r}$ when:

a $p = 7, q = 4$ and $r = 2$

c $p = -7, q = 4$ and $r = 4$

b $p = 7, q = -2$ and $r = 2$

d $p = -7, q = -2$ and $r = 2$

1B Like terms



Demitri has three pencil cases, each containing the same number, x , of pencils. So he has a total of $3x$ pencils. If he gets two more pencil cases with x pencils in each, then he has $3x + 2x = 5x$ pencils in total.

$3x$ and $2x$ are said to be **like terms**.

If Lee has x packets of chocolates each containing y chocolates, then she has $x \times y = xy$ chocolates. If David has twice as many chocolates as Lee, he has $2 \times xy = 2xy$ chocolates.

Together they have $2xy + xy = 3xy$ chocolates.

$2xy$ and xy are **like terms**. The pronumerals are the same and have the same indices. The **coefficients** of these terms are 2 and 1.

The terms $2x$ and $3y$ are not like terms.

Similarly, the terms $2x^2y$ and $3xy^2$ are not like terms since the indices of x and y differ.

Example 2

Which of the following are pairs of like terms?

- | | | |
|------------------------------|--------------------------|------------------------------|
| a $3x$ and $2x$ | b $3m$ and $2n$ | c $3x^2$ and $3x$ |
| d $2x^2y$ and $3yx^2$ | e $2mn$ and $3nm$ | f $5x^2y$ and $6y^2x$ |

Solution

- a** $3x$ and $2x$ are like terms.
- b** $3m$ and $2n$ are not like terms. (The pronumerals are different.)
- c** $3x^2$ and $3x$ are not like terms. (The indices of x are different.)
- d** $2x^2y$ and $3yx^2$ are like terms. (The multiplication order is not important.)
- e** $2mn$ and $3nm$ are like terms. (The multiplication order is not important.)
- f** $5x^2y$ and $6y^2x$ are not like terms. (x^2y is not the same as xy^2)



Like terms

- **Like terms** contain the same pronumerals, with each pronumeral having the same index.
- We can add and subtract like terms.
- Unlike terms cannot be added or subtracted to form a single term.

Consider the following examples.

Example 3

Simplify each expression, if possible.

a $4a + 7a$

b $3x^2y + 4x^2y - 2x^2y$

c $5m + 6n$

d $9b + 2c - 3b + 6c$

e $3z + 5yx - z - 6yx$

f $6x^3 - 4x^2 + 5x^3$

Solution

a $4a + 7a = 11a$

b $3x^2y + 4x^2y - 2x^2y = 5x^2y$

c $5m + 6n$. No simplification is possible because $5m$ and $6n$ are unlike terms.

d $9b + 2c - 3b + 6c = 9b - 3b + 2c + 6c$
 $= 6b + 8c$

e $3z + 5yx - z - 6yx = 3z - z + 5xy - 6xy$
 $= 2z - xy$

f $6x^3 - 4x^2 + 5x^3 = 6x^3 + 5x^3 - 4x^2$
 $= 11x^3 - 4x^2$

Example 4

Simplify each expression.

a $\frac{x}{2} + \frac{x}{3}$

b $\frac{3x}{4} - \frac{2x}{5}$

Solution

a $\frac{x}{2} + \frac{x}{3} = \frac{3x}{6} + \frac{2x}{6}$
 $= \frac{5x}{6}$

b $\frac{3x}{4} - \frac{2x}{5} = \frac{15x}{20} - \frac{8x}{20}$
 $= \frac{7x}{20}$

Exercise 1B

Example 2

1 Which of the following are pairs of like terms?

a $11a$ and $4a$

b $6b$ and $-2b$

c $12m$ and $5m$

d $14p$ and $5p$

e $7p$ and $-3q$

f $-6a$ and $7b$

- g** $4mn$ and $7mn$ **h** $3pq$ and $-2pq$ **i** $6ab$ and $-7b$
j $4ab$ and $5a$ **k** $11a^2$ and $5a^2$ **l** $6x^2$ and $-7x^2$
m $4b^2$ and $-3b$ **n** $2y^2$ and $2y$ **o** $-5a^2b$ and $9a^2b$
p $6mn^2$ and $11mn^2$ **q** $6\ell^2s$ and $17\ell s^2$ **r** $12d^2e$ and $14de^2$
s $-4x^2y^2$ and $12x^2y^2$ **t** $-5a^2bc^2$ and $8a^2bc^2$ **u** $6a^2bc$ and $6a^2bc^2$

Example 3a,b

2 Simplify each expression by collecting like terms.

- a** $7a + 2a$ **b** $8b + 3b$ **c** $11c - 6c$
d $15d - 8d$ **e** $6x^2 + 4x^2$ **f** $9a^2 - 6a^2$
g $4a + 5a - 6a$ **h** $7f - 3f + 9f$ **i** $6m - 4m - 9m$
j $8p + 10p - 17p$ **k** $9a^2 + 5a^2 - 12a^2$ **l** $17m^2 - 14m^2 + 8m^2$
m $6ab + 9ab - 12ab$ **n** $3mn - 7mn + 10mn$
o $15xy - 8xy - 20xy$ **p** $14a^2d - 10a^2d - 6a^2d$

3 Copy and complete:

- a** $2a + \dots = 7a$ **b** $5b - \dots = 2b$
c $8mn + \dots = 12mn$ **d** $11pq - \dots = 6pq$
e $4x^2 + \dots = 7x^2$ **f** $6m^2 - \dots = m^2$
g $8ab - \dots = -2ab$ **h** $5m^2n - \dots = -6m^2n$
i $-7a^2b + \dots = a^2b$ **j** $-6\ell m + \dots = \ell m$

Example 3d,e

4 Simplify each expression by collecting like terms.

- a** $2a + 4b + 3a + 7b$ **b** $5c - 2d + 8c + 6d$
c $7m + 2n + 5m - n$ **d** $8p + 6 + 3p - 2$
e $9 - 2m + 5 + 7m$ **f** $8m - 6n - 2n - 7m$
g $10ab + 11b - 12b + 3ab$ **h** $6a^2 - 2b + 7ab - 9b$
i $-3x^2 + 5x - 2x^2 + 7x$ **j** $3a^2 - 2a + 4a^2 + 5a$
k $5m^2 - 3m - 2m + 6m^2$ **l** $4p^2 - 3p - 8p - 3p^2$

Example 3f

5 Simplify each expression by collecting like terms.

- a** $45xy^2 + 32xy^2 + 16xy^2$ **b** $19xy + 6yx - 4xy$
c $8xy^2 + 9xy^2 - y^2x$ **d** $-4x^2 + 3x^2 - 3y - 7y$
e $6v^2z - 11z + 7v^2z - 14z$ **f** $6yz - 11x + 10zy + 15x$
g $7x^3 + 6x^2 - 4y^3 - x^2$ **h** $8x^2 - 12x^2 + y^2 + 12y^2$
i $2x^2 + x^2 - 5xy + 7yx$ **j** $-3ab^2 + 4a^2b - 5ab^2 + a^2b$

6 Simplify each expression.

a $\frac{x}{4} + \frac{x}{3}$

b $\frac{a}{2} + \frac{a}{5}$

c $\frac{b}{7} + \frac{b}{3}$

d $\frac{z}{5} + \frac{z}{3}$

e $\frac{c}{6} + \frac{c}{7}$

f $\frac{z}{5} + \frac{z}{8}$

g $\frac{z}{3} + z$

h $\frac{c}{5} + \frac{c}{10}$

i $\frac{x}{4} + x$

7 Simplify each expression.

a $\frac{z}{2} - \frac{z}{3}$

b $\frac{z}{3} - \frac{z}{5}$

c $\frac{x}{7} - \frac{x}{8}$

d $\frac{x}{5} - \frac{x}{10}$

e $\frac{x}{2} - \frac{x}{7}$

f $\frac{x}{3} - \frac{x}{7}$

g $x - \frac{x}{3}$

h $\frac{x}{4} - \frac{x}{8}$

i $c - \frac{c}{7}$

8 Simplify each expression.

a $\frac{2x}{3} + \frac{x}{4}$

b $\frac{5x}{7} + \frac{x}{3}$

c $\frac{3x}{4} + \frac{x}{2}$

d $\frac{5x}{3} + \frac{x}{2}$

e $\frac{7x}{11} - \frac{x}{2}$

f $\frac{2x}{3} - \frac{x}{2}$

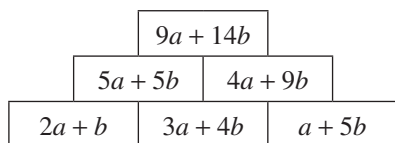
g $\frac{5x}{11} - \frac{2x}{3}$

h $\frac{7x}{3} + \frac{5x}{4}$

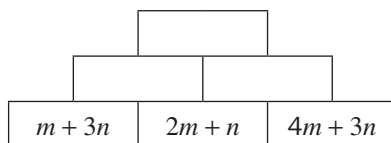
i $2x - \frac{7x}{4}$

9 In each part, the expression in each box is obtained by adding the expressions in the two boxes directly below it. Fill in the contents of each box. (The first one has been done for you.)

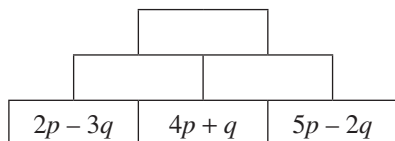
a



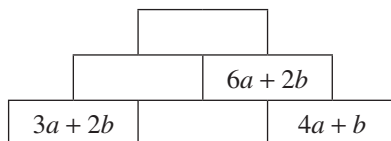
b



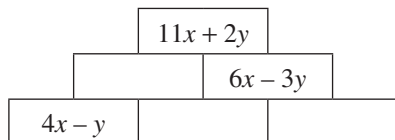
c



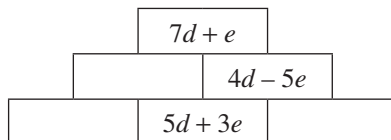
d



e



f





Any two terms, like or unlike, can be multiplied together to produce a single term. This is different from addition and subtraction, where only like terms can be combined into a single term.

For a single term, use alphabetical order when you simplify your answer.

For example, write $3abc$ instead of $3bca$, and write $4x^2yz$ instead of $4yx^2z$.

Example 5

Simplify each expression.

a $4 \times 3a$

b $2d \times 5e$

c $4m \times 5m$

d $3p \times 2pq$

e $3x \times (-6)$

f $-5ab \times (-3bc)$

Solution

a $4 \times 3a = 12a$

b $2d \times 5e = 10de$

c $4m \times 5m = 20m^2$

d $3p \times 2pq = 6p^2q$

e $3x \times (-6) = -18x$

f $-5ab \times (-3bc) = 15ab^2c$

Example 6

Simplify each expression.

a $24x \div 6$

b $3 \times 12a \div 4$

c $36a^2 \div 9$

d $-18x^2 \div (-3)$

Solution

a $24x \div 6 = 4x$

b $3 \times 12a \div 4 = 36a \div 4$
 $= 9a$

c $36a^2 \div 9 = 4a^2$

d $-18x^2 \div (-3) = 6x^2$

Example 7

Simplify each expression.

a $\frac{15a}{3}$

b $\frac{12x}{4}$

c $-\frac{24x^2}{6}$

(continued on next page)

Solution

$$\mathbf{a} \quad \frac{15a}{3} = 5a$$

$$\mathbf{b} \quad \frac{12x}{4} = 3x$$

$$\mathbf{c} \quad -\frac{24x^2}{6} = -4x^2$$



Multiplication and division

- To multiply fractions, multiply the numerators and multiply the denominators.
- When the numerator and denominator have a common factor, the factor should be cancelled; that is, the numerator and the denominator should be divided by the common factor.

Example 8

Rewrite each expression as a single fraction.

$$\mathbf{a} \quad \frac{2a}{5} \times \frac{a}{4}$$

$$\mathbf{b} \quad \frac{3x}{7} \times \frac{5y}{12}$$

$$\mathbf{c} \quad \frac{4p}{9} \times \frac{3}{2p}$$

$$\mathbf{d} \quad \frac{15}{x} \times \frac{2}{3x}$$

Solution

$$\mathbf{a} \quad \frac{2a}{5} \times \frac{a}{4} = \frac{\cancel{2} \times a \times a}{5 \times \cancel{4}_2}$$

(Multiply the numerators and the denominators.)

$$= \frac{a \times a}{5 \times 2}$$

(Divide 2 into the numerator and the denominator.)

$$= \frac{a^2}{10}$$

b $\frac{3x}{7} \times \frac{5y}{12} = \frac{\cancel{3} \times x \times 5 \times y}{7 \times \cancel{12}_4}$ (Multiply the numerators and the denominators.)
 $= \frac{5 \times x \times y}{7 \times 4}$
 $= \frac{5xy}{28}$

c $\frac{4p}{9} \times \frac{3}{2p} = \frac{\overset{2}{\cancel{4}} \times \cancel{p} \times \cancel{3}}{\underset{3}{\cancel{9}} \times \cancel{2} \times \cancel{p}}$ (cancel 2, 3 and p)
 $= \frac{2}{3}$

d $\frac{15}{x} \times \frac{2}{3x} = \frac{\overset{5}{\cancel{15}} \times 2}{x \times \cancel{3} \times x}$
 $= \frac{5 \times 2}{x \times x}$
 $= \frac{10}{x^2}$

Recall that to divide by a fraction, we multiply by its reciprocal.

Example 9

Rewrite each expression as a single fraction.

a $\frac{2x}{3} \div \frac{3x}{5}$

b $\frac{6a}{7b} \div \frac{2ab}{3}$

Solution

a $\frac{2x}{3} \div \frac{3x}{5} = \frac{\cancel{2x}}{3} \times \frac{5}{\cancel{3x}}$
 $= \frac{2 \times 5}{3 \times 3}$
 $= \frac{10}{9}$

b $\frac{6a}{7b} \div \frac{2ab}{3} = \frac{\overset{3}{\cancel{6}} \cancel{a}}{7b} \times \frac{3}{\cancel{2} \cancel{a} b}$
 $= \frac{3 \times 3}{7 \times b \times b}$
 $= \frac{9}{7b^2}$

Your answer should always be expressed in simplest form – that is, the highest common factor of the numerator and denominator is one.



Exercise 1C

Example 5

1 Simplify:

- | | | | |
|-----------------------------|--------------------------------|----------------------------|-------------------------------|
| a $2 \times 3a$ | b $4b \times 5$ | c $7c \times 3$ | d $8d \times (-9)$ |
| e $4a \times 3b$ | f $5c \times 2d$ | g $4f \times (-5g)$ | h $-3m \times 4n$ |
| i $-2p \times (-3q)$ | j $-6\ell \times (-5n)$ | k $a \times a$ | l $m \times m$ |
| m $2a \times 4a$ | n $5b \times 6b$ | o $-3a \times 5a$ | p $-2m \times (-4m)$ |
| q $7a \times 8ab$ | r $2p \times 3pq$ | s $-2mn \times 5n$ | t $-6cd \times (-2de)$ |

2 Copy and complete:

- | | | |
|-------------------------------------|---|--|
| a $8a \times \dots = 16a$ | b $9b \times \dots = 18b$ | c $8a \times \dots = 16ab$ |
| d $5m \times \dots = 15mn$ | e $3a \times \dots = 12a^2$ | f $6p \times \dots = 30p^2$ |
| g $-5b \times \dots = 10b^2$ | h $-3\ell \times \dots = 24\ell^2$ | i $4m \times \dots = 12m^2n$ |
| j $4d \times \dots = 28d^2e$ | k $-3ab \times \dots = 15ab^2c$ | l $-2de \times \dots = 10d^2ef$ |

Example 6

3 Simplify:

- | | | |
|---------------------------------|--------------------------------|---------------------------------|
| a $15x \div 5$ | b $27y \div 3$ | c $24a^2 \div 8$ |
| d $32m \div 16$ | e $3 \times 12t \div 9$ | f $7 \times 15p \div 21$ |
| g $21t \div 12 \times 4$ | h $24x \div 8 \times 3$ | i $18y \div 6 \times 2$ |
| j $-18x^2 \div 9$ | k $-16a^2 \div (-4)$ | l $-24x^2 \div (-8)$ |

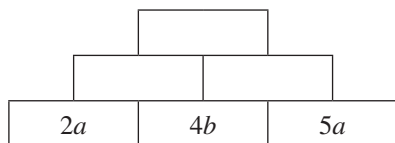
Example 7

4 Simplify each expression by cancelling common factors.

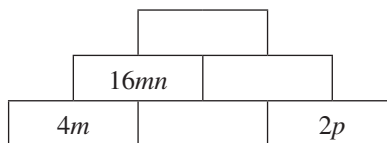
- | | | | | |
|----------------------------|---------------------------|----------------------------|----------------------------|-----------------------------|
| a $\frac{4x}{6}$ | b $\frac{3a}{9}$ | c $-\frac{12m}{18}$ | d $\frac{14p}{21}$ | e $\frac{22x^2}{33}$ |
| f $\frac{15xy}{20}$ | g $-\frac{3ab}{b}$ | h $\frac{12ab}{4a}$ | i $\frac{2xy}{6xy}$ | j $-\frac{4xy}{8x}$ |

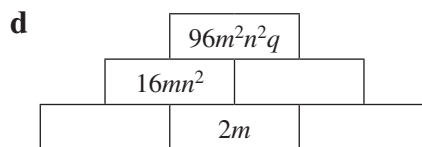
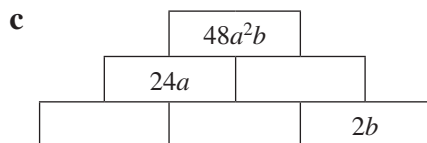
5 In each part, the expression in each box is obtained by multiplying together the terms in the two boxes directly below it. Fill in the empty boxes.

a



b





Example 8

6 Rewrite each expression as a single fraction. Cancel common factors first.

a $\frac{2b}{3} \times \frac{9}{4}$

b $\frac{3x}{5} \times \frac{2}{3}$

c $\frac{2y}{5} \times \frac{y}{4}$

d $\frac{3b}{4a} \times \frac{2ab}{9}$

e $\frac{2}{5a} \times \frac{1}{4a}$

f $\frac{p}{6q} \times \frac{9p}{4q}$

g $\frac{20x}{3y} \times \frac{6}{5xy}$

h $\frac{14mp}{9p} \times \frac{3np}{7p}$

i $\frac{2yz}{5xz} \times \frac{3xy}{4yz}$

Example 9

7 Rewrite each expression as a single fraction. Simplify your answer by cancelling common factors.

a $\frac{2b}{3} \div \frac{4}{9}$

b $\frac{3x}{5} \div \frac{3}{4}$

c $\frac{2y}{5} \div \frac{y}{4}$

d $\frac{p}{6} \div \frac{9p}{4}$

e $\frac{5}{6a} \div \frac{1}{4a}$

f $\frac{p}{6q} \div \frac{4p}{9q}$

g $\frac{8x}{5} \div 4$

h $\frac{9y}{2} \div 18$

i $\frac{3a}{4} \div \frac{5a}{2}$

j $\frac{4m}{5n} \div \frac{12mn}{7}$

k $\frac{5p}{6} \div \left(-\frac{10p}{3}\right)$

l $-\frac{9y}{4xy} \div \frac{3x}{16y}$



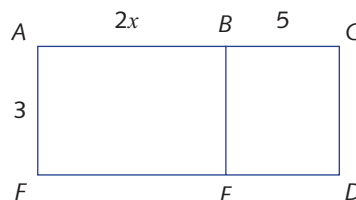
This section looks at how algebraic products that contain brackets can be rewritten in different forms. Removing brackets is called **expanding**.

We are familiar with the distributive law for numbers from Year 7 and Year 8.

The distributive law can be illustrated using a diagram, as shown below.

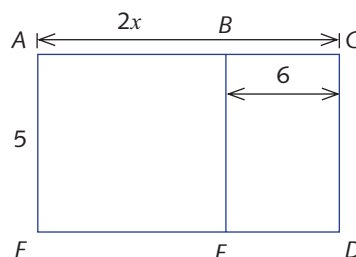
Area of rectangle $ACDF$ = area of rectangle $ABEF$ + area of rectangle $BCDE$

$$\begin{aligned}\text{That is, } 3(2x + 5) &= 3 \times 2x + 3 \times 5 \\ &= 6x + 15\end{aligned}$$



Area of rectangle $ABEF$ = area of rectangle $ACDF$ – area of rectangle $BCDE$

$$\begin{aligned}\text{That is, } 5(2x - 6) &= 5 \times 2x - 5 \times 6 \\ &= 10x - 30\end{aligned}$$



Simple expansion of brackets

- In general: $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$
- Each term in the brackets is multiplied by the term outside the brackets.

Example 10

Expand the brackets.

a $2(a + 3)$

b $3(x - 2)$

c $4(2m - 7)$

d $p(p + n)$

e $-2(x - 4)$

f $-5(3x + y)$

Solution

a $2(a + 3) = 2a + 6$

b $3(x - 2) = 3x - 6$

c $4(2m - 7) = 8m - 28$

d $p(p + n) = p^2 + np$

e $-2(x - 4) = -2x + 8$

f $-5(3x + y) = -15x - 5y$

Example 11

Expand and simplify each expression.

a $5(a + 1) + 6$

b $4(2b - 1) + 7$

c $6(d + 5) - 3d$

Solution

a $5(a + 1) + 6 = 5a + 5 + 6$
 $= 5a + 11$

b $4(2b - 1) + 7 = 8b - 4 + 7$
 $= 8b + 3$

c $6(d + 5) - 3d = 6d + 30 - 3d$
 $= 3d + 30$

The distributive law can be used to simplify expressions that initially appear quite complicated.

Example 12

Expand and collect like terms for each expression.

a $2(b + 5) + 3(b + 2)$

b $3(x - 2) - 2(x + 1)$

c $5(a + 1) - 2(a - 4)$

d $3a(2a + 3b) + 2a(3a - 2b)$

Solution

a $2(b + 5) + 3(b + 2) = 2b + 10 + 3b + 6$
 $= 5b + 16$

b $3(x - 2) - 2(x + 1) = 3x - 6 - 2x - 2$
 $= x - 8$

c $5(a + 1) - 2(a - 4) = 5a + 5 - 2a + 8$
 $= 3a + 13$

Note: When expanding the second bracket, be very careful.
The -2 is multiplied by each term in the second bracket, and
 $-2 \times (-4) = +8$.

d $3a(2a + 3b) + 2a(3a - 2b) = 6a^2 + 9ab + 6a^2 - 4ab$
 $= 12a^2 + 5ab$

Example 13

Expand and simplify each expression.

a $\frac{3}{5} \left(6x + \frac{7}{3} \right)$

b $\frac{4}{3} (6x + 11) + \frac{2}{3}$

Solution

$$\begin{aligned} \mathbf{a} \quad \frac{3}{5} \left(6x + \frac{7}{3} \right) &= \frac{3}{5} \times 6x + \frac{\cancel{3}^1}{5} \times \frac{7}{\cancel{3}_1} \\ &= \frac{18x}{5} + \frac{7}{5} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \frac{4}{3} (6x + 11) + \frac{2}{3} &= \frac{4}{3} \times 6x + \frac{4}{3} \times 11 + \frac{2}{3} \\ &= 8x + \frac{44}{3} + \frac{2}{3} \\ &= 8x + \frac{46}{3} \end{aligned}$$

Note: It is neither necessary nor desirable to write $\frac{46}{3}$ as $15\frac{1}{3}$ in an algebraic expression.

Exercise 1D

Example 10a,b,c

- 1** Expand each expression using the distributive law. For parts **a**, **e**, **i** and **m** check your answers by substituting values for the pronumeral.

a $5(x + 3)$	b $2(b + 7)$	c $4(a + 6)$	d $7(d + 3)$
e $3(2a + 1)$	f $5(3d + 7)$	g $4(3b + 2)$	h $6(2d + 5)$
i $4(a - 7)$	j $3(b - 5)$	k $6(d - 2)$	l $8(f - 4)$
m $2(4f - 5)$	n $3(2g - 6)$	o $5(3p - 2)$	p $6(5q - 1)$

Example 10e,f

- 2** Expand each expression using the distributive law. For parts **a**, **e**, **i** and **m** check your answers by substituting values for the pronumeral.

a $-2(a + 4)$	b $-3(b + 6)$	c $-4(m + 5)$	d $-5(p + 7)$
e $-6(2a + 7)$	f $-3(2p + 1)$	g $-3(4b + 9)$	h $-5(4m + 5)$
i $-2(3a - 1)$	j $-6(4b - 7)$	k $-5(2b - 7)$	l $-7(3b - 2)$
m $-4(3b - 5)$	n $-5(4b - 7)$	o $-9(3x + 2)$	p $-12(4y - 5)$

3 Expand each expression using the distributive law.

a $\frac{1}{2}(2x + 6)$

b $\frac{1}{3}(9x + 18)$

c $\frac{1}{4}(16a + 8)$

d $\frac{1}{2}(10a + 16)$

e $\frac{2}{3}(12p + 6)$

f $\frac{3}{4}(20q + 40)$

g $\frac{1}{3}(12t - 9)$

h $\frac{1}{2}(6x - 24)$

i $-\frac{1}{2}(10\ell - 6)$

j $-\frac{1}{4}(16p - 20)$

k $-\frac{2}{3}(9m - 15)$

l $-\frac{4}{5}(25n - 100)$

4 Expand:

a $\frac{2}{3}\left(6x + \frac{3}{4}\right)$

b $\frac{1}{2}\left(5y + \frac{2}{5}\right)$

c $\frac{3}{5}\left(\frac{x}{6} + \frac{1}{3}\right)$

d $\frac{2}{7}\left(\frac{3x}{4} + \frac{5}{6}\right)$

e $\frac{1}{3}\left(\frac{y}{2} - \frac{3}{4}\right)$

f $-\frac{3}{5}\left(\frac{a}{3} - \frac{2}{3}\right)$

Example 10d

5 Expand:

a $a(a + 4)$

b $b(b + 7)$

c $c(c - 5)$

d $d(d - 9)$

e $e(3e + 1)$

f $f(5f + 6)$

g $2g(3g - 5)$

h $4h(5h - 7)$

i $2i(5i + 7)$

j $3j(4j + 7)$

k $-k(5k - 4)$

l $-\ell(3\ell - 1)$

m $-2m(5m - 4)$

n $-3n(5n + 7)$

o $-4x(3x - 5)$

6 Expand:

a $3a(2a + b)$

b $4c(2c - d)$

c $5d(2d - 4e)$

d $2p(3q - 5r)$

e $-3x(2x + 5y)$

f $-2z(3z - 4y)$

g $2a(3 + 4ab)$

h $5m(2m - 4n)$

i $5x(2xy + 3)$

j $3p(2 - 5pq)$

k $-6y(2x - 3y)$

l $-10b(3a - 7b)$

7 A student expanded brackets and obtained the following answers. In each part, identify and correct the student's incorrect answer and write the correct expansion.

a $4(a + 6) = 4a + 6$

b $5(a + 1) = 5a + 6$

c $8(p - 7) = 8p + 56$

d $-3(p - 5) = -3p - 15$

e $a(a + b) = 2a + ab$

f $2m(3m + 5) = 6m + 10m$

g $4a(3a + 5) = 7a^2 + 20a$

h $3a(4a - 7) = 12a^2 - 7$

i $-6(x - 5) = 6x + 30$

j $3x(2x - 7y) = 6x^2 - 21y$

Example 11

8 Expand and collect like terms for each expression.

a $8(a + 2) + 7$

b $5(b + 3) + 10$

c $2(c + 7) - 9$

d $3(d + 5) - 12$

e $2(e - 5) + 15$

f $3(f - 4) - 6$

$$\begin{array}{lll}
 \mathbf{g} & 5(g+2)+8g & \mathbf{h} & 4(h+1)+3h & \mathbf{i} & 6(i-5)-3i \\
 \mathbf{j} & 2(j-3)-j & \mathbf{k} & 2a(4a+3)+7a & \mathbf{l} & 5b(2b-3)+6b \\
 \mathbf{m} & 2a(4a+3)+7a^2 & \mathbf{n} & 3b(3b-5)-7b^2 & \mathbf{o} & 5(2-4p)+20p \\
 \mathbf{p} & 4(1-3q)+15q & \mathbf{q} & 2a(3a+2b)-6a^2 & \mathbf{r} & 7m(4m-3n)+30mn
 \end{array}$$

- 9** Expand and collect like terms for each expression. Express the answer as a single fraction.

$$\begin{array}{lll}
 \mathbf{a} & \frac{2}{3}(x+3)+\frac{x}{6} & \mathbf{b} & \frac{1}{4}(x+2)+\frac{x}{3} & \mathbf{c} & \frac{3}{5}(x-1)+\frac{2x}{3} \\
 \mathbf{d} & \frac{5}{6}(x-4)+\frac{3x}{4} & \mathbf{e} & \frac{3}{7}(3x+5)+\frac{x}{3} & \mathbf{f} & \frac{1}{2}(4x-1)+\frac{2x}{5} \\
 \mathbf{g} & \frac{3}{4}(2x+1)-\frac{x}{3} & \mathbf{h} & -\frac{1}{2}(3x+2)-\frac{2x}{5} & \mathbf{i} & -\frac{2}{3}(4x-3)-\frac{x}{4}
 \end{array}$$

Example 12

- 10** Expand and collect like terms for each expression.

$$\begin{array}{ll}
 \mathbf{a} & 2(y+1)+3(y+4) \\
 \mathbf{b} & 3(x-1)+4(x+3) \\
 \mathbf{c} & 5(a-2)+2(a-1) \\
 \mathbf{d} & 2(3b-2)+5(2b-1) \\
 \mathbf{e} & 3(a+5)-2(a+7) \\
 \mathbf{f} & 5(b-2)-4(b+3) \\
 \mathbf{g} & 2(3y-4)-3(2y-1) \\
 \mathbf{h} & x(x-2)+3(x-2) \\
 \mathbf{i} & 2p(p+1)-5(p+1) \\
 \mathbf{j} & 4y(3y-5)-3(3y-5) \\
 \mathbf{k} & 2x(2x+1)-7(x+1) \\
 \mathbf{l} & 2p(3p+1)-4(2p+1) \\
 \mathbf{m} & 3z(z+4)-z(3z+2) \\
 \mathbf{n} & 3y(y-4)+y(y-4) \\
 \mathbf{o} & 4z(4z-2)-z(z+2) \\
 \mathbf{p} & 5y(y+6)+y(y+6)
 \end{array}$$

- 11 a** Copy and complete:

$$\mathbf{i} \quad 12 \times 99 = 12 \times (100 - 1) = 12 \times \dots - 12 \times \dots = \dots - \dots = \dots$$

$$\mathbf{ii} \quad 14 \times 53 = 14 \times (50 + \dots) = 14 \times \dots + 14 \times \dots = \dots + \dots = \dots$$

- b** Use a similar technique to the one used in part **a** to calculate the value of each product.

$$\mathbf{i} \quad 14 \times 21$$

$$\mathbf{ii} \quad 17 \times 101$$

$$\mathbf{iii} \quad 70 \times 29$$

$$\mathbf{iv} \quad 8 \times 121$$

$$\mathbf{v} \quad 13 \times 72$$

$$\mathbf{vi} \quad 17 \times 201$$

- 12** Expand:

$$\mathbf{a} \quad x(x^2+3)$$

$$\mathbf{b} \quad x(x^2+2x+1)$$

$$\mathbf{c} \quad 2x(x^2-3x)$$

$$\mathbf{d} \quad 2x^2(3-x)$$

$$\mathbf{e} \quad 5a(3a+1)$$

$$\mathbf{f} \quad 6a^2(1+2a-a^2)$$

1E Binomial products



In the previous section, we learnt how to expand a product containing one pair of brackets, such as $3(2x - 4)$. Now we are going to learn how to expand a product with two pairs of brackets, such as $(x + 2)(x + 5)$. Such expressions are called **binomial products**.

We first look at expanding the expression $(a + 2)(b + 5)$ using the distributive law.

The expression in the second pair of brackets, $(b + 5)$, is multiplied by both a and 2 , giving:

$$\begin{aligned}(a + 2)(b + 5) &= a(b + 5) + 2(b + 5) \\ &= ab + 5a + 2b + 10\end{aligned}$$

Each term in the second pair of brackets is multiplied by each term in the first, and the sum of these is taken.

The procedure can be illustrated with the following diagram.

	a	2
b	ab	$2b$
5	$5a$	10

Area of the rectangle = sum of the areas of the smaller rectangles.

That is, $(a + 2)(b + 5) = ab + 5a + 2b + 10$.



Binomial products

- In general:

$$\begin{aligned}(a + b)(c + d) &= a(c + d) + b(c + d) \\ &= ac + ad + bc + bd\end{aligned}$$

- Each term in the second pair of brackets is multiplied by each term in the first.

Example 14

Expand $(x + 4)(x + 5)$.

(continued on next page)

Solution

The expansion can be completed using a table.

	x	4
x	x^2	$4x$
5	$5x$	20

$$\begin{aligned}(x + 4)(x + 5) &= x(x + 5) + 4(x + 5) \\ &= x^2 + 5x + 4x + 20 \\ &= x^2 + 9x + 20\end{aligned}$$

The x^2 and $9x$ are unlike terms and cannot be added.

To perform a quick check, substitute $x = 1$:

$$\begin{aligned}(x + 4)(x + 5) &= (1 + 4)(1 + 5) \\ &= 5 \times 6 \\ &= 30\end{aligned}$$

$$\begin{aligned}\text{and } x^2 + 9x + 20 &= 1^2 + 9 \times 1 + 20 \\ &= 1 + 9 + 20 \\ &= 30\end{aligned}$$

Remember that this is not a proof that the expansion is correct. It means that it is probably correct. With practice you should be able to complete the expansion in one step.

Example 15

Expand and collect like terms for each expression.

a $(x + 3)(x - 2)$

b $(x - 2)(x + 5)$

c $(x - 4)(x - 3)$

d $(2y + 1)(3y - 4)$

Solution

$$\begin{aligned}\text{a } (x + 3)(x - 2) &= x^2 - 2x + 3x - 6 \\ &= x^2 + x - 6\end{aligned}$$

$$\begin{aligned}\text{b } (x - 2)(x + 5) &= x^2 + 5x - 2x - 10 \\ &= x^2 + 3x - 10\end{aligned}$$

$$\begin{aligned}\text{c } (x - 4)(x - 3) &= x^2 - 3x - 4x + 12 \\ &= x^2 - 7x + 12\end{aligned}$$

$$\begin{aligned}\text{d } (2y + 1)(3y - 4) &= 6y^2 - 8y + 3y - 4 \\ &= 6y^2 - 5y - 4\end{aligned}$$



Exercise 1E

Example 14

- 1 Expand and collect like terms. For parts **a** and **d**, substitute a value for the pronumeral as a check.

a $(x + 3)(x + 4)$ **b** $(x + 4)(x + 5)$ **c** $(x + 6)(x + 4)$

d $(x + 7)(x + 3)$ **e** $(a + 5)(a + 8)$ **f** $(a + 3)(a + 9)$

g $(3 + x)(5 + x)$ **h** $(2 + x)(x + 1)$ **i** $(a + 8)(9 + a)$

Example 15a,b,c

- 2 Expand and collect like terms. For parts **a** and **d**, substitute a value for the pronumeral as a check.

a $(x - 5)(x - 1)$ **b** $(x - 3)(x - 2)$ **c** $(a - 7)(a - 4)$

d $(a - 9)(a - 5)$ **e** $(m + 4)(m - 5)$ **f** $(p - 6)(p + 4)$

g $(x - 7)(x + 6)$ **h** $(x + 11)(x - 3)$ **i** $(x + 3)(x - 8)$

j $(x + 6)(x - 9)$ **k** $(x - 11)(x - 7)$ **l** $(x + 7)(x - 4)$

Example 15d

- 3 Expand and collect like terms.

a $(2x + 3)(3x + 4)$ **b** $(3x + 2)(5x + 4)$ **c** $(5x + 1)(x + 2)$

d $(2x - 1)(3x + 5)$ **e** $(4x - 3)(5x + 1)$ **f** $(3x - 5)(2x - 1)$

g $(2a - 5)(a - 3)$ **h** $(3a - 1)(2b + 5)$ **i** $(4m + 3)(2m - 1)$

j $(2p + 5)(3p - 2)$ **k** $(3x - 2)(4x + 7)$ **l** $(4x - 1)(2x + 7)$

m $(5x - 2)(3x - 8)$ **n** $(2x - 7)(3x - 1)$ **o** $(6x - 5)(x + 2)$

p $(7x + 9)(x - 5)$ **q** $(3a + 4)(a - 5)$ **r** $(2b + 3)(4b - 2)$

- 4 Expand and collect like terms.

a $(2a + b)(a + 3b)$ **b** $(m + 3n)(2m + n)$ **c** $(4c + d)(2c - 3d)$

d $(3m + n)(2m - 3n)$ **e** $(6p - 5q)(p + q)$ **f** $(5\ell + 3k)(\ell - 2k)$

g $(4x + 5y)(2x - y)$ **h** $(3x - 2a)(x + 5a)$ **i** $(3x - y)(2x + 5y)$

j $(6c - d)(c + 6d)$ **k** $(4m - 3n)(m + 3n)$ **l** $(2p - q)(3q + 4p)$

m $(2a + b)(3b - a)$ **n** $(2p + 5q)(3q - 2p)$ **o** $(2p - 5q)(3q - 2p)$

- 5 Write down each expansion, if possible, in one step.

a $(x + 2)(x + 5)$ **b** $(x + 3)(x - 7)$ **c** $(x - 6)(x - 4)$

d $(2x + 1)(x + 3)$ **e** $(3x + 2)(x + 5)$ **f** $(4x + 1)(3x - 1)$

g $(2x - 3)(3x - 5)$ **h** $(5x - 2)(3x + 7)$ **i** $(4x + 3)(2x - 1)$

j $(4x - 7)(3x + 5)$ **k** $(2x - 1)(2x + 1)$ **l** $(x - 4)(2x + 5)$

- 6 Zak expanded a product using the distributive law but, unfortunately, he erased the terms in the second pair of brackets in each case. Fill in the missing terms in the second pair of brackets.

a $(x + 6)(\dots) = x^2 + 7x + 6$

b $(x + 5)(\dots) = x^2 + 8x + 15$

c $(x + 7)(\dots) = x^2 + 5x - 14$

d $(x + 3)(\dots) = x^2 - 2x - 15$

e $(x + 4)(\dots) = 2x^2 + 9x + 4$

f $(x + 3)(\dots) = 3x^2 + 7x - 6$

g $(2x + 1)(\dots) = 6x^2 - x - 2$

h $(3x + 4)(\dots) = 3x^2 + x - 4$

i $(4x + 3)(\dots) = 8x^2 - 22x - 21$

j $(3x + 1)(\dots) = 15x^2 - 13x - 6$

- 7 Fill in the blanks.

a $(x + 5)(x + 7) = x^2 + \dots x + \dots$

b $(x + \dots)(x + 6) = x^2 + 9x + \dots$

c $(x + 4)(x - \dots) = x^2 - 2x - \dots$

d $(2x + 3)(x + \dots) = 2x^2 + 7x + \dots$

e $(4x - 1)(x - \dots) = 4x^2 - \dots x + 3$

f $(\dots x + 1)(3x - 5) = 6x^2 - \dots x - \dots$

g $(\dots x + \dots)(2x + 5) = 4x^2 + 12x + \dots$

h $(\dots x - 3)(\dots x + \dots) = 12x^2 - x - 6$

- 8 Expand and collect like terms.

a $(x + 3)(x - 3)$

b $(x + 3)(x + 3)$

c $(x - 5)(x - 5)$

d $(3x + 1)(3x - 1)$

e $(x + 6)(x + 6)$

f $(4x + 1)(4x + 1)$

g $(7x - 1)(7x + 1)$

h $(2x - 5)(x + 3)$

i $(7x + 1)(7x + 1)$

j $(2 - x)(x + 2)$

k $(2x + 3)(2x + 3)$

l $(5a - 1)(5a + 1)$

- 9 Expand and collect like terms.

a $\left(\frac{a}{2} + 2\right)\left(\frac{a}{3} + 1\right)$

b $\left(\frac{2b}{3} + 2\right)\left(\frac{b}{5} - 2\right)$

c $\left(\frac{2x}{5} + \frac{1}{2}\right)\left(\frac{x}{5} - 2\right)$

d $\left(\frac{y}{4} + 3\right)\left(\frac{y}{3} - \frac{3}{4}\right)$

e $\left(\frac{3m}{4} + 1\right)\left(\frac{2m}{3} - 3\right)$

f $\left(\frac{5b}{4} + \frac{1}{5}\right)\left(\frac{b}{5} - \frac{1}{2}\right)$

1F Perfect squares



A **perfect square** is an expression such as $(x + 3)^2$, $(x - 5)^2$ or $(2x + 7)^2$. The expansions of these have a special form.

$$\begin{aligned}(x + 3)^2 &= (x + 3)(x + 3) \\ &= x(x + 3) + 3(x + 3) \\ &= x^2 + 3x + 3x + 9 \\ &= x^2 + 6x + 9\end{aligned}$$

Note that the form of the answer is x^2 + twice $(3 \times x)$ + 3^2 .

$$\begin{aligned}(x - 5)^2 &= (x - 5)(x - 5) \\ &= x^2 - 5x - 5x + 25 \\ &= x^2 - 10x + 25 \\ &= x^2 + \text{twice } (-5 \times x) + (-5)^2\end{aligned}$$

Another example:

$$\begin{aligned}(3x + 7)^2 &= 3x(3x + 7) + 7(3x + 7) \\ &= 9x^2 + 21x + 21x + 7^2 \\ &= 9x^2 + 42x + 49 \\ &= (3x)^2 + \text{twice } (7 \times 3x) + 7^2\end{aligned}$$

Perfect squares

- In general:

$$(a + b)^2 = a^2 + 2ab + b^2$$

- Similarly:

$$(a - b)^2 = a^2 - 2ab + b^2$$

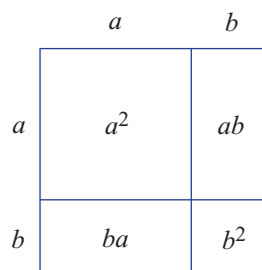
- To expand $(a + b)^2$, take the sum of the squares of the terms and add twice the product of the terms.

The result can be illustrated geometrically.

Area of square = $(a + b)^2$

This is the same as the sum of the areas of rectangles = $a^2 + ab + ba + b^2$

$$= a^2 + 2ab + b^2$$



Example 16

Expand:

a $(x + 3)^2$

b $(x - 6)^2$

(continued on next page)

Solution

$$\begin{aligned}\mathbf{a} \quad (x+3)^2 &= x^2 + 2 \times 3x + 9 \\ &= x^2 + 6x + 9\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad (x-6)^2 &= x^2 - 2 \times 6x + 36 \\ &= x^2 - 12x + 36\end{aligned}$$

Example 17

Expand each expression.

$$\mathbf{a} \quad (2x+3)^2$$

$$\mathbf{b} \quad (ax-b)^2$$

Solution

$$\begin{aligned}\mathbf{a} \quad (2x+3)^2 &= (2x)^2 + 2 \times 2x \times 3 + 3^2 \\ &= 4x^2 + 12x + 9\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad (ax-b)^2 &= (ax)^2 - 2 \times ax \times b + b^2 \\ &= a^2x^2 - 2abx + b^2\end{aligned}$$

The expansion of a perfect square can be performed mentally. With practice you will be able to write down the answer without a middle step.

Exercise 1F

Example 16a

- 1 Expand each expression. Check your answers to parts **a**, **c** and **e** by substitution.

$$\mathbf{a} \quad (x+1)^2$$

$$\mathbf{b} \quad (x+5)^2$$

$$\mathbf{c} \quad (x+3)^2$$

$$\mathbf{d} \quad (x+6)^2$$

$$\mathbf{e} \quad (x+10)^2$$

$$\mathbf{f} \quad (x+4)^2$$

$$\mathbf{g} \quad (x+7)^2$$

$$\mathbf{h} \quad (x+20)^2$$

$$\mathbf{i} \quad (a+8)^2$$

$$\mathbf{j} \quad (2+x)^2$$

$$\mathbf{k} \quad (9+x)^2$$

$$\mathbf{l} \quad (x+a)^2$$

Example 16b

- 2 Expand each expression. Check your answers to parts **a**, **c** and **e** by substitution.

$$\mathbf{a} \quad (x-4)^2$$

$$\mathbf{b} \quad (x-7)^2$$

$$\mathbf{c} \quad (x-6)^2$$

$$\mathbf{d} \quad (x-1)^2$$

$$\mathbf{e} \quad (x-5)^2$$

$$\mathbf{f} \quad (x-20)^2$$

$$\mathbf{g} \quad (x-11)^2$$

$$\mathbf{h} \quad (x-a)^2$$

Example 17a

- 3 Expand each expression.

$$\mathbf{a} \quad (3x+2)^2$$

$$\mathbf{b} \quad (2a+b)^2$$

$$\mathbf{c} \quad (2a+3b)^2$$

$$\mathbf{d} \quad (3a+4b)^2$$

$$\mathbf{e} \quad (2x+3y)^2$$

$$\mathbf{f} \quad (2a+3)^2$$

$$\mathbf{g} \quad (3x+a)^2$$

$$\mathbf{h} \quad (5x+4y)^2$$

4 Expand each expression.

a $(3x - 2)^2$ **b** $(4x - 3)^2$ **c** $(2a - b)^2$ **d** $(2a - 3b)^2$

e $(3a - 4b)^2$ **f** $(2x - 3y)^2$ **g** $(3c - b)^2$ **h** $(4x - 5)^2$

5 Expand:

a $\left(\frac{x}{2} + 3\right)^2$ **b** $\left(\frac{x}{3} - 2\right)^2$ **c** $\left(\frac{2x}{5} - 1\right)^2$ **d** $\left(\frac{3x}{4} + \frac{2}{3}\right)^2$

6 Maria has two pieces of paper. One is $10 \text{ cm} \times 10 \text{ cm}$ and the other is $6 \text{ cm} \times 6 \text{ cm}$.

a Is it possible for these two square pieces of paper to cover a square of size $16 \text{ cm} \times 16 \text{ cm}$ completely?

b How does the answer to part **a** show that $(10 + 6)^2 \neq 10^2 + 6^2$?

c How much more paper is needed to cover the $16 \text{ cm} \times 16 \text{ cm}$ square?

d How does your answer to part **c** reinforce the result that $(10 + 6)^2 = 10^2 + 2 \times 10 \times 6 + 6^2$?

7 Tom sees the following written on the board in a mathematics classroom:

‘Now, $6 + 4 = 10$

Squaring both sides gives

$$(6 + 4)^2 = 10^2$$

so, $6^2 + 4^2 = 10^2$

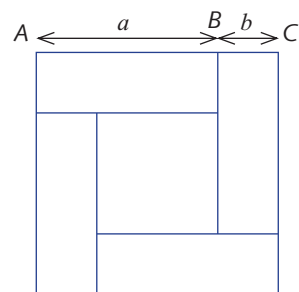
That is, $36 + 16 = 100$ ’

Tom, realising that $36 + 16 \neq 100$, tells the teacher there is a mistake. Can you explain what mistake had been made?

8 The figure opposite is made from rectangles and a square, with $AB = a$ and $BC = b$.

a Explain how the diagram geometrically proves that $(a - b)^2 + 4ab = (a + b)^2$.

b By expanding, show algebraically that $(a - b)^2 + 4ab = (a + b)^2$.



- 9** Evaluate the following squares, using the method shown in the example below.
(You may be able to do these mentally.)

$$\begin{aligned} 21^2 &= (20 + 1)^2 \\ &= 20^2 + 2 \times 20 \times 1 + 1^2 \\ &= 400 + 40 + 1 \\ &= 441 \end{aligned}$$

a 31^2 **b** 19^2 **c** 42^2 **d** 18^2 **e** 51^2
f 101^2 **g** 99^2 **h** 201^2 **i** 301^2 **j** 199^2

- 10** Evaluate the following using $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$:

a $(1.01)^2$ **b** $(0.99)^2$ **c** $(4.01)^2$ **d** $(4.02)^2$
e $(0.999)^2$ **f** $(1.02)^2$ **g** $(3.01)^2$ **h** $(0.98)^2$

- 11** Expand and collect like terms.

a $(x - 2)^2 + (x - 4)^2$ **b** $(x - 2)^2 + (x + 2)^2$
c $(2x - 3)^2 + (x - 1)^2$ **d** $(2x + 5)^2 + (2x - 5)^2$
e $(x - 4)^2 + (x + 5)^2$ **f** $(2x - 4)^2 + (x - 3)^2$

- 12** Expand and collect like terms.

a $x^2 + (x + 1)^2 + (x + 2)^2 + (x + 3)^2$
b $x^2 + (x - 1)^2 + (x - 2)^2 + (x - 3)^2$
c $x^2 - (x - 1)^2 + (x - 2)^2 - (x - 3)^2$

- 13** Expand:

a $\left(\frac{x}{2} + 1\right)^2 + \left(\frac{x}{2} - 1\right)^2$ **b** $\left(\frac{x}{3} + 3\right)^2 + \left(\frac{2x}{3} + 1\right)^2$
c $\left(\frac{3x}{4} + 2\right)^2 + \left(\frac{x}{2} + 3\right)^2$ **d** $\left(\frac{2x}{5} - \frac{1}{4}\right)^2 + \left(\frac{x}{5} + \frac{1}{2}\right)^2$



In this section, we look at a special type of expansion: one that produces the **difference of two squares**.

As an example, consider $(x + 2)(x - 2)$.

$$\begin{aligned}(x + 2)(x - 2) &= x^2 - 2x + 2x - 2^2 \\ &= x^2 - 2^2\end{aligned}$$

Similarly,

$$\begin{aligned}(2x + 3)(2x - 3) &= 4x^2 - 6x + 6x - 3^2 \\ &= (2x)^2 - 3^2\end{aligned}$$

Note that:

- the product $(x + 2)(x - 2)$ is of the form:
(sum of two terms) $\times$ (difference of those terms)
- the answer is of the form:
(first term) squared $-$ (second term) squared

In general:

$$\begin{aligned}(a + b)(a - b) &= a^2 - ab + ab - b^2 \\ &= a^2 - b^2\end{aligned}$$

Example 18

Expand:

a $(x - 5)(x + 5)$

b $(3x - 4)(3x + 4)$

Solution

a Using the result $(a + b)(a - b) = a^2 - b^2$, we get

$$(x - 5)(x + 5) = x^2 - 25$$

b $(3x - 4)(3x + 4) = (3x)^2 - 4^2$
 $= 9x^2 - 16$

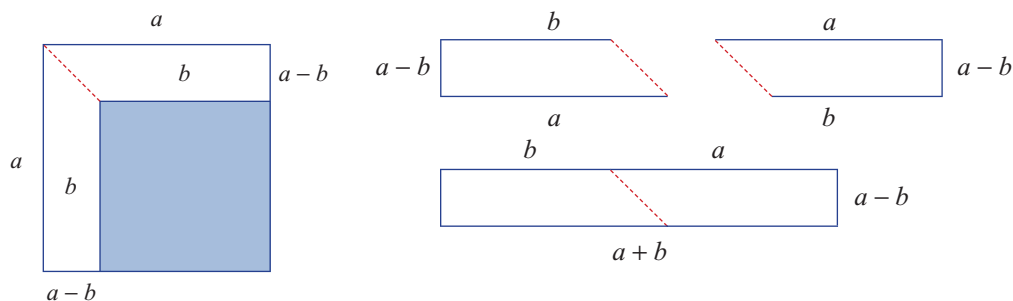


Difference of two squares

- In general:

$$(a + b)(a - b) = a^2 - b^2$$

The result is illustrated by the following diagrams:



The diagram on the left shows that the unshaded region has area $a^2 - b^2$.

The diagram on the bottom right shows that the unshaded region has area $(a - b)(a + b)$.

Exercise 1G

Example 18a

1 Expand by using the difference of two squares.

- | | | |
|---------------------------|-----------------------------|---------------------------|
| a $(x - 4)(x + 4)$ | b $(x - 7)(x + 7)$ | c $(a - 1)(a + 1)$ |
| d $(a - 9)(a + 9)$ | e $(c - 3)(c + 3)$ | f $(d - 2)(d + 2)$ |
| g $(z - 7)(7 + z)$ | h $(10 + x)(10 - x)$ | i $(x - 5)(x + 5)$ |

Example 18b

2 Expand:

- | | | |
|-----------------------------|-----------------------------|-----------------------------|
| a $(2x - 1)(2x + 1)$ | b $(3x - 2)(3x + 2)$ | c $(4a - 5)(4a + 5)$ |
| d $(3x - 5)(3x + 5)$ | e $(2x + 7)(2x - 7)$ | f $(5a + 2)(5a - 2)$ |

3 Expand:

- | | | |
|-------------------------------|-------------------------------|-------------------------------|
| a $(8 - 3a)(8 + 3a)$ | b $(3a + 2b)(3a - 2b)$ | c $(9x + y)(9x - y)$ |
| d $(2r - 3s)(2r + 3s)$ | e $(2x + 3y)(2x - 3y)$ | f $(5a + 2b)(5a - 2b)$ |

4 Expand by using the difference of two squares:

- | | |
|---|---|
| a $\left(\frac{2x}{3} + 1\right)\left(\frac{2x}{3} - 1\right)$ | b $\left(\frac{x}{2} + 3\right)\left(\frac{x}{2} - 3\right)$ |
| c $\left(\frac{x}{3} + \frac{1}{2}\right)\left(\frac{x}{3} - \frac{1}{2}\right)$ | d $\left(\frac{2x}{5} + \frac{3}{4}\right)\left(\frac{2x}{5} - \frac{3}{4}\right)$ |

5 The following are either difference of squares expansions or perfect square expansions. Which are which?

- | | | |
|----------------------|---------------------------|--------------------------|
| a $a^2 - 1$ | b $a^2 - 2a + 1$ | c $x^2 - 9$ |
| d $4x^2 - 25$ | e $4a^2 + 12a + 9$ | f $9a^2 - 6a + 1$ |

- 6 Evaluate the following products, using the method shown in the example below.
(You may be able to do these mentally.)

$$\begin{aligned} 51 \times 49 &= (50 + 1)(50 - 1) \\ &= 50^2 - 1^2 \\ &= 2499 \end{aligned}$$

- a** 21×19 **b** 31×29 **c** 18×22 **d** 32×28
e 17×23 **f** 59×61 **g** 101×99 **h** 102×98

- 7 Evaluate using the difference of two squares identity:

- a** 1.01×0.99 **b** 5.01×4.99 **c** 8.02×7.98
d 1.05×0.95 **e** 10.01×9.99 **f** 20.1×19.9

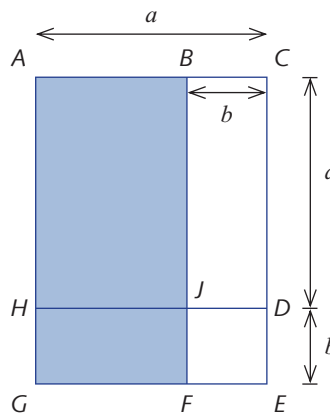
- 8 Look at the figure opposite.

- a** What is the length of AB ?
b What is the length of AG ?
c What is the area of the shaded region?
d Explain why:

$$\text{area } ABFG = \text{area } ACDH + \text{area } HDEG$$

$$- \text{area } BCDJ - \text{area } JDEF.$$

e What has this got to do with the difference of squares expansions?





In this section we present some harder practice questions.

Example 19

Expand and collect like terms:

a $(x + 2)^2 - (x + 1)^2$

b $(x + 6)^2 - (x - 6)^2$

Solution

a $(x + 2)^2 - (x + 1)^2 = x^2 + 4x + 4 - (x^2 + 2x + 1)$
 $= 2x + 3$

b $(x + 6)^2 - (x - 6)^2 = x^2 + 12x + 36 - (x^2 - 12x + 36)$
 $= 24x$

Exercise 1H

Example 19

1 Expand and collect like terms in each case.

a $(x + 2)^2 - (x + 4)^2$

b $(x + 6)^2 - (x + 5)^2$

c $(a - 4)^2 - (a - 4)(a + 4)$

d $(a + 6)(a - 5) - (a + 3)(a - 2)$

e $(3x + 2)(x + 1) - (x + 2)(x + 3)$

f $(2b - 5)(3b + 2) + (b + 5)(b + 6)$

g $(a + 2b)(2a - b) - (a + b)^2$

h $(2x + y)(y - 2x) + (2x - y)^2$

i $(4m - 1)(2m + 3) - (3m + 1)(m - 4)$

2 Evaluate $3m^3 - 2n^2$ when:

a $m = 2$ and $n = 3$

b $m = -4$ and $n = -5$

c $m = -\frac{1}{2}$ and $n = \frac{3}{4}$

d $m = \frac{1}{3}$ and $n = \sqrt{2}$

3 Evaluate $\frac{2x^2 + y}{y^2}$ when:

a $x = 3$ and $y = 4$

b $x = -\frac{1}{2}$ and $y = \frac{3}{2}$

c $x = \sqrt{3}$ and $y = -2$

d $x = \sqrt{6}$ and $y = -4$

4 Evaluate $a + 2b - c$ when:

a $a = \frac{1}{4}$, $b = \frac{1}{2}$ and $c = \frac{3}{8}$

b $a = \frac{b}{3}$, $b = -\frac{1}{4}$ and $c = \frac{5b}{6}$

c $a = \frac{1}{c}$, $b = \frac{3c}{2}$ and $c = \frac{3}{8}$

d $a = c^2$, $b = \frac{1}{c^2}$ and $c = \frac{2}{3}$

5 Evaluate $\frac{2p^3 - q^2}{p^2}$ when:

a $p = \frac{2}{3}$ and $q = \frac{3}{4}$

b $p = \frac{1}{2}$ and $q = -2$

c $p = -2$ and $q = -4$

6 Evaluate $(x^2 - 2)^2$ when:

a $x = 2$

b $x = -2$

c $x = \frac{2}{3}$

d $x = -\sqrt{2}$

e $x = \sqrt{3}$

7 Simplify each expression by collecting like terms.

a $\frac{3}{7}a^2 + \frac{2}{3}a^2$

b $\frac{3a}{b} + \frac{5a}{2b}$

c $\frac{2x}{y^3} + \frac{7x}{3y^3}$

d $\frac{2a^2b}{5} - \frac{2a^2b}{5} + \frac{6m}{n^2}$

8 Copy and complete:

a $\frac{2a^2b}{5} - \dots = \frac{3a^2b}{10}$

b $\frac{2x^2}{y} - \dots = \frac{5x^2}{3y}$

c $\frac{3m}{2n^2} + \dots = \frac{7m}{2n^2}$

d $\frac{p^3}{2n^2} - \dots = -\frac{p^3}{n^2}$

9 For each value of x , find x^2 and simplify.

a $x = \sqrt{5}$

b $x = 3a$

c $x = -2b$

d $x = a + b$

e $x = 3m - 2n$

10 For each value of x , find $x^2 + 3x$ and simplify.

a $x = 3a$

b $x = -2b$

c $x = a - b$

d $x = 2m + n$

11 For each value of x , find $(x - 1)^2$ and simplify.

a $x = 3a$

b $x = -2b$

c $x = 2a + 1$

d $x = -4b + 2$

12 Expand and collect like terms in each case.

a $x(2x^2 + 3x + 4)$

b $3a(a^2 - 4a + 1)$

c $(m + 2)(m^2 + 3m + 2)$

d $(p - 3)(2p^2 + 5p + 3)$

e $(m + 4)(m^2 - 6m + 3)$

f $(m - 5)(m^2 + 2)$

g $(x - 1)(x^2 + x + 1)$

h $(x + 1)(x^2 - x + 1)$

13 If $x = 1$, $y = 3$, $z = 5$ and $w = 0$, find the exact value of $\sqrt{3xy} + \sqrt{3xz} + \sqrt{3wy}$.

14 Expand and collect like terms.

a $(x + y - z)(x - y + z)$

b $(a - b + c)(a + b - c)$

c $(2x - y + z)(2x + y - z)$

d $(x + z + 1)^2$

e $(x + y + z)^2$

f $(x + y - z)^2$

15 Find $\frac{2ab}{a+b}$ if:

a $a = 2$ and $b = 3$ **b** $a = 10$ and $b = 50$ **c** $a = \frac{1}{2}$ and $b = \frac{1}{3}$

d $a = \frac{1}{3}$ and $b = \frac{1}{4}$ **e** $a = \frac{2}{3}$ and $b = \frac{3}{4}$ **f** $a = \frac{1}{x}$ and $b = \frac{1}{y}$

16 Find $\frac{a}{c}$ and $\frac{a-b}{b-c}$ if:

a i $a = 4, b = 2$ and $c = \frac{4}{3}$ **ii** $a = \frac{3}{4}, b = \frac{3}{5}$ and $c = \frac{1}{2}$

iii $a = 3, b = \frac{3}{2}$ and $c = 1$

b In each of the above, show that $\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}$.

17 Simplify by first removing brackets.

a $a - (b - c) + a + (b - c) + b - (c + a)$

b $3a - 2b - c - (4a - 3b + 5c)$

c $a - (b + a - (b + a))$

d $-(-(a - b - c))$

e $-(-2x - (3y - (2x - 3y) + (3x - 2y)))$

18 Expand and collect like terms.

a $2[(a+b)(a-b) + (a+c)(a-c) + a(b+c)] + (a-b)^2 + (a-c)^2 + (b-c)^2$

b $(a-b)(a^2 + a + b) + (a+b)(a^2 - a + b)$

c $\left(\frac{1}{a} - \frac{1}{b}\right)^2 \left(\frac{1}{a} + \frac{1}{b}\right)^2$

d $\left(a + \frac{1}{a}\right)^2 \left(a - \frac{1}{a}\right)^2$

e $\left(2a + \frac{1}{a}\right)^2 - \left(a - \frac{1}{a}\right)^2$

f $\left(a + \frac{1}{b}\right)\left(b - \frac{1}{a}\right)$

g $(a+b)^2 - [2b(a+b) - (a+b)(b-a)]$

Review exercise



1 Evaluate $8m - 3n$ when:

a $m = 3$ and $n = 5$

b $m = 2$ and $n = -5$

c $m = -1$ and $n = 2$

d $m = -3$ and $n = -2$

2 Evaluate $5x + 11y$ when:

a $x = 6$ and $y = 5$

b $x = -6$ and $y = 5$

c $x = 6$ and $y = -5$

d $x = -6$ and $y = -5$

3 Evaluate $2a + 3b - 3c$ when:

a $a = 3$, $b = 5$ and $c = 2$

b $a = 3$, $b = -5$ and $c = 2$

c $a = -3$, $b = 5$ and $c = 2$

d $a = 3$, $b = 5$ and $c = -2$

e $a = -3$, $b = -5$ and $c = 2$

f $a = 3$, $b = -5$ and $c = -2$

4 Evaluate $x^2 - 2x$ when:

a $x = 3$

b $x = -2$

c $x = -7$

d $x = \frac{2}{3}$

e $x = -\frac{2}{3}$

f $x = -1.1$

5 Simplify:

a $7a + 2a$

b $8b + 3b$

c $6x + 4x$

d $9a - 6a$

e $7a + 5a - 6a$

f $7x - 3x + 9x$

g $11a + 5a - 13a$

h $12m - 13m + 9m$

i $7a^2 + 3a^2 - 12a^2$

6 Copy and complete:

a $2a + \dots = 11a$

b $5b - \dots = 3b$

c $3mn + \dots = 16mn$

d $11pq - \dots = 5pq$

e $8ab - \dots = -5ab$

f $3m^2n - \dots = -8m^2n$

g $-11a^2b + \dots = 2a^2b$

h $-7\ell m + \dots = 4\ell m$

i $14xy^2 - \dots = -7xy^2$

7 Expand each expression and simplify if possible.

a $3(x + 2)$

b $4(b + 6)$

c $5(3b - 2)$

d $-6(2d + 5)$

e $2(4x - 15)$

f $3(4g - 6)$

g $5(3p - 12)$

h $16(5q - 2)$

i $-6(3x + 8)$

j $-3(2p + 2)$

k $-5(3b - 5)$

l $-5(4b - 7)$

m $15(g + 2) - 8g$

n $5(h + 1) + 6h$

o $16(x - 1) - 13x$

p $2(y - 1) - y$

q $5a(2a + 3) - 7a$

r $-5b(2b - 3) + 16b$

s $2(3z - 1) - 4(2y - 1)$

t $z(z - 1) + 3(z - 2)$

u $2y(y + 1) - 5(y + 1)$

v $6y(2y - 5) - 5(2y - 5)$



8 Expand each expression and collect like terms.

a $(x + 2)(x + 4)$	b $(x + 6)(x + 5)$	c $(a - 11)(a - 4)$
d $(a - 5)(a - 5)$	e $(m + 6)(m - 5)$	f $(p - 7)(p + 4)$
g $(5x + 2)(2x + 2)$	h $(2x - 1)(4x - 5)$	i $(4x - 6)(11x + 1)$
j $(2x - 5)(6x - 1)$	k $(4x + 3)(2x - 1)$	l $(7a - 1)(7a + 3)$

9 Expand:

a $(x + 11)^2$	b $(x + 6)^2$	c $(x - 15)^2$	d $(x - 10)^2$
e $(x - 2y)^2$	f $(2a + 5b)^2$	g $(5x + 2)^2$	h $(5x - 6)^2$

10 Expand:

a $(x - 6)(x + 6)$	b $(z - 7)(z + 7)$	c $(p - 1)(p + 1)$
d $(a - 11)(a + 11)$	e $(5x - 1)(5x + 1)$	f $(7x - 5)(7x + 5)$
g $(6a - 5)(6a + 5)$	h $(10 - 3a)(10 + 3a)$	i $(5a + 2b)(5a - 2b)$
j $(12x + y)(12x - y)$	k $(4x + 3y)(4x - 3y)$	l $(8x + 3a)(8x - 3a)$

11 Expand and collect like terms.

a $(3x + y)(2x - y)$	b $(3x - 4a)(3x + 2a)$	c $(3c + 4b)(5c + 4b)$
d $(3x + 5y)^2$	e $(a - 2b)^2$	f $(5\ell + 2m)^2$
g $(3x - y)(3x + y)$	h $(5m + 2n)(5m - 2n)$	i $(3x + 5a)^2$

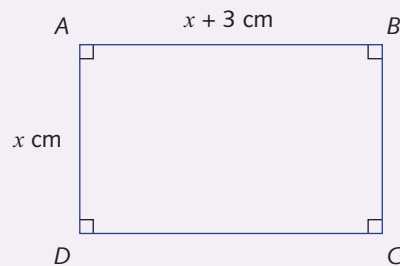
12 Expand and collect like terms.

a $\left(\frac{a}{2} + 1\right)\left(\frac{a}{3} - 2\right)$	b $\left(\frac{2x}{3} + 6\right)(x - 4)$	c $\left(\frac{a}{2} - 1\right)\left(\frac{a}{2} + 1\right)$
d $\left(\frac{3x}{5} - 1\right)\left(\frac{2x}{5} + 3\right)$	e $\left(\frac{5b}{6} - 3\right)(b + 2)$	f $(a - 6)\left(\frac{a}{3} + 4\right)$

Challenge exercise

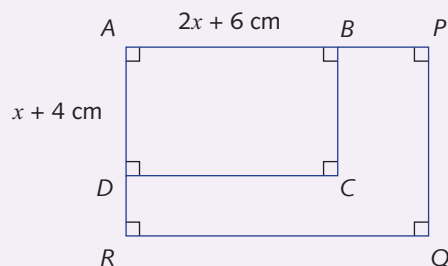


- 1 a Show that the perimeter of the rectangle opposite is $(4x + 6)$ cm.
- b Find the perimeter of the rectangle if $AD = 2$ cm.
- c Find the value of x if the perimeter is 36 cm.
- d Find the area of the rectangle $ABCD$ in terms of x .
- e Find the area of the rectangle if $AB = 6$ cm.



- 2 Rectangle $ABCD$ is $(x + 4)$ cm wide and $(2x + 6)$ cm long. A new rectangle, $APQR$, is formed by increasing the length of each side of $ABCD$ by 50%.

- a Find AP in terms of x .
- b Find AR in terms of x .
- c Find the area of $APQR$ in terms of x .
- d Show that the difference between the areas of the two rectangles is $\frac{5}{2}x^2 + \frac{35}{2}x + 30$.
- e What fraction of the area $ABCD$ is the answer in part d?



- 3 We know that $(a + b)(c + d) = ac + ad + bc + bd$.
Expand $(a + b + c)(d + e + f)$. Draw a diagram to illustrate your answer.
- 4 8, 9, 10 and 11 are four consecutive whole numbers.
 - a i What is the product of the outer pair (8×11)?
 - ii What is the product of the inner pair (9×10)?
 - b Find the difference between the answers to parts i and ii.
 - c If the smallest of the four consecutive whole numbers is denoted by n , express each of the other numbers in terms of n .
 - d Express the following in terms of n :
 - i The product of the outer pair
 - ii The product of the inner pair
 - iii The difference between the answers to part ii and part i.
- 5 6, 8, 10 and 12 are four consecutive even whole numbers.
 - a i What is the product of the outer pair (6×12)?
 - ii What is the product of the inner pair (8×10)?
 - b If the smallest of the four consecutive even numbers is denoted by n , express each of the other numbers in terms of n .



- c** Express the following in terms of n :
- i** The product of the outer pair
 - ii** The product of the inner pair
 - iii** The difference between the answers to part **ii** and part **i**.
- d** What happens if you take four consecutive odd whole numbers and complete parts **a**, **b** and **c** with these numbers?
- 6 a** 7, 8 and 9 are consecutive whole numbers. $8^2 - 7 \times 9 = 1$.
Prove that this result holds for any three consecutive numbers $n - 1$, n and $n + 1$.
That is, 'If the product of the outer numbers is subtracted from the square of the middle number, the result is one'.
- b** $a - d$, a and $a + d$ are three numbers. Show that if the product of the outer pair is subtracted from the square of the middle term, the result is d^2 .
- 7** Expand and collect like terms.
- a** $(x^2 + x + 1)(x^2 - x + 1)$ **b** $(x + 1)(x^2 - x + 1)$
- c** $(x^5 - 1)(x^5 + 1)$
- 8** Expand and collect like terms.
- a** $(x - 1)(x^2 + x + 1)$ **b** $(x - 1)(x^4 + x^3 + x^2 + x + 1)$
- c** What do you expect the result of expanding $(x - 1)(x^9 + x^8 + \dots + 1)$ will be?
- 9 a** Show that $(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$
- b** Hence write $(3^2 + 1^2)(5^2 + 2^2)$ as the sum of two squares.
- 10 a** Evaluate each of the following.
- i** $1 \times 2 \times 3 \times 4 + 1$ **ii** $2 \times 3 \times 4 \times 5 + 1$
- iii** $3 \times 4 \times 5 \times 6 + 1$ **iv** $4 \times 5 \times 6 \times 7 + 1$
- b** Expand $(n^2 + n - 1)^2$
- c** Show that $(n - 1)n(n + 1)(n + 2) + 1 = (n^2 + n - 1)^2$
- (Note that we have proved that the product of four consecutive integers plus one is a perfect square.)

Pythagoras' theorem and surds

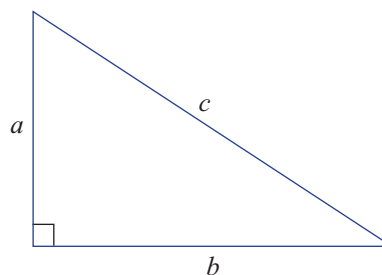
In *ICE-EM Mathematics Year 8 Book 1*, Chapter 8, you learnt about the remarkable relationship between the lengths of the sides of a right-angled triangle. This result is known as **Pythagoras' theorem**.

The theorem states that the square of the length of the hypotenuse equals the sum of the squares of the lengths of the other two sides. In symbols:

$$c^2 = a^2 + b^2$$

The converse of this theorem is also true. This means that if we have a triangle in which the square of one side equals the sum of the squares of the other two sides, then the triangle is right-angled, with the longest side being the hypotenuse.

Pythagoras' theorem leads to the discovery of certain irrational numbers, such as $\sqrt{2}$ and $\sqrt{3}$. These numbers are examples of **surds**. In this chapter, we investigate the arithmetic of surds.





In Year 8, we used Pythagoras' theorem to solve problems related to right-angled triangles.

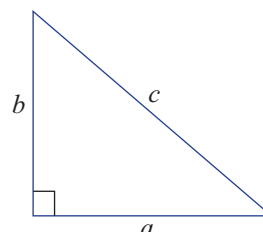


Pythagoras' theorem and its converse

- In any right-angled triangle, the square of the length of the hypotenuse equals the sum of the squares of the lengths of the other two sides.

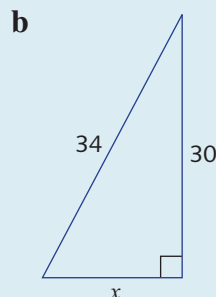
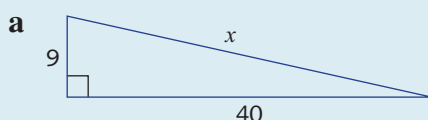
$$c^2 = a^2 + b^2$$

- A triangle with side lengths of a, b, c which satisfy $c^2 = a^2 + b^2$ is a right-angled triangle. The right angle is opposite the side of length c .



Example 1

Find the value of the unknown side in each triangle.



Solution

$$\begin{aligned} \mathbf{a} \quad x^2 &= 9^2 + 40^2 \\ &= 1681 \\ x &= \sqrt{1681} \\ &= 41 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad x^2 + 30^2 &= 34^2 \\ x^2 &= 34^2 - 30^2 \\ &= 256 \\ x &= \sqrt{256} \\ &= 16 \end{aligned}$$

Calculators and rounding

In Example 1 above, the values for x^2 are perfect squares and so we could take the square root easily. This is not always the case.

In Year 8, we found the approximate square roots of numbers that are not perfect squares by looking up a table of square roots.

Instead of doing this, from now on we are going to use a calculator.

When using a calculator to find a square root, try to have in mind a rough idea of what the answer should be. For example, $\sqrt{130}$ should be close to 11, since $11^2 = 121$ and $12^2 = 144$.

Note: $\sqrt{130} \approx 11.402$ correct to three decimal places.

A calculator gives the approximate value of a square root to a large number of decimal places, far more than we need. We often round off a decimal to a required number of decimal places.

The method for rounding to 2 decimal places is as follows.

- Look at the digit in the third decimal place.
- If the digit is less than 5, take the two digits to the right of the decimal point. For example, 1.764 becomes 1.76, correct to two decimal places.
- If the digit is more than 4, take the two digits to the right of the decimal point and increase the second of these by one. For example, 2.455 becomes 2.46, correct to 2 decimal places.

Example 2

- a** A calculator gives $\sqrt{3} \approx 1.732050808$. State the value of $\sqrt{3}$ correct to 2 decimal places.
- b** A calculator gives $\sqrt{5} \approx 2.236067977$. State the value of $\sqrt{5}$ correct to 2 decimal places.
- c** State the value of 1.697, correct to 2 decimal places.

Solution

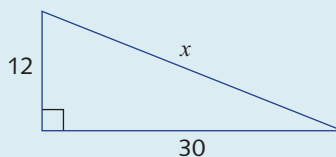
- a** $\sqrt{3} \approx 1.73$, since the digit, 2, in the third decimal place is less than 5.
- b** $\sqrt{5} \approx 2.24$, since the digit, 6, in the third decimal place is more than 4.
- c** $1.697 \approx 1.70$, since the digit, 7, in the third decimal place is more than 4 and 69 rounds to 70.

The symbol $\approx$ means ‘is approximately equal to’.

When approximating, we should state the number of decimal places to which we have rounded our answer.

Example 3

Find the length of the unknown side, correct to 2 decimal places.



(continued on next page)

Solution

$$\begin{aligned}x^2 &= 12^2 + 30^2 \\&= 1044 \\x &= \sqrt{1044} \\&\approx 32.31 \text{ (correct to 2 decimal places)}\end{aligned}$$

In the example above, $\sqrt{1044}$ is the exact length while 32.31 is an approximation to the length of the side.

Example 4

Determine whether or not the three side lengths given form the sides of a right-angled triangle.

a 24, 32, 40

b 14, 18, 23

Solution

a The square of the length of the longest side $= 40^2$
 $= 1600$

$$\begin{aligned}\text{The sum of the squares of the lengths of the other sides} &= 24^2 + 32^2 \\&= 576 + 1024 \\&= 1600\end{aligned}$$

Since $40^2 = 24^2 + 32^2$, the triangle is right-angled.

b The square of the length of the longest side $= 23^2$
 $= 529$

$$\begin{aligned}\text{The sum of the squares of the lengths of the other sides} &= 14^2 + 18^2 \\&= 196 + 324 \\&= 520\end{aligned}$$

Since $23^2 \neq 14^2 + 18^2$, the triangle is not right-angled.

Example 5

A door frame has height 1.7 m and width 1 m. Will a square piece of board 2 m by 2 m fit through the doorway?

Solution

Let d m be the length of the diagonal of the doorway.

Using Pythagoras' theorem,

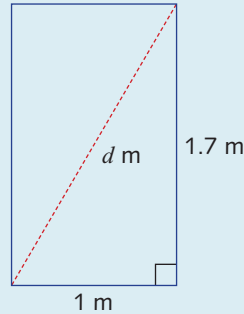
$$d^2 = 1^2 + 1.7^2$$

$$= 3.89$$

$$d = \sqrt{3.89}$$

$$\approx 1.97 \text{ (correct to 2 decimal places)}$$

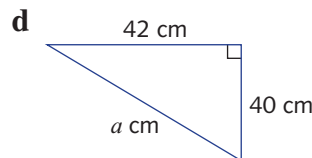
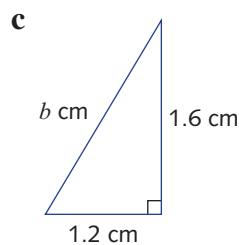
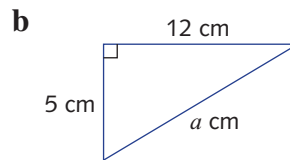
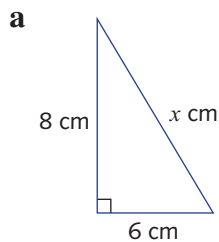
Hence the board will not fit through the doorway.



Exercise 2A

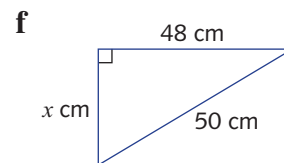
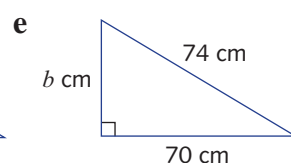
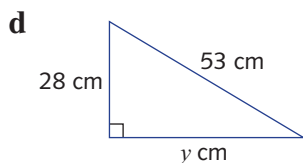
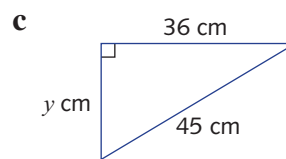
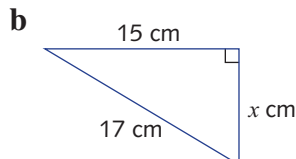
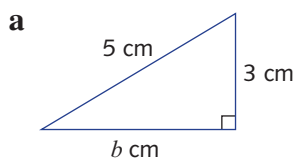
Example 1a

1 Use Pythagoras' theorem to find the value of the pronumeral.



Example 1b

2 Use Pythagoras' theorem to find the value of the pronumeral.



Example 2

- 3 Use a calculator to find, correct to 2 decimal places, approximations to these numbers.

a $\sqrt{19}$

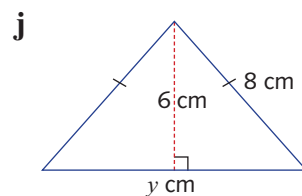
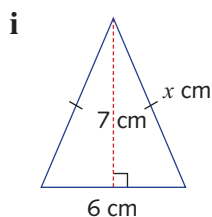
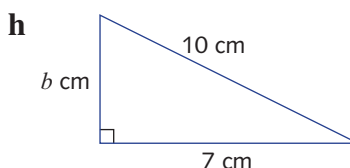
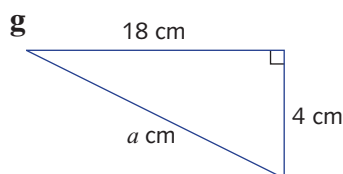
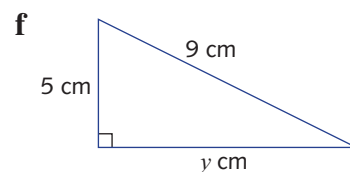
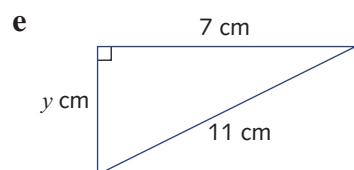
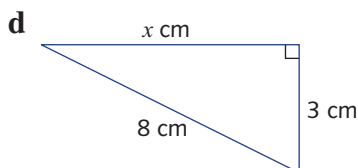
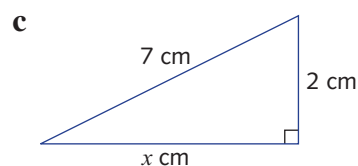
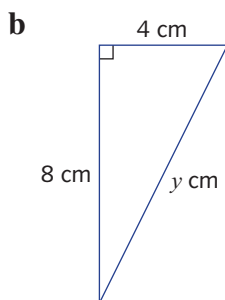
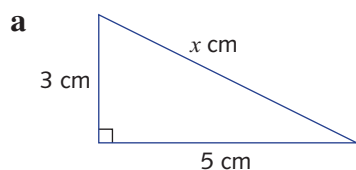
b $\sqrt{37}$

c $\sqrt{61}$

d $\sqrt{732}$

Example 3

- 4 Use Pythagoras' theorem to find the value of the pronumeral. Calculate your answer first as a square root and then correct to 2 decimal places.



Example 4

- 5 Determine whether or not the triangle with the three side lengths given is right-angled.

a 16, 30, 34

b 10, 24, 26

c 4, 6, 7

d 4.5, 7, 7.5

e 6, 10, 12

f 20, 21, 29

- 6 In the table below, the side lengths of right-angled triangles are listed. Copy and complete the table, giving answers as a whole number or square root.

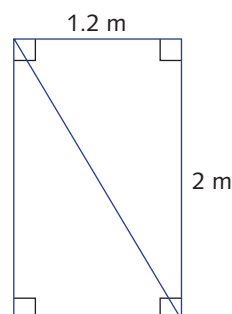
	Lengths of the two shortest sides of a right-angled triangle		Length of the hypotenuse
a	3 cm	4 cm	...
b	5 cm	6 cm	...
c	4 cm	...	9 cm
d	6 cm	10 cm	...
e	...	7 cm	10 cm

Example 5

- 7 A door frame has height 1.8 m and width 1 m. Will a piece of board 2.1 m wide fit through the opening?

- 8 A tradesman is making the wooden rectangular frame for a gate. In order to make the frame stronger and to keep it square, the tradesman will put a diagonal piece into the frame as shown in the diagram.

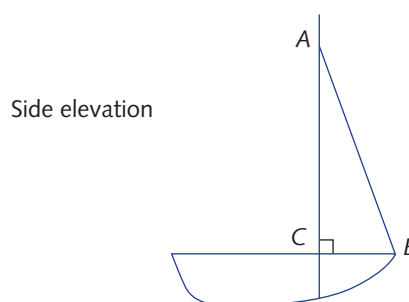
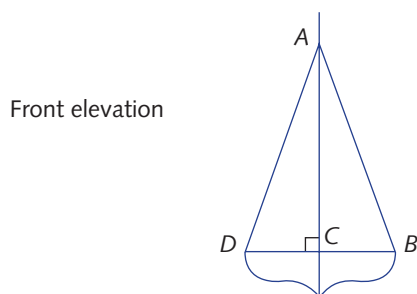
If the frame is 1.2 m wide and 2 m high, find the length of the diagonal piece of wood, in metres, correct to 3 decimal places.



- 9 A signwriter leans his ladder against a wall so that he can paint a sign. The wall is vertical and the ground in front of the wall is horizontal. The signwriter's ladder is 4 m long.

If the signwriter wants the top of the ladder to be 3.8 m above the ground when leaning against the wall, how far, correct to 1 decimal place, should the foot of the ladder be placed from the wall?

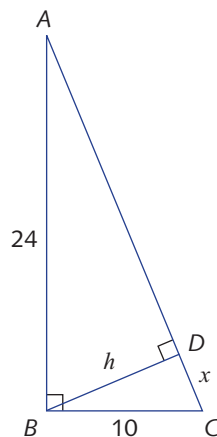
- 10 A boat builder needs to calculate the lengths of the stays needed to support a mast on a yacht. Two of the stays (AB and AD) will be the same length, as they go from a point A on the mast to each side of the boat, as shown in the diagram. The third stay (AE) will be different in length as it goes from the point A on the mast to the front of the boat.



If $AC = 5$ m, $CB = CD = 1.2$ m and $CE = 1.9$ m, find the length, to the nearest centimetre, of:

- a one of the side stays, AB or AD
- b the front stay, AE
- c stainless steel wire needed to make the three stays.

- 11 As part of a design, an artist draws a circle passing through the four corners (vertices) of a square.
 - a If the square has side lengths of 4 cm, what is the radius, to the nearest millimetre, of the circle?
 - b If the circle has a radius of 3 cm, what are the side lengths, to the nearest millimetre, of the square?
- 12 A parent is asked to make some scarves for the local Scout troop. Two scarves can be made from one square piece of material by cutting on the diagonal. If this diagonal side length is to be 100 cm long, what must be the side length of the square piece of material to the nearest mm?
- 13 A girl planned to swim straight across a river of width 25 m. After she had swum across the river, the girl found she had been swept 4 m downstream. How far did she actually swim? Calculate your answer, in metres, correct to 1 decimal place.
- 14 A yachtsman wishes to build a shed with a rectangular base to store his sailing equipment. If the shed is to be 2.6 m wide and must be able to house a 4.6 m mast, which is to be stored diagonally across the ceiling, how long must the shed be? Calculate your answer, in metres, correct to 1 decimal place.
- 15 In triangle ABC the line BD is drawn perpendicular to AC . h is the length of BD and x is the length of CD .
 - a Show that the length of AC is 26.
 - b Find the area of triangle ABC in two ways to show that $13h = 120$.
 - c Use Pythagoras' theorem to find x .



- 16** In diagrams 1 and 2, we have two squares with the same side length, $a + b$. The sides are divided up into lengths a and b as shown.

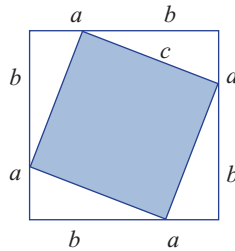


Diagram 1

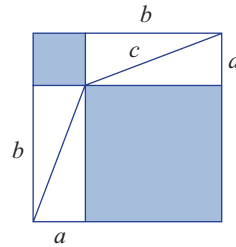


Diagram 2

- a** Prove that the shaded part of diagram 1 is a square.
b The two shaded parts of diagram 2 are also squares. Why?

By looking carefully at the two diagrams, show that the area of the shaded square in diagram 1 is the sum of the areas of the shaded squares in diagram 2.

- c** Use the result of part **b** to prove Pythagoras' theorem.

2B Simplifying surds



This section deals only with surds that are square roots. If a is a positive integer which is not a perfect square then $\sqrt{a}$ is called a **surd**. We will look at surds more generally in Section 2I.

We will now review the basic rules for square roots.

If a and b are positive numbers then:

$$(\sqrt{a})^2 = a$$

$$\sqrt{a^2} = a$$

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

$$\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$$

For example:

$$(\sqrt{11})^2 = 11$$

$$\sqrt{3^2} = 3$$

$$\begin{aligned}\sqrt{3} \times \sqrt{7} &= \sqrt{3 \times 7} \\ &= \sqrt{21}\end{aligned}$$

$$\begin{aligned}\sqrt{35} \div \sqrt{5} &= \sqrt{35 \div 5} \\ &= \sqrt{7}\end{aligned}$$

The first two of these rules remind us that, for positive numbers, squaring and taking a square root are **inverse processes**.

When we write $2\sqrt{3}$, we mean $2 \times \sqrt{3}$. As in algebra, we usually do not explicitly write the multiplication sign.

Example 6

Evaluate:

a $(\sqrt{6})^2$

b $(2\sqrt{6})^2$

Solution

a
$$\begin{aligned}(\sqrt{6})^2 &= \sqrt{6} \times \sqrt{6} \\ &= 6\end{aligned}$$

b
$$\begin{aligned}(2\sqrt{6})^2 &= 2\sqrt{6} \times 2\sqrt{6} \\ &= 2 \times 2 \times \sqrt{6} \times \sqrt{6} \\ &= 4 \times 6 \\ &= 24\end{aligned}$$

Example 7

Evaluate:

a $\sqrt{3} \times \sqrt{11}$

b $\sqrt{15} \div \sqrt{3}$

c $\frac{\sqrt{42}}{\sqrt{6}}$

d $\sqrt{35} \div \sqrt{10}$

Solution

a $\sqrt{3} \times \sqrt{11} = \sqrt{3 \times 11}$
 $= \sqrt{33}$

b $\sqrt{15} \div \sqrt{3} = \sqrt{15 \div 3}$
 $= \sqrt{5}$

c $\frac{\sqrt{42}}{\sqrt{6}} = \sqrt{\frac{42}{6}}$
 $= \sqrt{7}$

d $\sqrt{35} \div \sqrt{10} = \sqrt{\frac{35}{10}}$
 $= \sqrt{\frac{7}{2}}$

Example 8

Evaluate:

a $3 \times 7\sqrt{5}$

b $9\sqrt{2} \times 4$

Solution

a $3 \times 7\sqrt{5} = 21\sqrt{5}$

b $9\sqrt{2} \times 4 = 36\sqrt{2}$

Consider the surd $\sqrt{12}$. We can factor out the perfect square 4 from 12, and write:

$$\begin{aligned}\sqrt{12} &= \sqrt{4 \times 3} \\ &= \sqrt{4} \times \sqrt{3} && \text{(using } \sqrt{ab} = \sqrt{a} \times \sqrt{b} \text{)} \\ &= 2\sqrt{3}\end{aligned}$$

Hence $\sqrt{12}$ and $2\sqrt{3}$ are equal. We will regard $2\sqrt{3}$ as a simpler form than $\sqrt{12}$ since the number under the square root sign is smaller.

To **simplify** a surd (or a multiple of a surd) we write it so that the number under the square root sign has no factors that are perfect squares other than 1. For example,

$$\sqrt{12} = 2\sqrt{3}.$$

In mathematics, we are often instructed to leave our answers in **surd form**. This simply means that we should not approximate the answer using a calculator, but leave the answer, in simplest form, using square roots, cube roots, etc. This is called giving the **exact value** of the answer.

We can use our knowledge of factorising whole numbers to simplify surds.

Example 9

Simplify $\sqrt{18}$.

Solution

$$\begin{aligned}\sqrt{18} &= \sqrt{9 \times 2} \\ &= \sqrt{9} \times \sqrt{2} \\ &= 3\sqrt{2}\end{aligned}$$

We look for factors of the number under the square root sign that are perfect squares. Sometimes we may need to do this in stages.

Example 10

Simplify $\sqrt{108}$.

Solution

$$\begin{aligned}\sqrt{108} &= \sqrt{9 \times 4 \times 3} \\ &= \sqrt{9} \times \sqrt{4} \times \sqrt{3} \\ &= 3 \times 2\sqrt{3} \\ &= 6\sqrt{3}\end{aligned}$$

In some problems, we need to reverse this process.

Example 11

Express as the square root of a whole number.

a $3\sqrt{7}$

b $5\sqrt{3}$

Solution

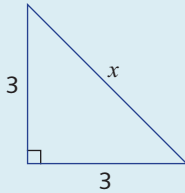
a $3\sqrt{7} = \sqrt{9 \times 7}$
 $= \sqrt{63}$

b $5\sqrt{3} = \sqrt{25 \times 3}$
 $= \sqrt{75}$

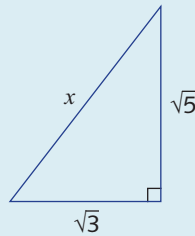
Example 12

Use Pythagoras' theorem to find the value of x . Give your answer as a surd which has been simplified.

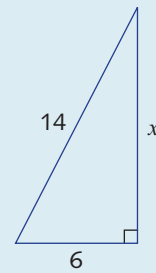
a



b



c



Solution

$$\begin{aligned}\mathbf{a} \quad x^2 &= 3^2 + 3^2 \\ &= 9 + 9 \\ &= 18 \\ x &= \sqrt{18} \\ &= \sqrt{9 \times 2} \\ &= 3\sqrt{2}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad x^2 &= (\sqrt{5})^2 + (\sqrt{3})^2 \\ &= 5 + 3 \\ &= 8 \\ x &= \sqrt{8} \\ &= \sqrt{4 \times 2} \\ &= 2\sqrt{2}\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad x^2 + 6^2 &= 14^2 \\ x^2 + 36 &= 196 \\ x^2 &= 160 \\ x &= \sqrt{160} \\ &= \sqrt{16 \times 10} \\ &= 4\sqrt{10}\end{aligned}$$



The arithmetic of surds

- If a and b are positive numbers, then:

$$(\sqrt{a})^2 = a$$

$$\sqrt{a^2} = a$$

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

$$\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$$

- A surd is in its **simplest form** if the number under the square root sign has no factors that are perfect squares, apart from 1.
- To **simplify** a surd, we write it so that the number under the square root sign has no factors that are perfect squares, apart from 1.

For example, $\sqrt{50} = 5\sqrt{2}$.

Exercise 2B

Example 6a

1 Evaluate:

a $(\sqrt{7})^2$

b $(\sqrt{3})^2$

c $(\sqrt{11})^2$

d $(\sqrt{231})^2$

Example 6b

2 Evaluate:

a $(2\sqrt{3})^2$

b $(4\sqrt{3})^2$

c $(5\sqrt{2})^2$

d $(3\sqrt{5})^2$

e $(2\sqrt{7})^2$

f $(7\sqrt{3})^2$

g $(11\sqrt{2})^2$

h $(2\sqrt{11})^2$

Example 7a

3 Express as a square root of a whole number.

a $\sqrt{3} \times \sqrt{5}$

b $\sqrt{2} \times \sqrt{6}$

c $\sqrt{5} \times \sqrt{6}$

d $\sqrt{11} \times \sqrt{3}$

e $\sqrt{7} \times \sqrt{6}$

f $\sqrt{3} \times \sqrt{10}$

g $\sqrt{2} \times \sqrt{7}$

h $\sqrt{2} \times \sqrt{13}$

i $\sqrt{17} \times \sqrt{3}$

j $\sqrt{3} \times \sqrt{15}$

k $\sqrt{3} \times \sqrt{14}$

l $\sqrt{10} \times \sqrt{19}$

Examples 7b,c,d

4 Express as a square root of a number.

a $\sqrt{6} \div \sqrt{2}$

b $\sqrt{10} \div \sqrt{5}$

c $\sqrt{35} \div \sqrt{7}$

d $\sqrt{22} \div \sqrt{2}$

e $\sqrt{56} \div \sqrt{8}$

f $\frac{\sqrt{39}}{\sqrt{3}}$

g $\frac{\sqrt{22}}{\sqrt{11}}$

h $\frac{\sqrt{46}}{\sqrt{2}}$

i $\frac{\sqrt{77}}{\sqrt{11}}$

j $\sqrt{40} \div \sqrt{18}$

k $\sqrt{6} \div \sqrt{42}$

l $\frac{\sqrt{14}}{\sqrt{21}}$

Example 8

5 Evaluate the product.

a $2 \times 3\sqrt{2}$

b $6 \times 2\sqrt{5}$

c $11 \times 3\sqrt{7}$

d $6 \times 5\sqrt{7}$

e $15 \times 3\sqrt{2}$

f $7 \times 5\sqrt{6}$

g $4\sqrt{11} \times 3$

h $7\sqrt{3} \times 4$

6 Evaluate:

a $(\sqrt{2})^3$

b $(\sqrt{5})^3$

c $(\sqrt{5})^2 - (\sqrt{2})^2$

d $(\sqrt{7})^2 - (\sqrt{3})^2$

e $(\sqrt{11})^2 + (\sqrt{2})^2$

f $(\sqrt{5})^2 + (\sqrt{11})^2$

g $\sqrt{2} \times \sqrt{18}$

h $\sqrt{3} \times \sqrt{12}$

i $\sqrt{32} \times \sqrt{2}$

Example 9

7 Simplify each of these surds.

a $\sqrt{8}$

b $\sqrt{12}$

c $\sqrt{45}$

d $\sqrt{24}$

e $\sqrt{27}$

f $\sqrt{44}$

g $\sqrt{50}$

h $\sqrt{54}$

i $\sqrt{20}$

j $\sqrt{98}$

k $\sqrt{63}$

l $\sqrt{60}$

m $\sqrt{126}$

n $\sqrt{68}$

o $\sqrt{75}$

p $\sqrt{99}$

q $\sqrt{28}$

r $\sqrt{242}$

Example 10

8 Simplify each of these surds.

- a** $\sqrt{72}$ **b** $\sqrt{32}$ **c** $\sqrt{80}$ **d** $\sqrt{288}$ **e** $\sqrt{48}$ **f** $\sqrt{180}$
g $\sqrt{112}$ **h** $\sqrt{216}$ **i** $\sqrt{96}$ **j** $\sqrt{252}$ **k** $\sqrt{160}$ **l** $\sqrt{128}$
m $\sqrt{320}$ **n** $\sqrt{176}$ **o** $\sqrt{192}$ **p** $\sqrt{200}$ **q** $\sqrt{162}$ **r** $\sqrt{243}$

Example 11

9 Express each of these surds as the square root of a whole number.

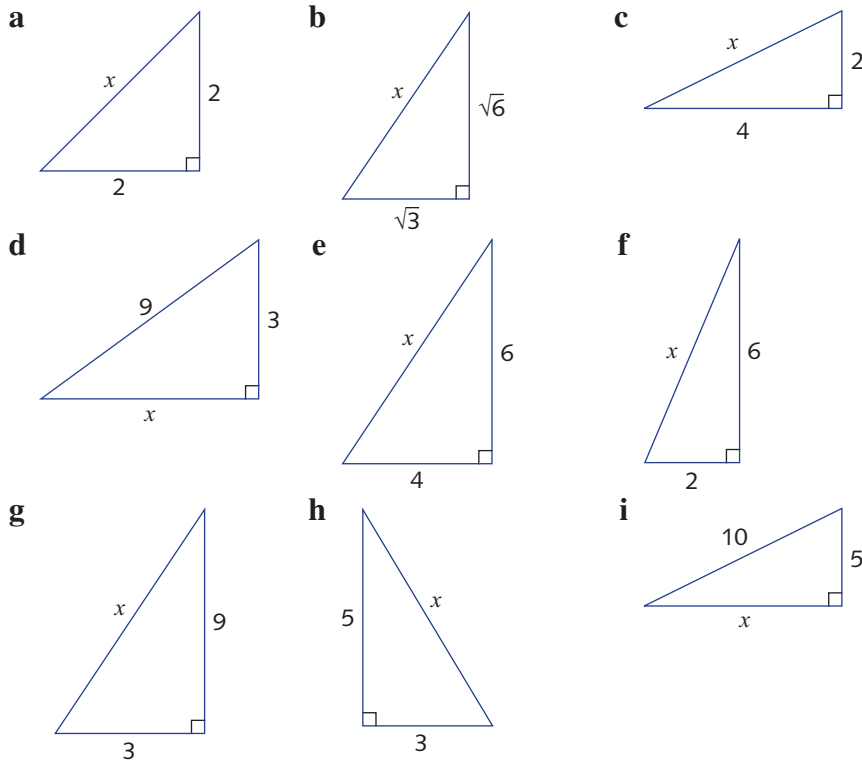
- a** $2\sqrt{3}$ **b** $4\sqrt{7}$ **c** $5\sqrt{11}$ **d** $6\sqrt{3}$ **e** $7\sqrt{2}$ **f** $3\sqrt{6}$
g $4\sqrt{5}$ **h** $5\sqrt{7}$ **i** $4\sqrt{3}$ **j** $9\sqrt{5}$ **k** $8\sqrt{3}$ **l** $2\sqrt{13}$
m $6\sqrt{11}$ **n** $10\sqrt{6}$ **o** $6\sqrt{5}$ **p** $12\sqrt{10}$ **q** $10\sqrt{7}$ **r** $4\sqrt{11}$

10 Evaluate:

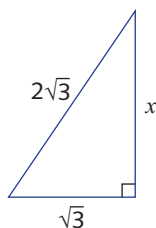
- a** $\left(\sqrt{\frac{2}{3}}\right)^2$ **b** $\sqrt{\frac{4}{25}}$ **c** $\sqrt{\frac{16}{25}}$ **d** $\sqrt{\frac{25}{36}}$
e $\left(\sqrt{\frac{5}{11}}\right)^2$ **f** $\sqrt{\frac{25}{121}}$ **g** $\left(\sqrt{\frac{7}{11}}\right)^2$ **h** $\sqrt{\frac{144}{169}}$

Example 12

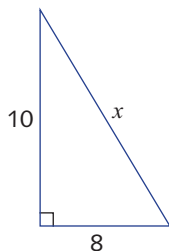
11 Use Pythagoras' theorem to find the value of x . Give your answer as a surd which has been simplified.



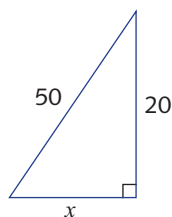
j



k



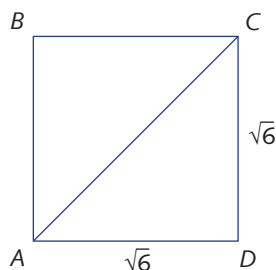
l



- 12** $ABCD$ is a square with side length $\sqrt{6}$.

Find:

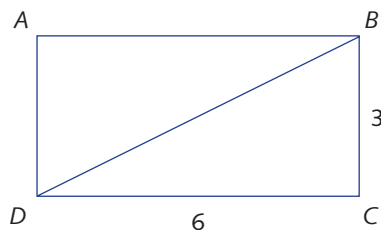
- a** the area of the square
b the length of the diagonal AC .



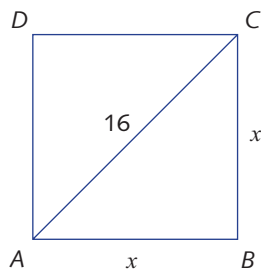
- 13** A square has side length $2\sqrt{3}$. Find:

- a** the area of the square **b** the length of the diagonal.

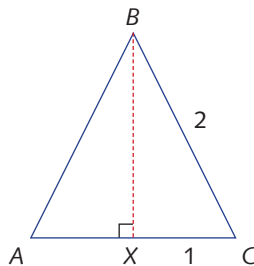
- 14** Find the length of the diagonal DB of the rectangle $ABCD$. Express your answer in simplest form.



- 15** $ABCD$ is a square. Find the value of x .



- 16** ABC is an equilateral triangle with side length 2. X is the midpoint of AC . Find the length BX .





We can sometimes simplify sums of surds. The sum $4\sqrt{7} + 5\sqrt{7}$ can be thought of as 4 lots of $\sqrt{7}$ plus 5 lots of $\sqrt{7}$ equals 9 lots of $\sqrt{7}$. This is very similar to algebra, where we write $4x + 5x = 9x$. We say $4\sqrt{7}$ and $5\sqrt{7}$ are **like surds** since they are both multiples of $\sqrt{7}$.

On the other hand, in algebra we cannot simplify $4x + 7y$, because $4x$ and $7y$ are not like terms. Similarly, it is not possible to write the sum of $4\sqrt{2}$ and $7\sqrt{3}$ in a simpler way. They are **unlike surds**, since one is a multiple of $\sqrt{2}$ while the other is a multiple of $\sqrt{3}$.

We can only simplify the sum or difference of like surds.

Example 13

Simplify:

a $4\sqrt{7} + 5\sqrt{7}$

b $6\sqrt{7} - 2\sqrt{7}$

c $2\sqrt{2} + 5\sqrt{2} - 3\sqrt{2}$

d $3\sqrt{5} + 2\sqrt{7} - \sqrt{5} + 4\sqrt{7}$

Solution

a $4\sqrt{7} + 5\sqrt{7} = 9\sqrt{7}$

b $6\sqrt{7} - 2\sqrt{7} = 4\sqrt{7}$

c $2\sqrt{2} + 5\sqrt{2} - 3\sqrt{2} = 4\sqrt{2}$

d $3\sqrt{5} + 2\sqrt{7} - \sqrt{5} + 4\sqrt{7} = 2\sqrt{5} + 6\sqrt{7}$

This cannot be simplified further since $\sqrt{5}$ and $\sqrt{7}$ are unlike surds.

When dealing with expressions involving surds, we should simplify the surds first and then look for like surds.

Example 14

Simplify:

a $\sqrt{8} + 4\sqrt{2} - \sqrt{18}$

b $\sqrt{27} + 2\sqrt{5} + \sqrt{20} - 2\sqrt{3}$

Solution

a $\sqrt{8} + 4\sqrt{2} - \sqrt{18} = 2\sqrt{2} + 4\sqrt{2} - 3\sqrt{2}$
 $= 3\sqrt{2}$

b $\sqrt{27} + 2\sqrt{5} + \sqrt{20} - 2\sqrt{3} = 3\sqrt{3} + 2\sqrt{5} + 2\sqrt{5} - 2\sqrt{3}$
 $= \sqrt{3} + 4\sqrt{5}$

This expression cannot be simplified further.

Example 15

Simplify:

a $\frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{5}$

b $\frac{\sqrt{8}}{3} - \frac{\sqrt{2}}{5}$

Solution

a $\frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{5} = \frac{5\sqrt{3}}{10} + \frac{6\sqrt{3}}{10}$
 $= \frac{11\sqrt{3}}{10}$

(Use a common denominator.)

b $\frac{\sqrt{8}}{3} - \frac{\sqrt{2}}{5} = \frac{2\sqrt{2}}{3} - \frac{\sqrt{2}}{5}$
 $= \frac{10\sqrt{2}}{15} - \frac{3\sqrt{2}}{15}$
 $= \frac{7\sqrt{2}}{15}$

(Simplify the surds.)

(Use a common denominator.)



Addition and subtraction of surds

- Simplify each surd first, then look for like surds.
- We can simplify sums and differences of like surds.
- We cannot simplify sums and differences of unlike surds.



Exercise 2C

Example 13

1 Simplify:

a $\sqrt{5} - \sqrt{5}$

b $6\sqrt{2} + 4\sqrt{2}$

c $\sqrt{3} + 4\sqrt{3}$

d $8\sqrt{2} - \sqrt{2}$

e $10\sqrt{7} - 5\sqrt{7}$

f $19\sqrt{5} + 24\sqrt{5}$

g $-13\sqrt{3} + 14\sqrt{3}$

h $3\sqrt{11} - 5\sqrt{11}$

i $5\sqrt{2} + 2\sqrt{2}$

j $6\sqrt{3} + 4\sqrt{3}$

k $7\sqrt{5} - 2\sqrt{5}$

l $11\sqrt{7} - 3\sqrt{7}$

m $4\sqrt{2} + 8\sqrt{2}$

n $9\sqrt{3} - 2\sqrt{3}$

o $-45\sqrt{2} + 50\sqrt{2}$

2 Simplify:

a $2\sqrt{2} + 3\sqrt{2} + \sqrt{2}$

b $-4\sqrt{6} - 3\sqrt{6} - 8\sqrt{6}$

c $10\sqrt{5} - \sqrt{5} - 6\sqrt{5}$

d $-2\sqrt{13} - 5\sqrt{13} + 16\sqrt{13}$

e $\sqrt{7} - 8\sqrt{7} + 5\sqrt{7}$

f $6\sqrt{19} + \sqrt{19} - 3\sqrt{19}$

g $-2\sqrt{10} + 9\sqrt{10} - 7\sqrt{10}$

h $-\sqrt{23} + 4\sqrt{23} + 10\sqrt{23}$

i $5\sqrt{3} + 2\sqrt{3} + 4\sqrt{3}$

j $-7\sqrt{2} + 2\sqrt{2} - 5\sqrt{2}$

k $3\sqrt{6} + 4\sqrt{2} - 2\sqrt{6} + 3\sqrt{2}$

l $4\sqrt{5} + 7\sqrt{3} - 2\sqrt{5} - 4\sqrt{3}$

3 Simplify:

a $3 - \sqrt{3} + 4 - 2\sqrt{3}$

b $\sqrt{3} - 2\sqrt{2} + 2\sqrt{3} + \sqrt{2}$

c $-4\sqrt{6} + 11 + 9 - 7\sqrt{6}$

d $5\sqrt{14} + 4\sqrt{6} + \sqrt{14} + 3\sqrt{6}$

e $6\sqrt{7} - 2\sqrt{14} + 4\sqrt{14} - 7\sqrt{7}$

f $\sqrt{5} - 3\sqrt{2} - 4\sqrt{5} + 7\sqrt{2}$

Example 14

4 Simplify:

a $\sqrt{8} + \sqrt{2}$

b $3\sqrt{2} - \sqrt{8}$

c $\sqrt{50} + 3\sqrt{2}$

d $\sqrt{18} + 2\sqrt{2}$

e $\sqrt{27} + 2\sqrt{3}$

f $\sqrt{48} + 2\sqrt{3}$

g $3\sqrt{5} + \sqrt{45}$

h $\sqrt{28} + \sqrt{63}$

i $\sqrt{8} + 4\sqrt{2} + 2\sqrt{18}$

5 Simplify:

a $\sqrt{72} - \sqrt{50}$

b $\sqrt{48} + \sqrt{12}$

c $\sqrt{8} + \sqrt{2} + \sqrt{18}$

d $\sqrt{12} + 4\sqrt{3} - \sqrt{75}$

e $\sqrt{32} - \sqrt{200} + 3\sqrt{50}$

f $4\sqrt{5} - 4\sqrt{20} - \sqrt{45}$

g $\sqrt{54} + \sqrt{24}$

h $\sqrt{27} - \sqrt{48} + \sqrt{75}$

i $\sqrt{45} + \sqrt{80} - \sqrt{125}$

j $\sqrt{2} + \sqrt{32} + \sqrt{72}$

k $\sqrt{1000} - \sqrt{40} - \sqrt{90}$

l $\sqrt{512} + \sqrt{128} + \sqrt{32}$

Example 15

6 Simplify:

a $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{3}$

b $\frac{2\sqrt{3}}{5} + \frac{\sqrt{3}}{2}$

c $\frac{\sqrt{7}}{2} - \frac{\sqrt{7}}{5}$

d $\frac{3\sqrt{11}}{7} + \frac{2\sqrt{11}}{21}$

e $\frac{7\sqrt{2}}{6} - \frac{\sqrt{2}}{3}$

f $\frac{5\sqrt{5}}{4} - \frac{3\sqrt{5}}{8}$

g $\frac{\sqrt{8}}{3} - \frac{\sqrt{2}}{2}$

h $\frac{\sqrt{24}}{2} + \frac{\sqrt{6}}{5}$

i $\frac{\sqrt{27}}{5} - \frac{\sqrt{3}}{2}$

j $\frac{\sqrt{32}}{7} - \frac{2\sqrt{2}}{5}$

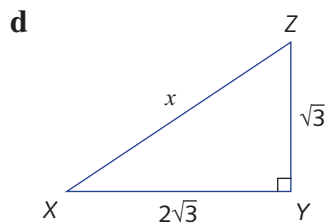
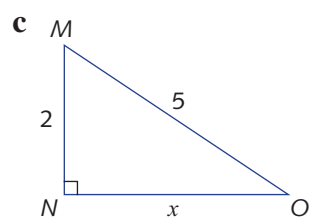
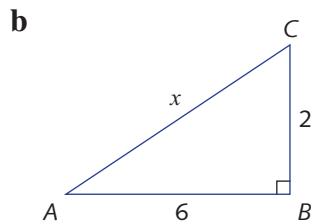
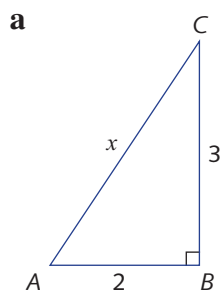
7 Find the value of x if:

a $\sqrt{63} - \sqrt{28} = \sqrt{x}$

b $\sqrt{80} - \sqrt{45} = \sqrt{x}$

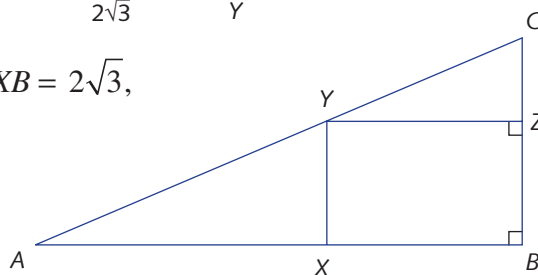
c $2\sqrt{24} - \sqrt{54} = \sqrt{x}$

8 Find the value of x and the perimeter.



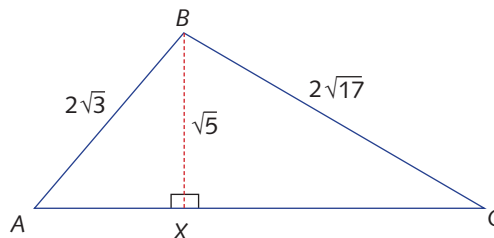
9 In the diagram to the right, $AB = 5\sqrt{3}$, $XB = 2\sqrt{3}$, $CB = \frac{5}{3}\sqrt{5}$, $YX = \sqrt{5}$. Find:

- a** AX **b** AY
c CZ **d** YC

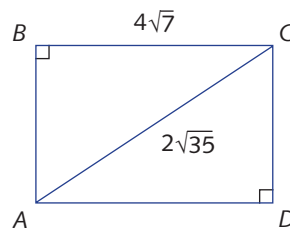


10 In the diagram to the right, find:

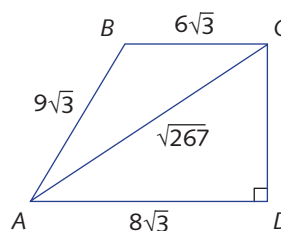
- a** AX
b XC
c AC



11 Find BA and the perimeter of the rectangle.



12 A quadrilateral $ABCD$ has side lengths $AB = 9\sqrt{3}$, $BC = 6\sqrt{3}$ and $AD = 8\sqrt{3}$. $\angle ADC = 90^\circ$ and the diagonal $AC = \sqrt{267}$. Find the perimeter of the quadrilateral.



2D Multiplication and division of surds



When we come to multiply two surds, we simply multiply the numbers outside the square root sign together, and similarly, multiply the numbers under the square root signs. A similar rule holds for division.

Example 16

Find $4\sqrt{7} \times 2\sqrt{2}$.

Solution

$$4\sqrt{7} \times 2\sqrt{2} = 8\sqrt{14} \quad (4 \times 2 = 8, \sqrt{7} \times \sqrt{2} = \sqrt{14})$$

We can state the procedure we just used as a general rule:

$$a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd},$$

where b and d are positive numbers.

Example 17

Find:

a $15\sqrt{35} \div 5\sqrt{7}$

b $49\sqrt{21} \div 14\sqrt{3}$

Solution

a $15\sqrt{35} \div 5\sqrt{7} = 3\sqrt{5} \quad (15 \div 5 = 3, \sqrt{35} \div \sqrt{7} = \sqrt{5})$

b $49\sqrt{21} \div 14\sqrt{3} = \frac{7\sqrt{7}}{2} \quad (49 \div 14 = \frac{7}{2}, \sqrt{21} \div \sqrt{3} = \sqrt{7})$

We can state the procedure we just used as a general rule:

$$a\sqrt{b} \div c\sqrt{d} = \frac{a}{c}\sqrt{\frac{b}{d}},$$

where b and d are positive numbers and $c \neq 0$.

As usual, we should always give the answer in simplest form.

Example 18

Find $5\sqrt{6} \times 3\sqrt{10}$.

Solution

$$\begin{aligned} 5\sqrt{6} \times 3\sqrt{10} &= 15\sqrt{60} \\ &= 15\sqrt{4 \times 15} \\ &= 30\sqrt{15} \end{aligned}$$

The distributive law

We can apply the distributive law to expressions involving surds, just as we do in algebra.

Example 19

Expand and simplify $2\sqrt{3}(4 + 3\sqrt{3})$.

Solution

$$\begin{aligned} 2\sqrt{3}(4 + 3\sqrt{3}) &= 2\sqrt{3} \times 4 + 2\sqrt{3} \times 3\sqrt{3} \\ &= 8\sqrt{3} + 6\sqrt{9} \\ &= 8\sqrt{3} + 18 \end{aligned}$$

In algebra, you learnt how to expand brackets such as $(a + b)(c + d)$. These are known as **binomial products**. You multiply each term in the second bracket by each term in the first, then add. This means you expand out $a(c + d) + b(c + d)$ to obtain $ac + ad + bc + bd$. We use this idea again when multiplying out binomial products involving surds.

Example 20

Expand and simplify:

a $(3\sqrt{7} + 1)(5\sqrt{7} - 4)$

b $(5\sqrt{2} - 3)(2\sqrt{2} - 4)$

c $(3\sqrt{2} - 4\sqrt{3})(5\sqrt{3} - \sqrt{2})$

d $(1 - \sqrt{2})(3 + 2\sqrt{2})$

Solution

$$\begin{aligned} \text{a } (3\sqrt{7} + 1)(5\sqrt{7} - 4) &= 3\sqrt{7}(5\sqrt{7} - 4) + 1(5\sqrt{7} - 4) \\ &= 15\sqrt{49} - 12\sqrt{7} + 5\sqrt{7} - 4 \\ &= 105 - 7\sqrt{7} - 4 \\ &= 101 - 7\sqrt{7} \end{aligned}$$

$$\begin{aligned}\text{b } (5\sqrt{2}-3)(2\sqrt{2}-4) &= 5\sqrt{2}(2\sqrt{2}-4)-3(2\sqrt{2}-4) \\ &= 20-20\sqrt{2}-6\sqrt{2}+12 \\ &= 32-26\sqrt{2}\end{aligned}$$

$$\begin{aligned}\text{c } (3\sqrt{2}-4\sqrt{3})(5\sqrt{3}-\sqrt{2}) &= 3\sqrt{2}(5\sqrt{3}-\sqrt{2})-4\sqrt{3}(5\sqrt{3}-\sqrt{2}) \\ &= 15\sqrt{6}-6-60+4\sqrt{6} \\ &= 19\sqrt{6}-66\end{aligned}$$

$$\begin{aligned}\text{d } (1-\sqrt{2})(3+2\sqrt{2}) &= 3+2\sqrt{2}-3\sqrt{2}-4 \\ &= -1-\sqrt{2}\end{aligned}$$

Multiplication and division of surds

- For positive numbers b and d , $a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$
- For positive numbers b and d , $a\sqrt{b} \div c\sqrt{d} = \frac{a}{c}\sqrt{\frac{b}{d}}$, provided $c \neq 0$
- Give the answers in simplest form.
- We expand binomial products involving surds just as we do in algebra:
 $(a+b)(c+d) = a(c+d) + b(c+d) = ac + ad + bc + bd$

Exercise 2D

Example 16

1 Simplify:

$$\begin{array}{llll}\text{a } \sqrt{2} \times \sqrt{3} & \text{b } \sqrt{5} \times \sqrt{2} & \text{c } \sqrt{3} \times 2\sqrt{7} & \text{d } 4\sqrt{7} \times \sqrt{2} \\ \text{e } 3\sqrt{2} \times 2\sqrt{3} & \text{f } 4\sqrt{5} \times 2\sqrt{2} & \text{g } 5\sqrt{7} \times 3\sqrt{2} & \text{h } 6\sqrt{3} \times 4\sqrt{7}\end{array}$$

Example 17

2 Simplify:

$$\begin{array}{lll}\text{a } \sqrt{5} \div \sqrt{5} & \text{b } 8\sqrt{10} \div \sqrt{10} & \text{c } 6\sqrt{3} \div 2\sqrt{3} \\ \text{d } \sqrt{21} \div \sqrt{3} & \text{e } 4\sqrt{15} \div \sqrt{5} & \text{f } 12\sqrt{33} \div 3\sqrt{3} \\ \text{g } \sqrt{35} \div \sqrt{7} & \text{h } 3\sqrt{2} \div \sqrt{2} & \text{i } 36\sqrt{15} \div 4\sqrt{3} \\ \text{j } 14\sqrt{40} \div 7\sqrt{5} & \text{k } 20\sqrt{30} \div 2\sqrt{6} & \text{l } 7\sqrt{28} \div 4\sqrt{7}\end{array}$$

3 Simplify:

$$\begin{array}{llll} \text{a } \sqrt{6} \times \sqrt{3} & \text{b } \sqrt{6} \times \sqrt{6} & \text{c } \sqrt{7} \times 2\sqrt{7} & \text{d } 4\sqrt{5} \times \sqrt{10} \\ \text{e } 2\sqrt{6} \times 2\sqrt{2} & \text{f } 3\sqrt{6} \times 2\sqrt{2} & \text{g } 7\sqrt{10} \times 3\sqrt{2} & \text{h } 3\sqrt{14} \times 2\sqrt{7} \end{array}$$

4 Expand and simplify:

$$\begin{array}{lll} \text{a } \sqrt{3}(\sqrt{6} + \sqrt{3}) & \text{b } \sqrt{2}(2\sqrt{2} - \sqrt{6}) & \text{c } 2\sqrt{5}(\sqrt{2} - \sqrt{5}) \\ \text{d } 3\sqrt{3}(2\sqrt{3} - 1) & \text{e } 5\sqrt{5}(4\sqrt{2} - 3) & \text{f } \sqrt{7}(2\sqrt{7} - \sqrt{14}) \\ \text{g } 4\sqrt{5}(2\sqrt{15} - 3\sqrt{3}) & \text{h } 2\sqrt{3}(3\sqrt{3} - 5) & \text{i } 3\sqrt{2}(3 + 4\sqrt{3}) \\ \text{j } 6\sqrt{5}(\sqrt{2} + 1) & \text{k } 3\sqrt{7}(2 - \sqrt{14}) & \text{l } 3\sqrt{5}(\sqrt{15} + 3) \end{array}$$

5 Expand and simplify:

$$\begin{array}{ll} \text{a } (4\sqrt{5} + 1)(3\sqrt{5} + 2) & \text{b } (4 + 2\sqrt{6})(2 + 5\sqrt{6}) \\ \text{c } (3\sqrt{2} + 2)(3\sqrt{2} - 1) & \text{d } (1 + \sqrt{5})(7 - 6\sqrt{5}) \\ \text{e } (2\sqrt{3} - 4)(3\sqrt{3} + 5) & \text{f } (3\sqrt{7} - 1)(5\sqrt{7} - 2) \\ \text{g } (7\sqrt{2} + 5)^2 & \text{h } (4\sqrt{3} - 2)^2 \end{array}$$

6 Expand and simplify:

$$\begin{array}{ll} \text{a } (3\sqrt{5} + 2)(\sqrt{2} + 3) & \text{b } (2\sqrt{3} - 3)(3\sqrt{2} + 1) \\ \text{c } (4 + 2\sqrt{3})(2\sqrt{7} - 5) & \text{d } (8 - 4\sqrt{6})(2 - \sqrt{7}) \\ \text{e } (\sqrt{7} - \sqrt{5})(\sqrt{7} + \sqrt{5}) & \text{f } (5\sqrt{2} - 3\sqrt{3})(2\sqrt{2} + 2\sqrt{3}) \\ \text{g } (4\sqrt{7} - \sqrt{5})(2\sqrt{5} + \sqrt{7}) & \text{h } (4\sqrt{2} - \sqrt{5})(2\sqrt{5} + \sqrt{2}) \end{array}$$

7 If $x = \sqrt{2} - 1$ and $y = \sqrt{3} + 1$, find:

$$\begin{array}{llll} \text{a } xy & \text{b } x + y & \text{c } x(x + 2) & \text{d } \sqrt{3}x - \sqrt{2}y \\ \text{e } 2xy & \text{f } x + 3y & \text{g } (x + 1)(y + 1) & \text{h } \frac{6\sqrt{10}}{x + 1} \\ \text{i } \frac{1}{x} + \frac{1}{\sqrt{2} + 1} & \text{j } \frac{1}{x} - \frac{1}{\sqrt{2} + 1} & \text{k } x + \frac{1}{x} & \end{array}$$

2E Special products



In algebra, you learned the following special expansions. They are called **identities** because they are true for all values of the pronumerals. These are especially important when dealing with surds.

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a - b)(a + b) = a^2 - b^2$$

The first two identities are called the **perfect square** identities. The last identity is known as the **difference of two squares** identity.

You will need to be very confident with these identities and be able to recognise them. Since they are true for all numbers, we can apply them to surds. Recall that $(\sqrt{a})^2 = a$ holds for any positive number a .

Example 21

Expand and simplify $(\sqrt{5} + \sqrt{3})^2$.

Solution

$$\begin{aligned}(\sqrt{5} + \sqrt{3})^2 &= (\sqrt{5})^2 + 2 \times \sqrt{5} \times \sqrt{3} + (\sqrt{3})^2 \\&= 5 + 2\sqrt{15} + 3 \\&= 8 + 2\sqrt{15}\end{aligned}$$

Example 22

Expand and simplify $(3\sqrt{2} - 2\sqrt{3})^2$.

Solution

$$\begin{aligned}(3\sqrt{2} - 2\sqrt{3})^2 &= (3\sqrt{2})^2 - 2 \times 3\sqrt{2} \times 2\sqrt{3} + (2\sqrt{3})^2 \\&= 18 - 12\sqrt{6} + 12 \\&= 30 - 12\sqrt{6}\end{aligned}$$

You should always express your answer in simplest form.

Example 23

Expand and simplify $(2\sqrt{3} + 4\sqrt{6})^2$.

Solution

$$\begin{aligned}(2\sqrt{3} + 4\sqrt{6})^2 &= (2\sqrt{3})^2 + 2 \times 2\sqrt{3} \times 4\sqrt{6} + (4\sqrt{6})^2 \\ &= 12 + 16\sqrt{18} + 96 \\ &= 108 + 48\sqrt{2}\end{aligned}$$

The following example shows an interesting application of the difference of two squares' identity. We will use this later in this chapter when simplifying surds with a binomial denominator.

Example 24

Expand and simplify:

a $(\sqrt{7} - 5)(\sqrt{7} + 5)$ **b** $(5\sqrt{6} - 2\sqrt{5})(5\sqrt{6} + 2\sqrt{5})$

Solution

a
$$\begin{aligned}(\sqrt{7} - 5)(\sqrt{7} + 5) &= (\sqrt{7})^2 - 5^2 \\ &= 7 - 25 \\ &= -18\end{aligned}$$

b
$$\begin{aligned}(5\sqrt{6} - 2\sqrt{5})(5\sqrt{6} + 2\sqrt{5}) &= (5\sqrt{6})^2 - (2\sqrt{5})^2 \\ &= 25 \times 6 - 4 \times 5 \\ &= 130\end{aligned}$$



Identities

The following identities are often used in calculations involving surds.

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a-b)(a+b) = a^2 - b^2$$



Exercise 2E

Example 21

1 Expand and simplify:

a $(5 + \sqrt{3})^2$

b $(\sqrt{2} + 6)^2$

c $(4 + 2\sqrt{5})^2$

d $(3\sqrt{3} + 1)^2$

e $(\sqrt{5} + \sqrt{7})^2$

f $(2\sqrt{3} + \sqrt{2})^2$

g $(5\sqrt{2} + 3\sqrt{3})^2$

h $(\sqrt{2} + 3\sqrt{5})^2$

i $(2\sqrt{5} + 4\sqrt{3})^2$

j $(\sqrt{x} + \sqrt{y})^2$

k $(a\sqrt{x} + b\sqrt{y})^2$

l $(\sqrt{xy} + 1)^2$

Example 22

2 Expand and simplify:

a $(\sqrt{7} - 2)^2$

b $(4 - \sqrt{3})^2$

c $(2\sqrt{5} - 1)^2$

d $(2 - 3\sqrt{3})^2$

e $(\sqrt{5} - \sqrt{3})^2$

f $(2\sqrt{3} - \sqrt{2})^2$

g $(\sqrt{2} - 4\sqrt{5})^2$

h $(4\sqrt{2} - 3\sqrt{7})^2$

i $(\sqrt{10} - 3\sqrt{6})^2$

j $(2\sqrt{14} - 4\sqrt{7})^2$

k $(\sqrt{x} - \sqrt{y})^2$

l $(3\sqrt{3} - 2\sqrt{6})^2$

Example 23

3 Expand and simplify:

a $(2\sqrt{10} + 4\sqrt{5})^2$

b $(5\sqrt{6} + 2\sqrt{3})^2$

c $(\sqrt{21} + \sqrt{3})^2$

d $(2\sqrt{35} + \sqrt{5})^2$

e $(2\sqrt{10} - \sqrt{2})^2$

f $(3\sqrt{2} - \sqrt{10})^2$

g $(\sqrt{70} - 3\sqrt{10})^2$

h $(\sqrt{50} - 3\sqrt{5})^2$

i $(\sqrt{11} - 2\sqrt{22})^2$

j $(\sqrt{10} - \sqrt{5})^2$

k $(2\sqrt{3} - \sqrt{6})^2$

l $(5\sqrt{14} - 3\sqrt{21})^2$

Example 24

4 Expand and simplify:

a $(3 - \sqrt{5})(3 + \sqrt{5})$

b $(\sqrt{6} - 1)(\sqrt{6} + 1)$

c $(2 + \sqrt{3})(2 - \sqrt{3})$

d $(\sqrt{7} + 4)(\sqrt{7} - 4)$

e $(7\sqrt{2} + 3)(7\sqrt{2} - 3)$

f $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$

g $(\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3})$

h $(2\sqrt{3} - \sqrt{2})(2\sqrt{3} + \sqrt{2})$

i $(\sqrt{2} - 3\sqrt{5})(\sqrt{2} + 3\sqrt{5})$

j $(4\sqrt{2} - 3\sqrt{5})(4\sqrt{2} + 3\sqrt{5})$

k $(6\sqrt{2} - 2\sqrt{7})(6\sqrt{2} + 2\sqrt{7})$

l $(2\sqrt{3} - 5\sqrt{6})(2\sqrt{3} + 5\sqrt{6})$

m $(2\sqrt{5} + 7\sqrt{10})(2\sqrt{5} - 7\sqrt{10})$

n $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$

5 If $x = \sqrt{2} - 1$ and $y = \sqrt{2} + 1$, find:

a x^2

b y^2

c $x^2 + y^2$

d $x^2 - y^2$

e xy

f x^2y

g y^2x

h $\frac{1}{x} + \frac{1}{y}$

6 Find:

a $(\sqrt{5} + \sqrt{3})^2 + (\sqrt{5} - \sqrt{3})^2$

b $(\sqrt{5} + \sqrt{3})^2 - (\sqrt{5} - \sqrt{3})^2$

c $(\sqrt{3} + \sqrt{2})^2 + (\sqrt{3} - \sqrt{2})^2$

d $(\sqrt{3} + \sqrt{2})^2 - (\sqrt{3} - \sqrt{2})^2$

e $(\sqrt{7} - \sqrt{2})^2 + (\sqrt{7} + \sqrt{2})^2$

f $(\sqrt{6} + 2)^2 - (\sqrt{6} - 2)^2$

7 Simplify $(a\sqrt{b} + c\sqrt{d})(a\sqrt{b} - c\sqrt{d})$

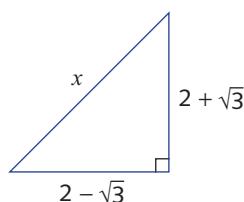
8 For the figures shown, find:

i the value of x

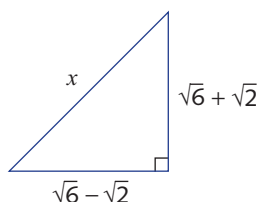
ii the area of the triangle

iii the perimeter of the triangle.

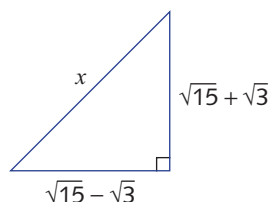
a



b



c

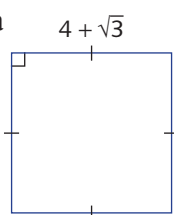


9 For the figures shown, find:

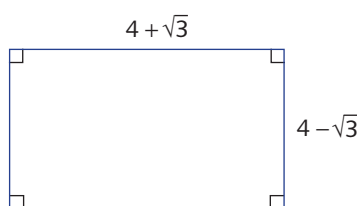
i the perimeter

ii the area.

a



b



10 An equilateral triangle has sides length $1 + \sqrt{3}$. Find:

a the perimeter of the triangle

b the area of the triangle.



The expression $\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{2}$ is untidy. The first term has a square root in the denominator.

Fractions involving surds are easiest to deal with when they are expressed in simplest form, with a rational denominator.

When we multiply the numerator and denominator of a fraction by the same number, we form an equivalent fraction. The same happens with a quotient involving surds.

Example 25

Express $\frac{4}{\sqrt{2}}$ with a rational denominator.

Solution

$$\begin{aligned}\frac{4}{\sqrt{2}} &= \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} && \text{(Multiply top and bottom by the same surd.)} \\ &= \frac{4\sqrt{2}}{2} \\ &= 2\sqrt{2}\end{aligned}$$

Example 26

Write $\frac{9}{4\sqrt{3}}$ as a quotient with a rational denominator.

Solution

$$\begin{aligned}\frac{9}{4\sqrt{3}} &= \frac{9}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} && \text{(Multiply top and bottom by } \sqrt{3} \text{.)} \\ &= \frac{9\sqrt{3}}{12} \\ &= \frac{3\sqrt{3}}{4}\end{aligned}$$

Example 27

Simplify $\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{2}$.

Solution

$$\begin{aligned}\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{2} &= \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} + \frac{\sqrt{3}}{2} && \text{(Rationalise the denominator of the first term.)} \\ &= \frac{2\sqrt{3}}{3} + \frac{\sqrt{3}}{2} \\ &= \frac{4\sqrt{3}}{6} + \frac{3\sqrt{3}}{6} && \text{(Use a common denominator.)} \\ &= \frac{7\sqrt{3}}{6}\end{aligned}$$



Rationalising the denominator

- Rationalising the denominator means converting the denominator into a rational number.
- We use the result

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}} \times \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}, \text{ where } a \text{ is positive,}$$

to rationalise the denominator of a quotient involving surds.

Exercise 2F

Example 25

1 Rationalise the denominator.

a $\frac{1}{\sqrt{5}}$	b $\frac{5}{\sqrt{6}}$	c $\frac{3}{\sqrt{3}}$	d $\frac{14}{\sqrt{7}}$	e $\frac{\sqrt{3}}{\sqrt{7}}$
f $\frac{\sqrt{18}}{\sqrt{6}}$	g $\frac{\sqrt{5}}{\sqrt{3}}$	h $\frac{\sqrt{6}}{\sqrt{3}}$	i $\frac{\sqrt{14}}{\sqrt{7}}$	j $\frac{2\sqrt{7}}{\sqrt{3}}$

Example 26

2 Rationalise the denominator.

a $\frac{1}{5\sqrt{3}}$	b $\frac{7}{3\sqrt{2}}$	c $\frac{4}{7\sqrt{2}}$	d $\frac{\sqrt{5}}{3\sqrt{7}}$	e $\frac{\sqrt{2}}{3\sqrt{10}}$
f $\frac{2\sqrt{5}}{3\sqrt{2}}$	g $\frac{4\sqrt{2}}{5\sqrt{7}}$	h $\frac{8\sqrt{18}}{2\sqrt{3}}$	i $\frac{\sqrt{3}}{4\sqrt{6}}$	j $\frac{\sqrt{15}}{3\sqrt{5}}$

- 3 Given that $\sqrt{2} \approx 1.412$ and $\sqrt{3} \approx 1.732$, find these values, correct to 2 decimal places, without using a calculator. (Where appropriate, first rationalise the denominator.)

a $\frac{1}{\sqrt{2}}$ **b** $\frac{1}{\sqrt{3}}$ **c** $\frac{3}{\sqrt{2}}$ **d** $\frac{5}{\sqrt{3}}$
e $\sqrt{12}$ **f** $\sqrt{18}$ **g** $\frac{1}{\sqrt{12}}$ **h** $\frac{1}{\sqrt{18}}$

Example 27

- 4 By first rationalising the denominators, simplify each expression.

a $\frac{1}{\sqrt{2}} + \sqrt{2}$ **b** $\frac{2}{\sqrt{3}} + \frac{3}{2\sqrt{3}}$ **c** $\frac{5\sqrt{2}}{3} - \frac{1}{\sqrt{3}}$
d $\frac{4\sqrt{5}}{\sqrt{2}} - \frac{2\sqrt{5}}{\sqrt{2}}$ **e** $\frac{\sqrt{72}}{\sqrt{3}} + \frac{3}{\sqrt{2}} - \frac{2}{2\sqrt{2}}$ **f** $\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{7}{2\sqrt{3}}$

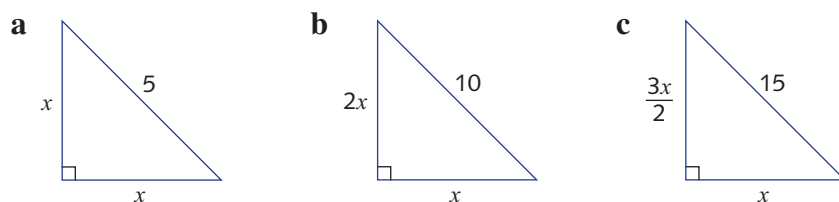
- 5 If $x = 2\sqrt{14}$ and $y = 4\sqrt{2}$, find and rationalise the denominator:

a $\frac{x}{y}$ **b** $\frac{y}{x}$ **c** $\frac{2x}{y}$ **d** $\frac{\sqrt{2}x}{\sqrt{3}y}$

- 6 If $x = 2\sqrt{3}$, find and rationalise the denominator:

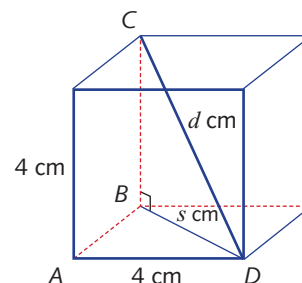
a $x + \frac{1}{x}$ **b** $x - \frac{1}{x}$ **c** $\left(x + \frac{1}{x}\right)^2$ **d** $\left(x - \frac{1}{x}\right)^2$

- 7 Find the value of x . Express your answer with a rational denominator.





How can we find the length of the diagonal CD of a cube whose side length is 4 cm? BD is a **face diagonal** and sometimes CD is called a **space diagonal**.



We can apply Pythagoras' theorem to triangle BAD to find the square of the length s cm of the diagonal BD .

$$\begin{aligned}s^2 &= 4^2 + 4^2 \\ &= 32\end{aligned}$$

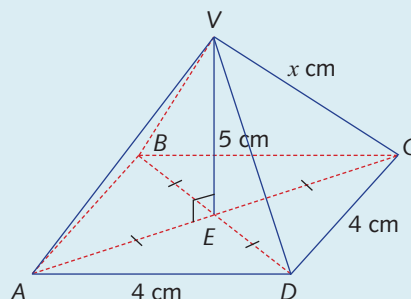
We can then apply Pythagoras' theorem again to triangle CBD since $\angle CBD$ is a right angle. The length d cm of the diagonal CD is given by:

$$\begin{aligned}d^2 &= s^2 + 4^2 \\ &= 32 + 16 \\ &= 48 \\ d &= \sqrt{48} = 4\sqrt{3} \\ &\approx 6.93 \quad (\text{correct to 2 decimal places})\end{aligned}$$

The length of the diagonal CD is 6.93 cm correct to 2 decimal places.

Example 28

A square pyramid has height 5 cm and square base with side length 4 cm. Find the length of the edge VC in this diagram.



Solution

The base is a square with side length 4 cm, so:

$$\begin{aligned} AC^2 &= 4^2 + 4^2 \\ &= 32 \\ AC &= \sqrt{32} \\ &= 4\sqrt{2} \end{aligned}$$

Hence $EC = 2\sqrt{2}$ (Leave in exact form to maintain accuracy.)

Triangle VEC is right-angled, so:

$$\begin{aligned} x^2 &= VE^2 + EC^2 \\ &= 5^2 + (2\sqrt{2})^2 \\ &= 25 + 8 \\ &= 33 \\ x &= \sqrt{33} \\ &\approx 5.74 \quad (\text{correct to 2 decimal places}) \end{aligned}$$

The length of VC is 5.74 cm correct to 2 decimal places.



Applications of Pythagoras' theorem in three dimensions

- Pythagoras' theorem can be used to find lengths in three-dimensional problems.
- Always draw a careful diagram identifying the appropriate right-angled triangle(s).
- To maintain accuracy, use exact values and only approximate using a calculator at the end of the problem if required.



Exercise 2G

Example 28

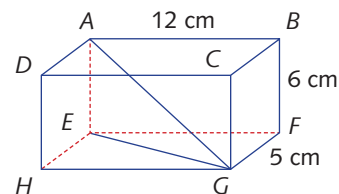
- 1 The rectangular prism in the diagram has a length of 12 cm, a width of 5 cm and a height of 6 cm.

a Consider triangle EFG . Find:

- i EF ii the size of $\angle EFG$.

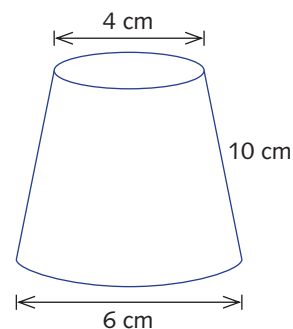
b Find EG .

c Find AG , correct to 1 decimal place.

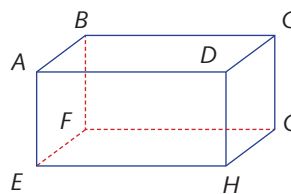


Note: AG is called the space diagonal of the rectangular prism.

- 2 Find the length of the space diagonal of the rectangular prism whose length, width and height are:
- a** 12 cm, 9 cm, 8 cm **b** 12 cm, 5 cm, 8 cm
c 10 cm, 7 cm, 4 cm **d** 8 cm, 6 cm, 4 cm
e 7 cm, 2 cm, 3 cm **f** a cm, b cm, c cm
- 3 Find the length of the longest pencil that can fit inside a cylindrical pencil case of length 15 cm and radius 2 cm.
- 4 A hemisphere of radius length 5 cm is partially filled with water. The top of the hemisphere is horizontal and the surface of the water is a circle of radius 4 cm. Find the depth of the water.
- 5 A builder needs to carry lengths of timber along a corridor in order to get them to where he is working. There is a right-angled bend in the corridor along the way. The corridor is 3 m wide and the ceiling is 2.6 m above the floor. What is the longest length of timber that the builder can take around the corner in the corridor? (Hint: Draw a diagram.)
- 6 A bobbin for an industrial knitting machine is in the shape of a truncated cone. The diameter of the top is 4 cm, the diameter of the base is 6 cm and the length of the slant is 10 cm. Find the height of the bobbin.



- 7 For the rectangular prism shown opposite, $EH = 4$ cm and $HG = 2$ cm.
- a** Find the exact length of EG , giving your answer as a surd in simplest form.
- b** If $AE = \frac{1}{2}EG$, find the exact value of AE .
- c** Find the length of:
- i** BE **ii** BH
- d** What type of triangle is triangle BEH ?
- e** Show that if $EH = 2a$ cm, $HG = a$ cm and $AE = \frac{1}{2}EG$, then the sides of the triangle BEH are in the ratio 3: 4: 5.



2H Binomial denominators



Consider the expression $\frac{1}{\sqrt{7}-\sqrt{5}}$.

How can we write this as a quotient with a rational denominator?

In the section on special products, we saw that:

$$\begin{aligned}(\sqrt{7}-\sqrt{5})(\sqrt{7}+\sqrt{5}) &= (\sqrt{7})^2 - (\sqrt{5})^2 \\&= 7 - 5 \\&= 2, \text{ which is rational,}\end{aligned}$$

so

$$\begin{aligned}\frac{1}{\sqrt{7}-\sqrt{5}} &= \frac{1}{\sqrt{7}-\sqrt{5}} \times \frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}+\sqrt{5}} \\&= \frac{\sqrt{7}+\sqrt{5}}{7-5} \\&= \frac{\sqrt{7}+\sqrt{5}}{2}.\end{aligned}$$

Similarly,

$$\begin{aligned}(5\sqrt{2}-4)(5\sqrt{2}+4) &= (5\sqrt{2})^2 - 4^2 \\&= 34, \text{ which again is rational,}\end{aligned}$$

so

$$\begin{aligned}\frac{3}{5\sqrt{2}-4} &= \frac{3}{5\sqrt{2}-4} \times \frac{5\sqrt{2}+4}{5\sqrt{2}+4} \\&= \frac{15\sqrt{2}+12}{34}.\end{aligned}$$

Using the difference of two squares identity in this way is an important technique.

Example 29

Simplify $\frac{2\sqrt{5}}{2\sqrt{5}-2}$.

Solution

$$\begin{aligned}\frac{2\sqrt{5}}{2\sqrt{5}-2} &= \frac{2\sqrt{5}}{2\sqrt{5}-2} \times \frac{2\sqrt{5}+2}{2\sqrt{5}+2} \\&= \frac{20+4\sqrt{5}}{20-4} \\&= \frac{20+4\sqrt{5}}{16} \\&= \frac{4(5+\sqrt{5})}{16} \\&= \frac{5+\sqrt{5}}{4}\end{aligned}$$

Example 30

Simplify $\frac{\sqrt{3}+\sqrt{2}}{3\sqrt{2}+2\sqrt{3}}$.

Solution

$$\begin{aligned}\frac{\sqrt{3}+\sqrt{2}}{3\sqrt{2}+2\sqrt{3}} &= \frac{\sqrt{3}+\sqrt{2}}{3\sqrt{2}+2\sqrt{3}} \times \frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}} \\&= \frac{(\sqrt{3}+\sqrt{2})(3\sqrt{2}-2\sqrt{3})}{18-12} \\&= \frac{(3\sqrt{6}-6+6-2\sqrt{6})}{6} \\&= \frac{\sqrt{6}}{6}\end{aligned}$$



Binomial denominators

To rationalise a denominator which has two terms, we use the difference of two squares identity:

- In an expression such as $\frac{3}{5 + \sqrt{3}}$, multiply top and bottom by $5 - \sqrt{3}$.
- In an expression such as $\frac{\sqrt{2}}{7 - 3\sqrt{2}}$, multiply top and bottom by $7 + 3\sqrt{2}$.

The surd $7 - \sqrt{3}$ is called the **conjugate** of $7 + \sqrt{3}$ and $7 + \sqrt{3}$ is the conjugate of $7 - \sqrt{3}$.



Exercise 2H

1 Simplify:

a $(\sqrt{7} - \sqrt{5})(\sqrt{7} + \sqrt{5})$

b $(2\sqrt{5} - \sqrt{3})(2\sqrt{5} + \sqrt{3})$

c $(7\sqrt{2} + 4\sqrt{3})(7\sqrt{2} - 4\sqrt{3})$

d $(4 - \sqrt{3})(4 + \sqrt{3})$

2 Rationalise the denominator in each expression.

a $\frac{1}{\sqrt{6} + 1}$

b $\frac{3}{\sqrt{2} - 1}$

c $\frac{2}{3 + \sqrt{5}}$

d $\frac{2}{3 - \sqrt{5}}$

e $\frac{4}{\sqrt{5} + \sqrt{2}}$

f $\frac{1}{\sqrt{7} - \sqrt{5}}$

g $\frac{\sqrt{3}}{\sqrt{6} + \sqrt{5}}$

h $\frac{\sqrt{2}}{\sqrt{2} - \sqrt{3}}$

i $\frac{\sqrt{2}}{2\sqrt{5} + \sqrt{2}}$

j $\frac{\sqrt{5}}{2\sqrt{5} - \sqrt{3}}$

k $\frac{\sqrt{5}}{3\sqrt{2} + 4\sqrt{3}}$

l $\frac{\sqrt{3}}{2\sqrt{3} - 3\sqrt{6}}$

m $\frac{2\sqrt{6}}{4\sqrt{2} - 3\sqrt{7}}$

n $\frac{3\sqrt{2}}{6\sqrt{3} + 11\sqrt{5}}$

o $\frac{2\sqrt{3}}{3\sqrt{2} + 10\sqrt{3}}$

Example 29

- 3** Rationalise the denominators in these expressions and use the decimal approximations $\sqrt{2} \approx 1.414$ and $\sqrt{3} \approx 1.732$ to evaluate them correct to 2 decimal places.

a $\frac{1}{\sqrt{2}-1}$

b $\frac{1}{\sqrt{3}+2}$

c $\frac{1}{\sqrt{3}+\sqrt{2}}$

d $\frac{1}{\sqrt{3}-\sqrt{2}}$

- 4** Find the integers p and q such that $\frac{\sqrt{5}}{\sqrt{5}-2} = p + q\sqrt{5}$.

- 5** Simplify:

a $\frac{3}{\sqrt{5}-2} + \frac{2}{\sqrt{5}+2}$

b $\frac{2}{6-3\sqrt{3}} - \frac{1}{2\sqrt{3}+3}$

Example 30

- 6** Simplify:

a $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} + \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$

b $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} - \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$

c $\frac{5}{(\sqrt{7}-\sqrt{2})^2}$

2 Irrational numbers and surds



We first recall some facts about fractions and decimals.

Fractions and decimals

We know that some fractions can be written as decimals that **terminate**. For example,

$$\frac{1}{4} = 0.25, \quad \frac{3}{16} = 0.1875.$$

Some fractions have decimal representations that do not terminate. For example,

$$\frac{1}{3} = 0.333\,33\ldots \text{ which we write as } 0.\dot{3}.$$

The dot above the 3 indicates that the digit 3 is repeated forever.

Some fractions have decimal representations that eventually repeat. For example,

$$\frac{1}{6} = 0.166\,6\ldots \text{ which is written as } 0.1\dot{6}.$$

Other fractions have decimal representations with more than one repeating digit. For example,

$$\frac{1}{11} = 0.090\,909\ldots \text{ which we write as } 0.\dot{0}\dot{9},$$

with a dot above both of the repeating digits.

Another example is $0.12\dot{3}\,45\dot{6} = 0.123\,456\,345\,6\ldots$

Converting decimals to fractions

It is easy to write terminating decimals as fractions by using a denominator which is a power of 10.

For example,

$$0.14 = \frac{14}{100} = \frac{7}{50}.$$

Decimals that have repeated digits can also be written as fractions.

For example,

$$0.\dot{1}\dot{2}\dot{3} = \frac{41}{333} \text{ and } 0.67\dot{1}\dot{2} = \frac{443}{660}.$$

A method for doing this is shown in the next two examples.

Example 31

Write $0.\dot{2}$ as a fraction.

Solution

$$\begin{aligned}\text{Let} \quad S &= 0.\dot{2} \\ \text{So} \quad S &= 0.2222 \dots \\ \text{Then} \quad 10S &= 2.2222 \dots \quad (\text{Multiply by } 10.) \\ 10S &= 2 + 0.222 \dots \\ \text{Therefore } 10S &= S + 2 \\ \text{Hence} \quad 9S &= 2 \\ \text{so,} \quad S &= \frac{2}{9} \\ \text{Thus} \quad 0.\dot{2} &= \frac{2}{9}.\end{aligned}$$

Example 32

Write $0.\dot{1}\dot{2}$ as a fraction.

Solution

$$\begin{aligned}\text{Let } S &= 0.\dot{1}\dot{2} \\ \text{So} \quad S &= 0.12121212 \dots \\ \text{Then} \quad 100S &= 12.121212 \dots \\ \text{Therefore } 100S &= 12 + S \\ \text{Hence} \quad 99S &= 12 \\ \text{so,} \quad S &= \frac{12}{99} \\ &= \frac{4}{33} \\ \text{Thus} \quad 0.\dot{1}\dot{2} &= \frac{4}{33}.\end{aligned}$$

Rational numbers

A **rational number** is a number that can be written as a fraction $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

All integers are rational numbers. For example,

$$-3 = \frac{-3}{1}.$$

Example 33

Explain why each of these numbers is rational.

a $5\frac{3}{7}$

b 2.1237

c $0.\dot{2}\dot{4}$

Solution

We will express each number in the form $\frac{p}{q}$.

a $5\frac{3}{7} = \frac{38}{7}$

b $2.1237 = 2\frac{1237}{10000}$
 $= \frac{21237}{10000}$

c Let $S = 0.\dot{2}\dot{4}$
 $= 0.24242 \dots$

$$100S = 24.24242 \dots \quad (\text{Multiply by } 100.)$$

$$100S = 24 + S$$

Hence $99S = 24$

$$S = \frac{24}{99}$$
$$= \frac{8}{33}$$

Thus $0.\dot{2}\dot{4} = \frac{8}{33}$.

Each number has been expressed as a fraction $\frac{p}{q}$, where p and q are integers, so each number is rational.

Irrational numbers

Mathematicians up to about 600 BCE thought that all numbers were rational. However, when we apply Pythagoras' theorem, we encounter numbers such as $\sqrt{2}$ that are not rational. The number $\sqrt{2}$ is an example of an **irrational number**.

An irrational number is one that is not rational. Hence an irrational number cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Nor can it be written as a terminating or repeating decimal.

The decimal expansion of $\sqrt{2}$ goes on forever but does not repeat. The value of $\sqrt{2}$ can be approximated using a calculator. The same is true of other irrational numbers such as $\sqrt{3}$, $\sqrt{14}$ and $\sqrt{91}$. Try finding these numbers on your calculator and see what you get.

Here is a proof that $\sqrt{2}$ is irrational. It goes back to Euclid, 300BCE. See if you can follow the logical steps in the proof.

Proof:

Suppose $\sqrt{2}$ was rational, so we could write it as $\frac{p}{q}$, where p and q are whole numbers. We can assume that our fraction $\frac{p}{q}$ is in simplest form, so that p and q have no common factors.

$$\text{Then } \sqrt{2} = \frac{p}{q}$$

$$2 = \frac{p^2}{q^2} \quad (\text{Square both sides.})$$

$$2q^2 = p^2 \quad (\text{Multiply both sides by } q^2.)$$

The last line tells us that p^2 is an even number, since it is twice q^2 . Hence the number p is also even, since an odd number squared is odd.

We can write $p = 2k$, where k is some integer.

Substituting back in for p , we have

$$2q^2 = (2k)^2$$

$$2q^2 = 4k^2$$

$$q^2 = 2k^2$$

The last line tells us that q^2 is even and so q is even. We have found that both p and q are even. This contradicts our assumption that the fraction $\frac{p}{q}$ has been cancelled down. Hence our assumption that $\sqrt{2}$ was rational is incorrect.

That is, $\sqrt{2}$ is irrational.

Other examples of irrational numbers include

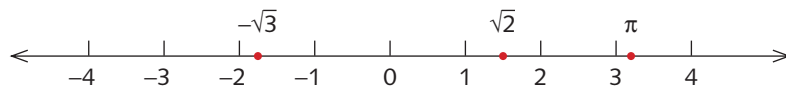
$$\sqrt{3}, \sqrt[3]{2}, \pi, \pi^3 \text{ and } \sqrt{\pi}.$$

The fact that π is irrational is not obvious. It is hard to prove. The first proof was given by J. H. Lambert in 1768.

There are infinitely many rational numbers because every whole number is rational. We can also easily write down infinitely many other fractions. For example, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

There are infinitely many irrational numbers too. For example, $\sqrt{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{4}, \dots$

Every number, whether rational or irrational, is represented by a point on the number line. Conversely, we can think of each point on the number line as a number.



Example 34

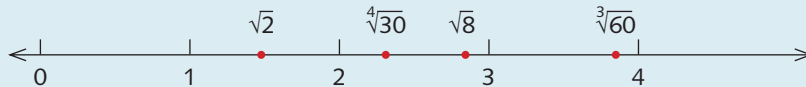
Arrange these irrational numbers in order of size on the number line.

$$\sqrt{8} \qquad \sqrt{2} \qquad \sqrt[3]{60} \qquad \sqrt[4]{30}$$

Solution

Find an approximation of each number to 2 decimal places using a calculator.

$$\begin{aligned} \sqrt{8} &\approx 2.83 & \sqrt{2} &\approx 1.41 \\ \sqrt[3]{60} &\approx 3.91 & \sqrt[4]{30} &\approx 2.34 \end{aligned}$$



Surds

In Year 8, we looked at squares, square roots, cubes and cube roots. For example:

$$\begin{aligned} 5^2 &= 25 & \text{and} & & \sqrt{25} &= 5 \\ 5^3 &= 125 & \text{and} & & \sqrt[3]{125} &= 5 \end{aligned}$$

The use of this notation can be extended. For example:

$$5^4 = 625 \quad \text{and} \quad \sqrt[4]{625} = 5$$

This is read as ‘The fourth power of 5 is 625 and the fourth root of 625 is 5’. We also have fifth roots, sixth roots and so on.

In general, we can take the n th root of any positive number a . The **n th root of a** is the positive number whose n th power is a .

The statement $\sqrt[n]{a} = b$ is equivalent to the statement $b^n = a$.

Your calculator will give you approximations to the n th root of a .

Note: For n , a positive integer, $0^n = 0$ and $\sqrt[n]{0} = 0$.

An irrational number which can be expressed as $\sqrt[n]{a}$, where a is a positive whole number, is called a **surd**.

For example, $\sqrt{2}$, $\sqrt{7}$ and $\sqrt[3]{5}$ are all examples of surds, while $\sqrt{4}$ and $\sqrt{9}$ are not surds since they are whole numbers. The number π , although it is irrational, is not a surd. This is also difficult to prove.

Constructing some surds geometrically

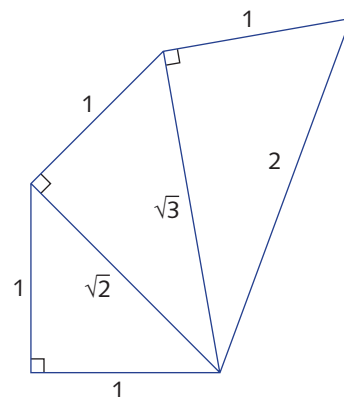
We can use Pythagoras' theorem to construct intervals of length $\sqrt{2}$, $\sqrt{3}$ and so on.

Using a ruler, draw a length 1 unit.

Using your set square, draw a right angle at the end of your interval and mark off 1 unit.

Joining the endpoints, we have a length of $\sqrt{2}$ units, by Pythagoras' theorem.

If we now draw an interval of length 1 unit perpendicular to the hypotenuse, as shown in the diagram, and form another right-angled triangle, then the new hypotenuse is $\sqrt{3}$ units in length.



We can continue this process as shown to construct intervals of length $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$ and so on.

The word **surd** has an interesting history. During the period known as the Dark Ages in Western Europe, much of the scientific and mathematical knowledge of the ancient Greeks was translated into Arabic in the Middle East. The Arabic equivalent for the Greek word for *irrational* also means *deaf*. When mathematics was revived in the West and works were translated from Arabic to Latin, the translators used the word *surdus*, meaning 'deaf', to translate what was originally meant as 'irrational'.



Irrational numbers and surds

- Every fraction can be written as a terminating, or eventually repeating, decimal.
- Every terminating, or eventually repeating, decimal can be written as a fraction.
- A **rational** number is a number which can be expressed as $\frac{p}{q}$, where p and q are integers and $q \neq 0$. That is, a rational number is an integer or a fraction.
- There are numbers such as π and $\sqrt{2}$ that are **irrational** (not rational).
- Each rational and each irrational number represents a point on the number line.
- Every point on the number line represents a rational or an irrational number.
- An irrational number which can be expressed as $\sqrt[n]{a}$, where n and a are positive whole numbers, is called a **surd**. π is not a surd.



Exercise 2I

Example 31

- 1 Write each repeating decimal as a fraction.

a $0.\dot{5}$ b $0.\dot{7}$ c $0.\dot{9}$ d $0.1\dot{4}$ e $0.2\dot{3}$ f $0.6\dot{2}$

Example 32

- 2 Write each repeating decimal as a fraction.

a $0.\dot{1}\dot{3}$ b $0.0\dot{7}$ c $0.9\dot{1}$ d $0.2\dot{4}\dot{1}$ e $0.\dot{6}\dot{1}\dot{3}$
 f $0.0\dot{1}\dot{6}$ g $0.3\dot{2}\dot{4}$ h $0.51\dot{2}\dot{6}$ i $0.00\dot{1}\dot{2}$

Example 33

- 3 Show that each number is rational by writing it as a fraction.

a $3\frac{2}{3}$ b 5.15 c $0.\dot{4}$ d $0.\dot{1}\dot{5}$ e $5\frac{1}{7}$ f 1.3

- 4 Which of these numbers are irrational?

a $\sqrt{7}$ b $\sqrt{25}$ c $0.\dot{6}$ d $\frac{\pi}{3}$

Example 34

- 5 By approximating correct to 2 decimal places, place these real numbers on the same number line.

a $\sqrt{7}$ b 2.7 c $\sqrt[3]{18}$ d 2π e $\sqrt{2}$

- 6 Which of these numbers are surds?

a 3 b $\sqrt{5}$ c 4 d $\sqrt{9}$ e $\sqrt{7}$ f $\sqrt{16}$
 g $\sqrt{10}$ h $\sqrt{1}$ i $\sqrt{3}$ j $\sqrt{15}$ k 5 l $\sqrt{25}$

- 7 a Use the fact that $1^2 + 2^2 = 5$ to construct a length $\sqrt{5}$.

b Use the fact that $4^2 + 5^2 = 41$ to construct a length $\sqrt{41}$.

- 8 How would you construct an interval of length:

a $\sqrt{73}$ b $\sqrt{12}$ c $\sqrt{21}$

- 9 a Show that $\frac{138}{19} = 7 + \frac{1}{3 + \frac{1}{1 + \frac{1}{4}}}$

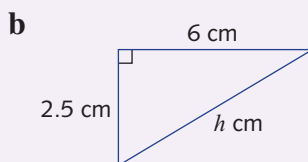
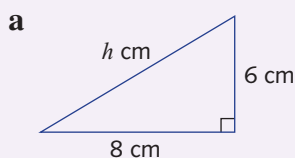
The expression on the right is called a **continued fraction**.

- b Express $\frac{153}{11}$ as a continued fraction with all numerators 1.

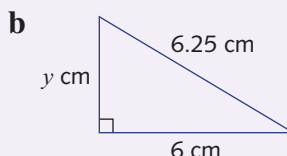
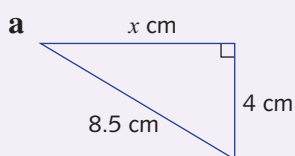


Review exercise

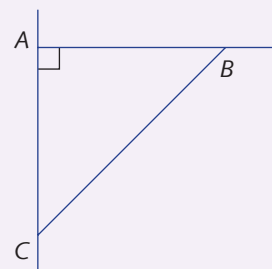
- 1 For each right-angled triangle, find the value of the pronumeral.



- 2 For each right-angled triangle, find the value of the pronumeral.



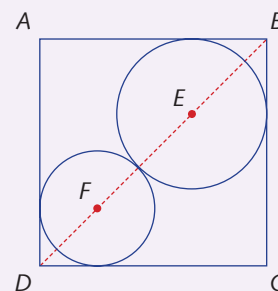
- 3 A support bracket is to be placed under a shelf, as shown in the diagram. If $AB = AC = 20$ cm, find, correct to the nearest millimetre, the length of BC .



- 4 A road runs in an east–west direction joining towns A and B , which are 40 km apart. A third town, C , is situated 20 km due north of B . A straight road is built from C to the road between A and B and meets it at D , which is equidistant from A and C . Find the length of road CD .

- 5 Circles of radius 6 cm and 3 cm are placed in a square, as shown in the diagram opposite. Find, correct to 2 decimal places:

- a** EB **b** FD
c the length of the diagonal BD
d the side length of the square.



- 6 Simplify:

a $3\sqrt{2} + 2\sqrt{2}$ **b** $\sqrt{32} - \sqrt{18}$

- 7 Simplify:

a $2\sqrt{3} \times 5\sqrt{6}$ **b** $3\sqrt{5} \times 2\sqrt{10}$

- 8 Simplify:

a $2\sqrt{3}(3 + \sqrt{3})$ **b** $5\sqrt{2}(3\sqrt{2} - 2)$

- 9 Expand and simplify:

a $(2\sqrt{2} + 1)(3\sqrt{2} - 2)$ **b** $(5\sqrt{3} - 2)(2\sqrt{3} - 1)$



10 Simplify:

a $\sqrt{80}$ **b** $\sqrt{108}$ **c** $\sqrt{125}$ **d** $\sqrt{72}$
e $\sqrt{2048}$ **f** $\sqrt{448}$ **g** $\sqrt{800}$ **h** $\sqrt{112}$

11 Simplify:

a $\sqrt{45} + 2\sqrt{5} - \sqrt{80}$ **b** $\sqrt{28} + 2\sqrt{63} - 5\sqrt{7}$
c $\sqrt{44} + \sqrt{275} - 4\sqrt{11}$ **d** $\sqrt{162} - \sqrt{200} + \sqrt{288}$

12 Express with a rational denominator.

a $\frac{3}{\sqrt{11}}$ **b** $\frac{1}{5\sqrt{15}}$ **c** $\frac{4}{7\sqrt{7}}$ **d** $\frac{3}{\sqrt{17}}$
e $\frac{3}{\sqrt{3}}$ **f** $\frac{3}{5\sqrt{15}}$ **g** $\frac{14}{\sqrt{7}}$ **h** $\frac{11}{\sqrt{3}}$

13 Express with a rational denominator.

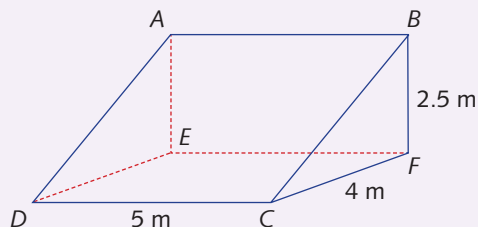
a $\frac{3}{3-\sqrt{3}}$ **b** $\frac{22}{2+3\sqrt{5}}$ **c** $\frac{24}{1-\sqrt{7}}$ **d** $\frac{24}{1+\sqrt{17}}$
e $\frac{3}{2-\sqrt{3}}$ **f** $\frac{30}{1-\sqrt{11}}$ **g** $\frac{15}{2-\sqrt{7}}$ **h** $\frac{10}{2-\sqrt{3}}$

14 **a** Expand and simplify $(2\sqrt{5} - \sqrt{3})^2$.

b Simplify $\frac{2}{\sqrt{3}-2} + \frac{2}{\sqrt{3}}$, expressing your answer with a rational denominator.

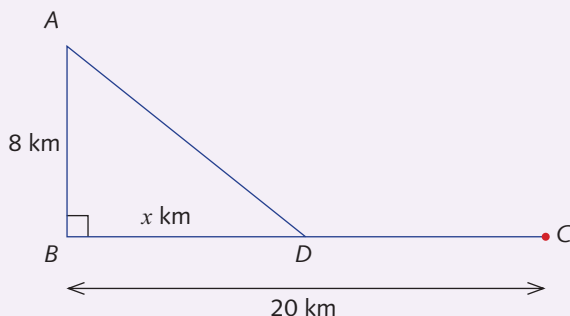
15 The diagram opposite shows part of a skate board ramp. (It is a prism whose cross-section is a right-angled triangle). Use the information in the diagram to find:

a BC **b** AC



16 Find the length of the space diagonal of a cube with side length 5 cm.

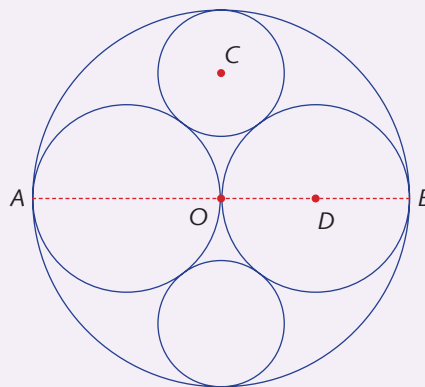
17 A motorist departs from town B , which is 8 km due south from another town, A , and drives due east towards town C , which is 20 km from B . After driving a distance of x km, he notices that he is the same distance away from both towns A and C .





- a Express the motorist's distance from A in terms of x (that is, AD).
- b Express the motorist's distance from C in terms of x .
- c Find the distance he has driven from B .

- 18** The diagram opposite shows the logo of a particular company. The large circle has centre O and radius 12 cm and AB is a diameter. D is the centre of the middle-sized circle with diameter OB . Finally, C is the centre of the smallest circle.

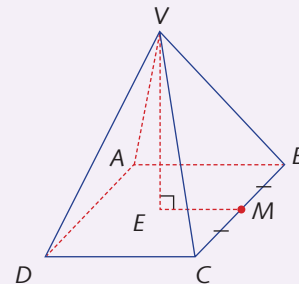


- a What is the radius of the circle with centre D ?
- b If the radius of the circle with centre C is r cm, express these in terms of r :

i OC ii DC

- c Find the value of r .

- 19** In the diagram opposite, $VABCD$ is a square-based pyramid with $AB = BC = CD = DA = 10$ and $VA = VB = VC = VD = 10$.



- a What type of triangle is triangle VBC ?
- b If M is the midpoint of CB , find the exact values of:
 - i VM
 - ii VE , the height of the pyramid.

- 20** Write each repeating decimal as a fraction.

- | | | | |
|----------------|----------------|-----------------|------------------|
| a $0.\dot{8}$ | b $0.1\dot{5}$ | c $0.7\dot{2}$ | d $0.0\dot{3}$ |
| e $0.8\dot{1}$ | f $0.9\dot{6}$ | g $0.10\dot{1}$ | h $0.001\dot{5}$ |

- 21** Write $\frac{67}{29}$ as a continued fraction. (See Exercise 2I, question 9)

- 22** Write $0.47928\dot{1}$ as a fraction.

- 23** How would you construct intervals of the following lengths? Discuss with your teacher.

- | | | | |
|--------------|------------------------|--------------|------------------------|
| a $\sqrt{3}$ | b $\frac{1}{\sqrt{2}}$ | c $\sqrt{8}$ | d $\frac{1}{\sqrt{8}}$ |
|--------------|------------------------|--------------|------------------------|

- 24** Find the length of the long diagonal of the rectangular prism whose length, width and height are:

- | | |
|---|---|
| a $3 + \sqrt{2}$, $3 - \sqrt{2}$, $3\sqrt{3}$ | b $5 + \sqrt{3}$, $5 - \sqrt{3}$, $2\sqrt{2}$ |
|---|---|

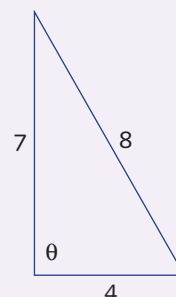
- 25** For $x = \sqrt{2} + 1$ and $y = \sqrt{5} - 2$, find in simplest form.

- | | | | |
|-----------------|-------------------------------|-----------------------------------|-------------------------------|
| a $\frac{1}{x}$ | b $\frac{1}{x} + \frac{1}{y}$ | c $\frac{1}{x^2} + \frac{1}{y^2}$ | d $\frac{1}{x} - \frac{1}{y}$ |
|-----------------|-------------------------------|-----------------------------------|-------------------------------|

Challenge exercise



- 1 Prove that $\sqrt[3]{2}$ is irrational.
- 2 In the triangle to the right, $4^2 + 7^2 = 65 > 8^2$.
Is the angle θ greater or less than 90° ? Explain your answer.

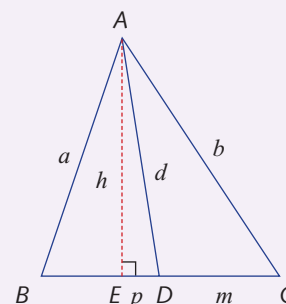
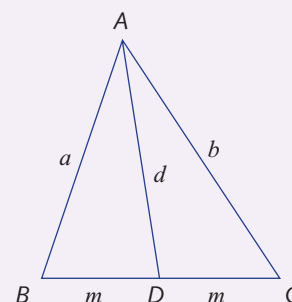


- 3 **Apollonius' theorem** states that in any triangle, if we join a vertex to the midpoint of the opposite side, and the length of that line is d , then $a^2 + b^2 = 2(d^2 + m^2)$ where $2m = c$.

In words it says: In any triangle, the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median.

Prove this result as follows:

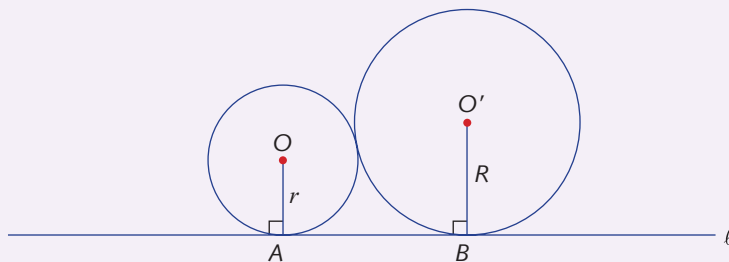
- a Drop a perpendicular AE of length h from A to BC and let $ED = p$.
 - i Write BE in terms of m and p .
 - ii Write EC in terms of m and p .
- b By applying Pythagoras' theorem to the three right-angled triangles in the diagram, complete these statements:
 $a^2 = \dots\dots\dots$
 $b^2 = \dots\dots\dots$
 $d^2 = \dots\dots\dots$
- c Add the first two equations in part **b** above and simplify.
- d Use the third equation in part **b** to deduce Apollonius' theorem.
- e What happens when the angle at A is a right angle?



- 4 A rectangle $ABCD$ has $AB = 15$ cm, $AD = 10$ cm.
A point P is located inside the rectangle such that $AP = 14$ cm and $PB = 11$ cm.
Find PD in exact form.
- 5 E is any point inside the rectangle $ABCD$. Let $AE = a$, $EB = b$, $EC = c$ and $ED = x$.
Prove that $x^2 = a^2 - b^2 + c^2$.



- 6 We use the fact that every integer can be uniquely factorised as a product of primes to give another proof that $\sqrt{2}$ is irrational. We begin by supposing the opposite – that is, we suppose that we can write $\sqrt{2} = \frac{p}{q}$, where p and q are whole numbers with no common factors except 1.
- a Explain why q^2 is a factor of p^2 .
 - b Explain why q is a factor of p .
 - c Explain how this shows that $\sqrt{2}$ is irrational.
- 7 Prove that $\sqrt{6}$ is irrational.
- 8 Expand and simplify:
- a $(\sqrt{2} + 1)^4$
 - b $(\sqrt{3} - 1)^4$
- 9 Prove that there are infinitely many irrational numbers between 0 and 1.
- 10 Prove that between any two numbers, there are infinitely many rational numbers and infinitely many irrational numbers.
- 11 Rationalise the denominator of:
- a $\frac{1}{(1 + \sqrt{2}) + \sqrt{5}}$
 - b $\frac{1}{\sqrt{7} + \sqrt{3} - 2}$
- 12 Show that $\sqrt[3]{7} < \sqrt[5]{5} < \sqrt{2} < \sqrt[3]{3}$ without using a calculator.
- 13



Two circles with centres O and O' and radii r and R meet at a single point. A and B are the points where the circles touch line ℓ . Find the distance between points A and B .



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CHAPTER

3

Number and Algebra

Consumer arithmetic

This chapter deals with some important practical financial topics such as investing and borrowing money, income tax and GST, inflation, depreciation, profits and losses, discounts and commissions. Mention is also made of topics such as population, rainfall and the composition of materials.

When buying a car, using a credit card, or deciding how good a particular 'bargain' is, we all need to be competent in our understanding of consumer arithmetic. Otherwise, we run the serious risk of wasting our money or, even worse, being at the mercy of unscrupulous individuals. This is particularly important when we come to the 'big ticket items' such as purchasing a car or a home, or planning superannuation. It is also important when completing our tax returns.

Using a calculator and approximations

Everything in this chapter involves calculations with percentages. We are now assuming that you are using a calculator, so we have made little attempt to set questions where the numbers work out nicely. However, some of the problems can be done mentally. This chapter gives you the opportunity to learn how to efficiently use a calculator.

Nevertheless, you should always look over your work and check that the answers to your calculations are reasonable and sensible.

When the calculator displays numbers with many decimal places, you will need to round the answer in some way that is appropriate in the context of the question.

Whenever you round a number, you should be careful to use the symbol $\approx$, which means *approximately equals*, rather than the symbol $=$, which means *exactly equals*.

For example, if we are told that a grocer received \$1000 in cash payments for goods that he had sold and that he banked one third of this, we would write:

$$\begin{aligned}\text{amount banked} &= 1000 \div 3 \\ &\approx \$333.33\end{aligned}$$

not

$$\begin{aligned}\text{amount banked} &= 1000 \div 3 \\ &= \$333.3333 \dots\end{aligned}$$

because the grocer could not have banked fractions of a cent.



A percentage such as 35% is a rational number and can be rewritten as a decimal or as a fraction, as follows:

$$35\% = 35 \div 100 = 0.35 \quad \text{or} \quad 35\% = \frac{35}{100} = \frac{7}{20}$$

In this chapter, we encourage you to write percentages as decimals rather than fractions. Decimals are more commonly used than fractions when dealing with money.



Converting a percentage to a decimal or a fraction

To convert a percentage to a decimal or a fraction, divide by 100.

For example: $42\% = 42 \div 100$

$$= 0.42$$

$$37\% = 37 \times \frac{1}{100}$$

$$= \frac{37}{100}$$

Conversely, to convert a decimal such as 1.24, or a fraction such as $\frac{3}{8}$, to a percentage, multiply by 100%.

$$1.24 = 1.24 \times 100\% = 124\%$$

$$\text{and} \quad \frac{3}{8} = \frac{3}{8} \times \frac{100}{1}\% = 37\frac{1}{2}\%$$



Converting a decimal or a fraction to a percentage

To convert a decimal or a fraction to a percentage, multiply by 100%.

Percentages of a quantity

To calculate a percentage of a quantity:

- convert the percentage to a decimal or a fraction
- multiply the quantity by the fraction or decimal.

Example 1

The Brilliant Light Bulb Company estimates that 3.5% of its light bulbs are defective. If a shop owner buys 1250 light bulbs to light the shop, how many would he expect to be defective?

Solution

$$\begin{aligned}\text{Number of defective bulbs} &= 1250 \times 3.5\% \\ &= 1250 \times 0.035 \\ &\approx 44 \quad (\text{Round } 43.75 \text{ to } 44.)\end{aligned}$$

Calculating a percentage

We can express one quantity as a percentage of another. Remember that both quantities must be expressed in the same unit of measurement before calculating a percentage.

Example 2

A typical computer weighs about 25 kg. When it is broken down as waste, it yields about 3 g of arsenic. What percentage of the total is this?

Solution

Using grams, the computer weighs 25 000 g and the arsenic weighs 3 g.

$$\begin{aligned}\text{Hence percentage of arsenic} &= \frac{3}{25\,000} \times 100\% \\ &= \frac{3}{250} \% \\ &= 0.012\%\end{aligned}$$



Calculating a percentage

To calculate the percentage that one quantity, a , is of another quantity, b :

- first convert both quantities to the same unit of measurement
- then form the fraction $\frac{a}{b}$ and multiply it by 100%.



Exercise 3A

1 Express each percentage as a decimal.

- a** 72% **b** 7.6% **c** 98% **d** 16%
e 8% **f** 6.25% **g** 175% **h** 0.6%
i $77\frac{3}{4}\%$ **j** 0.1% **k** 142.6% **l** $\frac{1}{4}\%$

2 Express each percentage as a fraction in lowest terms.

- a** 35% **b** 56% **c** 75% **d** $37\frac{1}{2}\%$
e $33\frac{1}{3}\%$ **f** $16\frac{2}{3}\%$ **g** 7.25% **h** 6.4%
i 210% **j** 125% **k** $112\frac{1}{2}\%$ **l** 136%

3 Express each fraction or decimal as a percentage.

- a** $\frac{3}{5}$ **b** $\frac{3}{8}$ **c** $\frac{9}{16}$ **d** $2\frac{1}{4}$
e $\frac{7}{20}$ **f** $\frac{2}{3}$ **g** $\frac{4}{3}$ **h** $\frac{3}{400}$
i 0.43 **j** 0.225 **k** 0.04 **l** 0.015
m 1.2 **n** 2.03 **o** 1.175 **p** 0.0075

4 Copy and complete this table.

	Percentage	Fraction	Decimal
a	54%		
b		$\frac{2}{5}$	
c			0.32
d	18.5%		
e			0.06
f		$\frac{7}{8}$	
g			1.02
h	108.6%		
i		$1\frac{2}{5}$	

Example 1

5 Evaluate these amounts, correct to 2 decimal places where necessary.

- a 15% of 40 b 57% of 1000 c 26% of 264
 d 120% of 538 e 15.8% of 972 f 138.5% of 650
 g 2.8% of 318 h 0.1% of 6000 i 150% of 846

6 Evaluate these amounts, correct to the nearest cent where necessary.

- a 62% of \$10 b 23.7% of \$960 c 3.2% of \$1500
 d 110% of \$1280 e 0.25% of \$800 f $6\frac{1}{2}\%$ of \$200
 g $7\frac{3}{4}\%$ of \$1000 h $\frac{1}{4}\%$ of \$840 i 7.25% of \$1600

Example 2

7 Find what percentage the first quantity is of the second quantity, correct to 1 decimal place where necessary.

- a 7 km, 50 km b \$4, \$200 c 14 kg, 400 kg
 d 70 m, 50 m e 15 weeks, 60 weeks f 60 weeks, 15 weeks

8 Find what percentage the first quantity is of the second quantity, correct to 2 decimal places where necessary. You will need to express both quantities in the same unit first.

- a 68 cents, \$5.00 b 3.4 cm, 2 m
 c 7 g, 3 kg d 8 hours, 2 weeks
 e 15 days, 3 years f 250 m, 4 km
 g 4 km, 250 m h 1 day, 1 year
 i 1 year, 1 day j 33 weeks, 1 century
 k 56 cm, 2.4 km l 5 apples, 16 dozen apples

9 Find what percentage the first quantity is of the second quantity, correct to 4 decimal places where necessary.

- a 48 mm, 1 km b 1.5 hours, 3 years
 c 7.8 g, 60 kg d 3.5 cents, \$1400

10 There are 740 students at a primary school, 5% of whom have red hair. Calculate the number of students in the school who have red hair.

11 A sample of a certain alloy weighs 1.6 g.

- a Aluminium makes up 48% of the alloy. What is the weight of the aluminium in the sample?
 b The percentage of lead in the alloy is 0.23%. What is the weight of the lead in the sample?

12 A soccer match lasted 92 minutes (including injury time). If team A was in possession for 55% of the match, for how many minutes and seconds was team A in possession?

- 13** A football club with 13 000 members undertook a membership drive, and the membership increased by 112%.
- a** How many new members joined the club?
 - b** What is the size of the club's membership now?
- 14** Carbon dioxide makes up 0.059% of the mass of the Earth's atmosphere. The total mass of the atmosphere is about 5 million megatonnes. What is the total mass of the carbon dioxide in the atmosphere?
- 15** The label on a Sunnyvale tomato paste bottle says that in every 25 g serving, there are 3.6 g of carbohydrate, 0.1 g of fat, and 105 mg of sodium.
- a** Express as a percentage of the 25 g serving:
 - i** the mass of carbohydrate
 - ii** the mass of fat
 - iii** the mass of sodium.
 - b** The Sunnyvale website claims that the percentage of protein is 3.2%. What mass of protein is that per 25 g serving?
- 16** Mt Kosciusko has a height of 2228 m, while the height of Mt Everest is 8848 m. Calculate your answers to this question correct to 3 decimal places.
- a** What percentage is the height of Mt Everest of the height of Mt Kosciusko?
 - b** The Earth's radius is about 6400 km. What percentage of the radius of the Earth is the height of Mt Everest?
- 17** In the Federal Parliament, there are 150 members in the House of Representatives, of whom 37 are from Victoria.
- a** Correct to 1 decimal place, what percentage of members are from Victoria?
 - b** The population of Australia is about 22.6 million. What percentage of Australians are members of the House of Representatives?
- 18** The distance by air from Melbourne to Darwin is 3346 km, and from Melbourne to Singapore it is 6021 km. What percentage, correct to the nearest percent, is:
- a** the Melbourne–Darwin distance of the Melbourne–Singapore distance?
 - b** the Melbourne–Singapore distance of the Melbourne–Darwin distance?

3B Using percentages



Percentages are used extensively in finance, and are common in many other practical situations. The rest of this chapter will give examples of a few well-known situations, concentrating on financial applications. If you read a newspaper for a few days, you will find a great variety of further interesting uses of percentages.



First, however, we will introduce another important method that will be used with percentages throughout this chapter.

Reversing the process to find the original amount

Suppose that 15% of the total mass of a chicken roll is actually chicken. What mass of chicken rolls can be made with 600 g of chicken?

$$\text{Mass of chicken} = \text{mass of rolls} \times 15\%$$

Reversing this:

$$\begin{aligned}\text{mass of rolls} &= \text{mass of chicken} \div 15\% \\ &= 600 \div 0.15 \\ &= 4000 \text{ g}\end{aligned}$$

Hence 4 kg of chicken rolls can be made with 600 g of chicken.



Finding the original amount

- To find, for example, 15% of a given amount, multiply by 15%.
- Conversely, to find the original amount given 15% of it, divide by 15%.

Example 3

Joshua saves 12% of his after-tax salary every week. If he saves \$90 a week, what is his after-tax salary?

Solution

$$\begin{aligned}\text{Savings} &= \text{after-tax salary} \times 12\% \\ \text{Reversing this:} \\ \text{after-tax salary} &= \text{savings} \div 12\% \\ &= 90 \div 0.12 \\ &= \$750\end{aligned}$$



Example 4

Sterling silver is an alloy that is made up of 92.5% by mass silver and 7.5% copper.

- a How much sterling silver can be made with 5 kg of silver and unlimited supplies of copper?
- b How much sterling silver can be made with 5 kg of copper and unlimited supplies of silver?

Solution

- a Mass of silver = mass of sterling silver $\times$ 92.5%

Reversing this:

$$\begin{aligned}\text{mass of sterling silver} &= \text{mass of silver} \div 92.5\% \\ &= 5 \div 0.925 \\ &\approx 5.405 \text{ kg}\end{aligned}$$

- b Mass of copper = mass of sterling silver $\times$ 7.5%

Reversing this:

$$\begin{aligned}\text{mass of sterling silver} &= \text{mass of copper} \div 7.5\% \\ &= 5 \div 0.075 \\ &\approx 66.667 \text{ kg}\end{aligned}$$

Note: One reason for the choice of rounding to 3 decimal places is that there are 1000 g in a kilogram. That is, we are calculating to the nearest gram.

Commission

If a person takes a painting to a gallery to be sold, the gallery will usually charge the vendor or seller a percentage of the selling price as the fee for exhibiting, advertising and selling the painting. Such a fee is called a **commission**, and it applies whenever an agent sells goods or services such as a house or a car on behalf of someone else.

Example 5

The Eureka Gallery charges a commission of 9.2%.

- a The Australian painting *Showing the Flag at Bakery Hill* was sold recently for \$180 000. How much did the Gallery receive, and how much was left for the seller?
- b The Gallery received a commission of \$7912 for selling the painting *Ned at the Glen*. What was the selling price of the painting, and what did the seller actually receive?

(continued on next page)

Solution

$$\begin{aligned}\mathbf{a} \quad \text{Commission} &= 180\,000 \times 9.2\% \\ &= 180\,000 \times 0.092 \\ &= \$16\,560\end{aligned}$$

$$\begin{aligned}\text{Amount received by seller} &= 180\,000 - 16\,560 \\ &= \$163\,440\end{aligned}$$

$$\mathbf{b} \quad \text{Commission} = \text{selling price} \times 9.2\%$$

Reversing this:

$$\begin{aligned}\text{selling price} &= \text{commission} \div 9.2\% \\ &= 7912 \div 0.092 \\ &= \$86\,000\end{aligned}$$

$$\begin{aligned}\text{Amount received by seller} &= 86\,000 - 7912 \\ &= \$78\,088\end{aligned}$$

Profit and loss as percentages

Is an annual profit of \$20 000 a great performance or a poor performance? For a business with annual turnover of \$100 000, such a profit would be considered very large. For a business with annual turnover of \$100 000 000, however, it would be considered a very poor performance.

For this reason, it is often relevant to express profit or loss as a percentage of the total costs or the annual turnover.

Example 6

The Budget Shoe Shop spent \$6 600 000 last year buying shoes and paying salaries and other expenses. They made a 2% profit on these costs.

- a** What was their profit last year?
- b** What was the total of their sales?
- c** In the previous year, their costs were \$5 225 000 and their sales were only \$5 145 000. What percentage loss did they make on their costs?

Solution

$$\begin{aligned}\mathbf{a} \quad \text{Profit} &= 6\,600\,000 \times 2\% \\ &= 6\,600\,000 \times 0.02 \\ &= \$132\,000\end{aligned}$$

b Total sales = total costs + profit

$$= 6\,600\,000 + 132\,000$$

$$= \$6\,732\,000$$

c Last year, loss = total costs – total sales

$$= 5\,225\,000 - 5\,145\,000$$

$$= \$80\,000$$

Percentage loss = $\left(\frac{80\,000}{5\,225\,000} \times \frac{100}{1} \right) \%$

$$\approx 1.53\%$$

Example 7

Joe's tile shop made a profit of 5.8% on total costs last year. If the actual profit was \$83 000, what were the total costs, and what were the total sales?

Solution

$$\text{Profit} = \text{costs} \times 5.8\%$$

Reversing this, costs = profit $\div$ 5.8%

$$= 83\,000 \div 0.058$$

$$\approx \$1\,431\,034, \text{ correct to the nearest dollar.}$$

Hence, total sales = profit + costs

$$\approx 83\,000 + 1\,431\,034$$

$$= \$1\,514\,034$$

Income tax

Income tax rates are often **progressive**. This means that the more you earn, the higher the rate of tax on each extra dollar you earn.

Australian income tax rates change every year, so here is an example using the rates of the island nation of Immutatia, where taxation rates have not changed for many years.

Example 8

Income tax in the nation of Immutatia is calculated as follows.

- There is no tax on the first \$12 000 that a person earns in any one year.
- From \$12 001 to \$30 000, the tax rate is 15c for each dollar over \$12 000.
- From \$30 001 to \$75 000, the tax rate is 25c for each dollar over \$30 000.
- Over \$75 000, the tax rate is 35c for each dollar over \$75 000.

Find the income tax payable by a person whose taxable income for the year is:

- a** \$10 600 **b** \$25 572 **c** \$62 300 **d** \$455 000

Solution

a There is no tax on an income of \$10 600.

b Tax on first \$12 000 = \$0

$$\begin{aligned}\text{Tax on remaining } \$13\,572 &= 13\,572 \times 0.15 \\ &= \$2035.80\end{aligned}$$

This is the total tax payable.

c Tax on first \$12 000 = \$0

$$\begin{aligned}\text{Tax on next } \$18\,000 &= 18\,000 \times 0.15 \\ &= \$2700\end{aligned}$$

$$\begin{aligned}\text{Tax on remaining } \$32\,300 &= 32\,300 \times 0.25 \\ &= \$8075\end{aligned}$$

$$\begin{aligned}\text{Total tax} &= 2700 + 8075 \\ &= \$10\,775\end{aligned}$$

d Tax on first \$12 000 = \$0

$$\text{Tax on next } \$18\,000 = \$2700 \quad (\text{see part c})$$

$$\begin{aligned}\text{Tax on next } \$45\,000 &= 45\,000 \times 0.25 \\ &= \$11\,250\end{aligned}$$

$$\begin{aligned}\text{Tax on remaining } \$380\,000 &= 380\,000 \times 0.35 \\ &= \$133\,000\end{aligned}$$

$$\begin{aligned}\text{Total tax} &= 2700 + 11\,250 + 133\,000 \\ &= \$146\,950\end{aligned}$$



Exercise 3B

Note that some of the questions can be done using the unitary method. For example, question 5 is one such question.

Example 3, 4

- 1
 - a Five per cent of a particular amount is \$12. Find the amount by dividing \$12 by $5\% = 0.05$.
 - b Check your answer to part a by taking 5% of it.
- 2
 - a Twenty-two per cent of a sand pile is 284 kg. Find the mass of the sand pile.
 - b Check your answer to part a by taking 22% of it.
- 3
 - a Seven per cent of a period of time is 6 weeks. Find the period of time.
 - b Check your answer to part a by taking 7% of it.
- 4 Find the quantity, given that:
 - a 2% of it is \$12
 - b 6% of it is 750 g
 - c 30% of it is 36 minutes
 - d 90% of it is 54 cm
 - e 75% of it is 39 hectares
 - f 7% of it is \$6.86.
- 5 In each part, find the price if:
 - a a deposit of \$360 is 30% of the price
 - b a deposit of \$168 is 15% of the price
 - c a deposit of \$640 is 40% of the price
 - d a deposit of \$348 is 25% of the price.
- 6 Find the original quantity, correct to a suitable number of decimal places, if:
 - a 23% of it is 100 kg
 - b 0.2% of it is 4 mm
 - c 0.92% of it is 1.86 hectares
 - d 12% of it is 72 mm
 - e 97% of it is \$700
 - f 68% of it is 3 hours.
- 7 Cameron and Wendy together earn \$1156 per week after tax. Of this, they pay \$460 off their mortgage, \$185 for groceries, and \$260 for their car and transport, and they save \$124.
 - a Express each amount as a percentage of their weekly income, correct to the nearest 1%.
 - b Find how much is unaccounted for in the list above, and what percentage it is of their weekly income.

8 What percentage of the total cost is a deposit of:

- a** \$33 on a television valued at \$550
- b** \$252 on a lounge suite valued at \$2100
- c** \$2835 on a car valued at \$18 900
- d** \$124.10 on a stove valued at \$1460?

Example 5a

9 A book dealer sells rare books and charges a commission of 8% on the selling price. Find, correct to the nearest cent, the commission charged on a book that sells for these prices, and the amount that the seller eventually receives.

- a** \$400 **b** \$1300 **c** \$575 **d** \$142.50

Example 5b

10 Shara the stockbroker charges 0.15% commission on all shares that she sells for clients. In each case, find the price at which a parcel of shares was sold if her commission was:

- a** \$30.00 **b** \$67.35 **c** \$384.75 **d** \$36.51

11 Jeff works as a salesman selling second-hand tractors. He is paid a salary of \$35 000 a year, together with a 6% commission on all the sales he makes. Find his total annual income if his sales for the year were:

- a** \$20 000 **b** \$1 000 000 **c** \$126 000 **d** \$3 458 000

12 A salesperson is paid a commission on her monthly sales. What is the percentage commission if she receives a payment of:

- a** \$168 on sales of \$1200 **b** \$540 on sales of \$6000
- c** \$1530 on sales of \$18 000 **d** \$1596 on sales of \$42 000

13 Find the selling price if the commission and the commission rate are as given.

- a** Commission \$35, rate 7% **b** Commission \$646, rate 3.4%
- c** Commission \$16 586.96, rate 5.2% **d** Commission \$3518.61, rate 11.4%

14 The Great Deals Homeware Store sells large domestic items, such as fridges and computers, by charging a deposit of 35% on delivery and allowing the customer three months to pay the rest of the cost. Find, correct to the nearest dollar, the deposit payable on these items, and the amount still owing.

- a** A fridge costing \$1600 **b** A computer costing \$2800
- c** A microwave costing \$468 **d** A monitor costing \$772

15 Find, to one decimal place, the percentage profit or loss on costs in these situations:

- a** Costs \$16 000 and sales \$18 000
- b** Costs \$162 000 and sales \$150 000
- c** Costs \$2 800 000 and sales \$3 090 000
- d** Costs \$289 000 000 and sales \$268 000 000.

Example 6

16 The Secure Locksmith Company had sales last year of \$568 000 and costs of \$521 000.

- a** What was their profit?
- b** What was the profit as a percentage of the cost price? (Calculate the percentage correct to 2 decimal places.)
- c** In the previous year, they made a loss of 4% of their costs of \$250 000. Find their loss and their sales.

Example 7

17 a A company made a profit of \$18 000, which was a 2.4% profit on its costs. Find the costs and the total sales.

b A company made a loss of \$657 000, which was a 4.5% loss on its costs. Find the costs and the total sales.

c A company made a loss of \$250 800, which was a 3.8% loss on its costs. Find the costs and the total sales.

Example 8

18 This question uses the income tax rates in the nation of Immutatia, which are as follows.

- There is no tax on the first \$12 000 that a person earns in any one year.
- From \$12 001 to \$30 000, the tax rate is 15c for each dollar over \$12 000.
- From \$30 001 to \$75 000, the tax rate is 25c for each dollar over \$30 000.
- Over \$75 000, the tax rate is 35c for each dollar over \$75 000.

a Find the income tax payable on:

- i** \$8000 **ii** \$14 000 **iii** \$36 000 **iv** \$200 000

b What percentage, to two decimal places, of each person's income was paid in income tax in parts **i–iv** of part **a**?

c Find the income if the income tax on it was:

- i** \$1260 **ii** \$3420 **iii** \$13 950 **iv** \$14 650

3C Simple interest



When money is lent by a bank, whoever borrows the money normally makes a payment, called **interest**, for the use of the money.



The amount of interest paid depends on:

- the **principal**, which is the amount of money borrowed
- the **rate** at which interest is charged
- the **time** for which the money is borrowed.

Conversely, if a person invests money in a bank or elsewhere, the bank pays the person interest because the bank uses the money to finance its own investments.

This section will deal only with **simple** interest. In simple interest transactions, interest is paid on the original amount borrowed.

Formula for simple interest

How much simple interest will I pay altogether if I borrow \$4000 for 10 years at an interest rate of 7% per annum? (The phrase *per annum* is Latin for ‘for each year’; it is often abbreviated to p.a.)

Last year you probably learned to set out the working for simple interest in two successive steps, something like this:

$$\begin{aligned}\text{Interest paid at the end of each year} &= 4000 \times 7\% \\ &= 4000 \times 0.07 \\ &= \$280\end{aligned}$$

$$\begin{aligned}\text{Total interest paid over 10 years} &= 280 \times 10 \\ &= \$2800\end{aligned}$$

This working can all be done in one step if we can develop a suitable formula.

Suppose I borrow \$ P for T years at an interest rate R . Using the same two-step approach as before:

$$\text{Interest paid at the end of each year} = P \times R$$

$$\begin{aligned}\text{Total interest paid over } T \text{ years} &= P \times R \times T \\ &= PRT\end{aligned}$$

This gives us the well-known **simple interest formula**:

$$I = PRT \quad (\text{Interest} = \text{principal} \times \text{rate} \times \text{time})$$

Using this formula, the calculation can now be set out in one step:

$$\begin{aligned}I &= PRT \\ &= 4000 \times 7\% \times 10 \quad (\text{Note: The interest rate } R \text{ is } 7\%, \text{ not } 7.) \\ &= 4000 \times 0.07 \times 10 \\ &= \$2800\end{aligned}$$

Note: The interest rate is given per year, so the time must also be written in years. In some books R is written as $r\%$.

Example 9

Find the simple interest on \$8000 for eight years at 9.5% p.a.

Solution

$$\begin{aligned}I &= PRT \\&= 8000 \times 9.5\% \times 8 \\&= 8000 \times 0.095 \times 8 \\&= \$6080\end{aligned}$$

Reverse use of the simple interest formula

There are four pronumerals in the formula $I = PRT$. If the values of any three are known, then substituting into the simple interest formula allows the fourth value to be found.

Example 10

Jessie borrows \$3000 from her parents to help buy a car. They agree that she should only pay simple interest. Five years later she pays them back \$3600, which includes simple interest on the loan. What was the interest rate?

Solution

The total interest paid was \$600, the principal was \$3000 and the time was 5 years

$$\begin{aligned}I &= PRT \\600 &= 3000 \times R \times 5 \\600 &= 15\,000 \times R \\R &= \frac{600}{15\,000} \times 100\% \quad (\text{Interest rates are normally written as percentages.}) \\&= 4\%\end{aligned}$$



Simple interest formula

- Suppose that a principal P is invested for T years at an interest rate R p.a. Then the total interest I is given by:

$$I = PRT$$

Remember that R is a percentage. If the interest rate is 5%, then $R = 0.05$.

- If the interest rate R is given per year, the time T must be given in years.
- The formula has four pronumerals. If any three are known, the fourth can be found by substitution.



Exercise 3C



Example 9

- 1 \$12 000 is invested at 7% p.a. simple interest for five years.
 - a How much interest will be earned each year?
 - b Find how much interest will be earned over the five-year period.
- 2 \$2000 is invested at 6.75% p.a. simple interest for three years.
 - a How much interest will be earned each year?
 - b Find how much interest will be earned over the three-year period.
- 3 Find the total simple interest earned in each of these investments.
 - a \$400 for three years at 6% p.a.
 - b \$850 for six years at 4.5% p.a.
 - c \$7500 for 10 years at 6.3% p.a.
 - d \$15 000 for 12 years at 8.4% p.a.
- 4 Find the time T involved in each of these simple interest investments.
 - a Interest of \$2000 on \$8000 at 5% p.a.
 - b Interest of \$192 000 on \$600 000 at 8% p.a.
- 5 Find the rate R in each of these simple interest investments.
 - a Interest of \$7200 on \$8000 for 12 years.
 - b Interest of \$3 400 000 on \$12 500 000 for four years.
- 6 Find the principal P in each of these simple interest investments.
 - a Interest of \$3500 at 7% for 10 years.
 - b Interest of \$4080 at 4.8% for six years.

Example 10

- 7 Calculate the missing entries for these simple interest investments.

	Principal	Rate p.a.	Time in years	Total interest
a	\$10 000	8%		\$3200
b	\$4 400 000	$7\frac{1}{2}\%$		\$3 960 000
c	\$5000		6	\$1350
d	\$260 000		8	\$83 200
e		6%	5	\$900
f		3.6%	4	\$115.20

- 8 Sartoro invested \$80 000 in a building society that pays 6.5% p.a. simple interest. Over the years, the investment has paid him \$57 200 in interest. How many years has he had the investment?
- 9 Madeline has received \$168 000 in total simple interest payments on an investment of \$400 000 that she made six years ago. What rate of interest has the bank been paying?
- 10 An investor wishes to earn \$240 000 interest over a five-year period from an account that earns 12.5% p.a. simple interest. How much does the investor have to deposit into the account?
- 11 Regan has arranged to borrow \$10 000 at 9.5% p.a. for four years. She will pay simple interest to the bank every year for the loan, with the principal remaining unchanged. How much interest will Regan pay over the four years of the loan?
- 12 Tyler intends to live on the interest on an investment with the bank at 8.6% p.a. simple interest. She will receive \$68 000 simple interest every year from the investment. How much money has she invested?



When a quantity is increased or decreased, the change is often expressed as a percentage of the original amount.

This section introduces a concise method of solving problems about percentage increase and decrease. This method will be applied in various ways throughout the remaining sections of the chapter.

Percentage increase

This evening's news reported that shares in the Consolidated Nail Factory Pty Ltd were selling at \$12.00 yesterday, but rose 14.5% today.

Rather than calculating the price increase and adding it on, the calculation can be done in one step by using the fact that the new price is $100\% + 14.5\% = 114.5\%$ of the old price.

$$\begin{aligned}\text{New price} &= \text{old price} \times 114.5\% \\ &= 12 \times 114.5\% \\ &= 12 \times 1.145 \\ &= \$13.74\end{aligned}$$

Example 11

The number of patients admitted to St Spyridon's Hospital this year suffering from pneumonia is 56% greater than the number admitted for this condition last year. If 245 pneumonia patients were admitted last year, how many were admitted this year?

Solution

This year's total is $100\% + 56\% = 156\%$ of last year's total.

$$\begin{aligned}\text{This year's total} &= 245 \times 156\% \\ &= 245 \times 1.56 \\ &\approx 382 \text{ (correct to the nearest whole number)}\end{aligned}$$

Inflation

The prices of goods and services in Australia usually increase by a small amount every year.

This gradual rise in prices is called **inflation**, and is measured by taking the average percentage increase in the prices of a large range of goods and services.

Other things such as salaries and pensions are often adjusted automatically every year to take account of inflation.

Example 12

The war-ravaged nation of Zerbai is experiencing inflation of 35% p.a. as a result of overspending on its navy and air force. Inflation of 35% means that, on average, prices are increasing by 35% every year.

- a** If the price of water is adjusted in line with inflation, what will an annual bill of \$600 become in the next year?
- b** What should an annual salary of \$169 000 in one year increase to in the following year if it is adjusted to keep pace with inflation?

Solution

Next year's prices are $100\% + 35\% = 135\%$ of last year's prices.

- a** Next year's bill = 600×1.35
 $= \$810$
- b** Next year's salary = $169\,000 \times 1.35$
 $= \$228\,150$

Percentage decrease

The same method can be used to calculate percentage decreases. For example, suppose that 35% of a farmer's sheep station, which has an area of 7500 hectares, went under water during the recent floods.

We can calculate how much land remained above the water for his stock to graze:

$$100\% - 35\% = 65\% \text{ of his land remained above water.}$$

$$\begin{aligned}\text{Area remaining above water} &= 7500 \times 65\% \\ &= 7500 \times 0.65 \\ &= 4875 \text{ hectares}\end{aligned}$$

Example 13

The company that Yuri Ivanov works for is going through hard times and has decreased all its salaries by 12%. Yuri is attempting to cut every one of his expenses by the same percentage.

- a** His family's weekly grocery bill averages 450 roubles. What should he try to reduce the weekly price of his groceries to?
- b** His monthly rental is 18 000 roubles. If he moves apartments, what monthly rental should he try to find?

(continued on next page)

Solution

Yuri's new salary is $100\% - 12\% = 88\%$ of his original salary.

a New weekly grocery bill = 450×0.88
= 396 roubles

b New monthly rental = $18\,000 \times 0.88$
= 15 840 roubles



Percentage increase and decrease

- To increase an amount by, for example, 15%, multiply by $1 + 0.15 = 1.15$.
- To decrease an amount by, for example, 15%, multiply by $1 - 0.15 = 0.85$.

Finding the rate of increase or decrease

Example 14

Suppose that the rainfall has increased from 480 mm p.a. to 690 mm p.a. What rate of increase is this?

Solution

Method 1

Find the actual increase by subtraction, and then express the increase as a percentage of the original rainfall.

Increase = 210 mm

$$\begin{aligned}\text{Percentage increase} &= \frac{\text{increase}}{\text{original rainfall}} \times 100\% \\ &= \frac{210}{480} \times 100\% \\ &= 43.75\%\end{aligned}$$

Method 2

Express the new value as a percentage of the original value.

$$\begin{aligned}\frac{\text{new rainfall}}{\text{old rainfall}} &= \frac{690}{480} \times 100\% \\ &= 143.75\%\end{aligned}$$

So the rainfall has increased by 43.75%.

Reversing the process to find the original amount

Example 15

The Wind Energy Company recently announced that this year's profit of \$1 400 000 constituted a 35% increase on last year's profit. What was last year's profit?

Solution

This year's profit is $100\% + 35\% = 135\%$ of last year's profit.

Hence this year's profit = last year's profit $\times 1.35$

Reversing this, last year's profit = this year's profit $\div 1.35$

$$= 1\,400\,000 \div 1.35$$

$$\approx \$1\,037\,037, \text{ correct to the nearest dollar.}$$

Thus, to find the original amount, we divide by 1.35, because dividing by 1.35 is the reverse of multiplying by 1.35.

Exactly the same principle applies when an amount has been decreased by a percentage.

Example 16

The price of shares in the Fountain Water Company has decreased by 15% over the last month to \$52.70. What was the price a month ago?

Solution

The new share price is $100\% - 15\% = 85\%$ of the old share price.

Hence new price = old price $\times 0.85$

Reversing this, old price = new price $\div 0.85$

$$= 52.70 \div 0.85$$

$$= \$62$$

Thus, to find the original price, we divide by 0.85, because dividing by 0.85 is the reverse of multiplying by 0.85.



Finding the original amount

- To find the original amount after an increase of, for example, 15%, divide by 1.15.
- To find the original amount after a decrease of, for example, 15%, divide by 0.85.

Example 17

Gordon has had mixed results with the shares that he bought six months ago. Shares in the GRAB Bank rose 23% to \$38.23, but shares in the DRAB Blind Company fell 34% to \$5.30. Find the prices he paid for these two shares, correct to the nearest cent.

Solution

GRAB shares are now $100\% + 23\% = 123\%$ of their previous value.

Thus $\text{new value} = \text{original price} \times 1.23$

Reversing this, $\text{original price} = 38.23 \div 1.23$
 $\approx \$31.08$

DRAB shares are now $100\% - 34\% = 66\%$ of their original value.

Thus $\text{new value} = \text{original price} \times 0.66$

Reversing this, $\text{original price} = 5.30 \div 0.66$
 $\approx \$8.03$

Discounts

It is very common for a shop to **discount** the price of an item. This is done to sell stock of a slow-moving item more quickly, or simply to attract customers into the shop.

Discounts are normally expressed as a percentage of the original price.

Example 18

The Elegant Shirt Shop is closing down and has discounted all its prices by 35%.

a What is the discounted price of a shirt whose original price is:

- i** \$120 **ii** \$75?

b What was the original price of a shirt whose discounted price is \$92.30?

Solution

a The discounted price of each item is $100\% - 35\% = 65\%$ of the old price.

i Hence discounted price = old price $\times 0.65$
 $= 120 \times 0.65$
 $= \$78$

ii discounted price = 75×0.65
 $= \$48.75$

b From part **a**, discounted price = old price $\times 0.65$

Reversing this, old price = discounted price $\div 0.65$
 $= 92.30 \div 0.65$
 $= \$142$

The GST

In 1999, the Australian Government introduced a Goods and Services Tax, or GST for short. This tax applies to nearly all goods and services in Australia.

The current rate is 10% on the pre-tax price of the goods or service.

- When GST applies, it is added to the pre-tax price. This is done by multiplying the pre-tax price by 1.10.
- Conversely, if a quoted price already includes GST, the pre-tax price is obtained by dividing the quoted price by 1.10.

Example 19

The current GST rate is 10% of the pre-tax price.

- a** If a domestic plumbing job costs \$630 before GST, how much will it cost after adding GST, and how much tax is paid to the Government?
- b** I paid \$70 for petrol recently. What was the price before adding GST, and what tax was paid to the Government?

Solution

The after-tax price is 110% of the pre-tax price.

a After-tax price = 630×1.10
 $= \$693$
Tax = $693 - 630$
 $= \$63$

Note: 10% of \$630 is \$63.

(continued on next page)

b Pre-tax price = $70 \div 1.10$ (Divide by 1.10 to reverse the process.)
 $\approx \$63.64$
 Tax $\approx 70 - 63.64$
 $= \$6.36$

Exercise 3D

Example 11

- 1** Traffic on all roads has increased by an average of 8% during the past 12 months. By multiplying by $108\% = 1.08$, estimate the number of vehicles now on a road given the number of vehicles the road carried a year ago was:

a 10 000 per day **b** 80 000 per day
c 6300 per day **d** 148 000 per day

Example 12

- 2** Prices have increased with inflation by an average of 3.8% since the same time last year. Find today's price for an item that one year ago cost:

a \$200 **b** \$1.68 **c** \$345 000 **d** \$9430

Example 13

- 3** Rainfall across one state has decreased over the last five years by about 24%. By multiplying by $76\% = 0.76$, estimate, correct to the nearest 10 mm, the annual rainfall this year at a place where the rainfall five years ago was:

a 1000 mm **b** 250 mm **c** 680 mm **d** 146 mm

Example 14

- 4** Admissions to different wards of St Luke's Hospital mostly rose from 2006 to 2007, but by quite different percentage amounts. Find the percentage increase or decrease in wards where the numbers during 2006 and 2007, respectively, were:

a 50 and 68 **b** 120 and 171 **c** 92 and 77 **d** 24 and 39

- 5** In another state, the percentage decrease in rainfall over the last five years has been quite variable, and in some cases, rainfall has actually increased. Find the rate of decrease or increase if the annual rainfall five years ago and now are, respectively:

a 500 mm and 410 mm **b** 920 mm and 960 mm
c 140 mm and 155 mm **d** 420 mm and 530 mm

Example 15

- 6** Radix Holdings Pty Ltd recently issued bonus shares to its shareholders. Each shareholder received an extra 12% of the number of shares currently held. Find the original holding of a shareholder who now holds:

a 672 shares **b** 4816 shares
c 1000 shares **d** 40 200 shares

Example 18a

- 7 A clothing store is offering a 15% discount on all its summer stock. What is the discounted price of an item with original price:

a \$80 b \$48 c \$680 d \$1.60?

Example 18b

- 8 A shoe store is offering a 35% discount at its end-of-year sale. Find the original price of an item whose discounted price is:

a \$1820 b \$279.50 c \$1.56 d \$20.80

- 9 A research institute is trying to find out how much water Lake Grendel had 1000 years ago. The lake now contains 24 000 megalitres, but there are various conflicting theories about the percentage change over the last 1000 years. Find how much water the lake had 1000 years ago, correct to the nearest 10 megalitres, if in the last 1000 years the volume has:

a risen by 80% b fallen by 28%
c risen by 140% d fallen by 4%

Examples 15, 16, 17

- 10 Mr Brown has a spreadsheet showing the value at which he bought his various parcels of shares, the value at 31 December last year, and the percentage increase or decrease in their value. (Decreases are shown with a negative sign.) Unfortunately, a virus has corrupted one entry in each row of his spreadsheet. Help him by calculating the missing values, correct to the nearest cent, and the missing percentages, correct to 2 decimal places.

	Value at purchase	Value at 31 December	Percentage increase
a	\$12 000		30%
b	\$28 679.26		−62%
c	\$5267.70		289.14%
d		\$72 000	20%
e		\$26 000	−22%
f		\$112 000	346.5%
g		\$15 934	−91.38%
h	\$60 000	\$81 000	
i	\$98 356.68	\$14 321.57	
j	\$14 294.12	\$2314.65	

- 11** The rate of income tax for companies is calculated quite differently from the income tax rate for individuals. Income tax for a company is simply 30% of the company's taxable income.

a Find the after-tax income of a company whose taxable income is:

i \$8000 **ii** \$560 000 **iii** \$3 790 000 000 **iv** \$68 245

b Find the taxable income of a company whose after-tax income is:

i \$28 000 **ii** \$600 000 **iii** \$2 320 000 000 **iv** \$462

c Find the taxable income of a company whose income tax is:

i \$1800 **ii** \$24 780 **iii** \$570 789 213 **iv** \$1 675 368

Example 19

- 12** The GST is a tax on most goods and services at the rate of 10% of the pre-tax price.

a Find the after-tax price on goods or services whose pre-tax price is:

i \$170 **ii** \$4624 **iii** \$68 920 **iv** \$6.80

b Find the pre-tax price on goods or services whose after-tax price is:

i \$550 **ii** \$7821 **iii** \$192 819 **iv** \$5.28

c Find the after-tax price on goods or services on which the GST is:

i \$60 **ii** \$678.20 **iii** \$54 000 **iv** \$0.93

- 13 a** A shirt originally priced at \$45 was increased in price by 100%. What percentage discount will restore it to its original price?

b The daily passenger total of the Route 58 bus was 460, and in one year, it increased by 24%. What percentage decrease next year would restore it to its original passenger total?

c Shafqat had savings of \$6000, but he spent 35% of this last year. By what percentage of the new amount must he increase his savings to restore them to their original value?

d The profit of the Arborville Gelatine Factory was \$86 400, but it then decreased by 42%. By what percentage must the profit increase to restore it to its original value?

- 14 a** Find, correct to 2 decimal places, the percentage decrease necessary to restore a quantity to its original value if it has been increased by:

i 10% **ii** 22% **iii** 240% **iv** 2.3%

b Find, correct to 2 decimal places, the percentage increase necessary to restore a quantity to its original value if it has been decreased by:

i 10% **ii** 22% **iii** 75% **iv** 2.3%

- 15** Mr Smith's shoe shop attracts wealthier clients than Mr Jones' shoe shop, and Mr Smith's shoes are more expensive. Mr Smith decides to hold a sale and wants to match Mr Jones' prices. What is the percentage discount necessary, correct to the nearest 1%, if his shoes are the following percentage more expensive:

a 5% **b** 12% **c** 15% **d** 20%



The method introduced in the last section becomes very useful when two or more successive increases or decreases are applied, because the original amount can simply be multiplied successively by two or more factors. Here is a typical example.

Repeated increase

Example 20

The population of Abelsburg in the census three years ago was 46 430. In the three years after the census, however, its population has risen by 6.2%, 8.5% and 13.1%, respectively.

- a** What was its population one year after the census?
- b** What was its population two years after the census?
- c** What is its population now, three years after the census?
- d** What was the percentage increase in population over the three years, correct to the nearest 0.1%?

Solution

- a** One year after the census, the population was 106.2% of its original value.
Hence population after one year = $46\,430 \times 1.062$
 $\approx 49\,309$, correct to the nearest whole number

- b** Two years afterwards, the population was 108.5% of its value one year afterwards.
Hence population after two years = $46\,430 \times 1.062 \times 1.085$
 $\approx 53\,500$, correct to the nearest whole number

Note: Do not use the approximation from part **a** to calculate part **b**. Either start the calculation again, or use the unrounded value from part **a**.

- c** Three years afterwards, the population was 113.1% of its value two years afterwards.
Hence population after three years = $46\,430 \times 1.062 \times 1.085 \times 1.131$
 $\approx 60\,508$, correct to the nearest whole number

- d** Population three years afterwards

$$= \text{original population} \times 1.062 \times 1.085 \times 1.131$$

$$\approx \text{original population} \times 1.303$$

$$\approx \text{original population} \times 130.3\%$$

Hence the population has increased over the three years by about 30.3%.

Note: The percentage increase of 30.3% is significantly larger than the sum of the three percentage increases,

$$6.2\% + 8.5\% + 13.1\% = 27.8\%.$$

Note: The answer to part **d** does not depend on what the original population was.

Repeated decrease

The same method can be applied just as easily to percentage decreases, as demonstrated in the next example.

Example 21

Teresa invested \$75 000 from her inheritance in a mining company that has not been very successful. In the first year, she lost 55% of the money, and in the second year, she lost 72% of what remained.

- a** How much does she have left after one year?
- b** How much does she have left after two years?
- c** What percentage of the original inheritance has she lost over the two years?

Solution

- a** One year later, the percentage remaining was $100\% - 55\% = 45\%$.

$$\begin{aligned}\text{Hence amount left after one year} &= 75\,000 \times 0.45 \\ &= \$33\,750\end{aligned}$$

- b** Two years later, she had $100\% - 72\% = 28\%$ of what she had after one year.

$$\begin{aligned}\text{Hence amount left after two years} &= 75\,000 \times 0.45 \times 0.28 \\ &= \$9450\end{aligned}$$

- c** Amount left after two years = original investment $\times 0.45 \times 0.28$
 $= \text{original investment} \times 0.126$
 $= \text{original investment} \times 12.6\%$

So she has lost $100\% - 12.6\% = 87.4\%$ of her investment over the two years.

Combinations of increases and decreases

Some problems involve both increases and decreases. They can be solved in exactly the same way.

Example 22

The volume of water in the Welcome Dam has varied considerably over the last three years. During the first year the volume rose by 27%, then it fell 43% during the second year, and it rose 16% in the third year.

- a** What is the percentage increase or decrease over the first two years, correct to the nearest 1%?

- b** What is the percentage increase or decrease over the three years, correct to the nearest 1%?
- c** If there were 366 500 megalitres of water in the dam three years ago, how much water is in the dam now, correct to the nearest 500 megalitres?

Solution

a Final volume = original volume $\times 1.27 \times 0.57$
 $\approx$ original volume $\times 0.72$

Since $0.72 < 1$, the volume has decreased over the first two years. The decrease is about $100\% - 72\% = 28\%$.

b Final volume = original volume $\times 1.27 \times 0.57 \times 1.16$
 $\approx$ original volume $\times 0.84$

Since $0.84 < 1$, the volume has decreased. The percentage decrease is about $100\% - 84\% = 16\%$ over the three years.

c Final volume = original volume $\times 1.27 \times 0.57 \times 1.16$
 $= 366\,500 \times 1.27 \times 0.57 \times 1.16$
 $\approx 308\,000$ megalitres, correct to the nearest 500 megalitres.

This time the sum of the percentages is $27\% - 43\% + 16\% = 0\%$, but the volume of water has changed.



Repeated increases and decreases

- To apply successive increases of, for example, 15%, 24% and 38% to a quantity, multiply the quantity by $1.15 \times 1.24 \times 1.38$.
- To apply successive decreases of, for example, 15%, 24% and 38% to a quantity, multiply the quantity by $0.85 \times 0.76 \times 0.62$.

Reversing the process to find the original amount

As shown before, division reverses the process and allows us to find the original amount, as in the following example.

Example 23

A clothing shop discounted a shirt by 45% a month ago, and has now discounted the reduced price by 20%.

- a** What was the total discount on the shirt?
- b** If the shirt is now selling for \$61.60, what was the original price of the shirt?

Solution

- a** After the first discount, the price was $100\% - 45\% = 55\%$ of the original price.
After the second discount, the price was $100\% - 20\% = 80\%$ of the reduced price.
Thus final price = original price $\times 0.55 \times 0.80$
 $= \text{original price} \times 0.44$
So the total discount is $100\% - 44\% = 56\%$.
- b** Reversing this, original price = final price $\div 0.44$
 $= 61.60 \div 0.44$
 $= \$140$



Reversing repeated increases and decreases

- To find the original quantity after successive increases of, for example, 15%, 24% and 38%, divide the final quantity by $(1.15 \times 1.24 \times 1.38)$.
- To find the original quantity after successive decreases of, for example, 15%, 24% and 38%, divide the final quantity by $(0.85 \times 0.76 \times 0.62)$.

Successive divisions

Calculations involving brackets can be tricky to handle when using the calculator.

For example, working with the figures in the summary box above, suppose that a population has increased by 15%, 24% and 38% in three successive years and is now 50 000. Then:

$$\begin{aligned}\text{Original population} &= 50\,000 \div (1.15 \times 1.24 \times 1.38) \\ &\approx 25\,408\end{aligned}$$

We suggest that it is easier to avoid brackets and divide 50 000 successively by 1.15, 1.24 and 1.38. In effect, the working then goes like this:

$$\begin{aligned}\text{Original population} &= 50\,000 \div (1.15 \times 1.24 \times 1.38) \\ &= 50\,000 \div 1.15 \div 1.24 \div 1.38 \\ &\approx 25\,408\end{aligned}$$

Try the calculation both ways and see which you find more natural.

Using the power button on the calculator

When a quantity is repeatedly increased or decreased by the same percentage, the power button makes calculations quicker. Make sure that you can use it correctly by experimenting with simple calculations like $3^4 = 81$ and $2^5 = 32$.

Example 24

The drought in Paradise Valley has been getting worse for years. Each year for the last five years, the rainfall has been 8% less than the previous year's rainfall.

- a** What is the percentage decrease in rainfall over the five years?
- b** If the rainfall this year was 458 mm, what was the rainfall five years ago?

Solution

Each year the rainfall is 92% of the previous year's rainfall.

- a** Final rainfall = original rainfall $\times 0.92 \times 0.92 \times 0.92 \times 0.92 \times 0.92$
 $= \text{original rainfall} \times (0.92)^5$
 $\approx \text{original rainfall} \times 0.659$

So rainfall has decreased by about $100\% - 65.9\% = 34.1\%$ over the five years.

- b** From part **a**, final rainfall = original rainfall $\times (0.92)^5$
Reversing this, original rainfall = final rainfall $\div (0.92)^5$
 $= 458 \div (0.92)^5$
 $\approx 695 \text{ mm}$



Exercise 3E

Example 20

- 1** Oranges used to cost \$2.80 per kg, but the price has increased by 5%, 10% and 12% in three successive years. Multiply by $1.05 \times 1.1 \times 1.12$ to find their price now.
- 2** The dividend per share in the Electron Computer Software Company has risen over the last four years by 10%, 15%, 5% and 12%, respectively. Find the latest dividend received by a shareholder whose dividend four years ago was:
 - a** \$1000 **b** \$1678 **c** \$28.46 **d** \$512.21
- 3** Rates in Bullimbamba Shire have risen 7% every year for the last seven years. Find the rates now payable by a landowner whose rates seven years ago were:
 - a** \$1000 **b** \$346 **c** \$2566.86 **d** \$788.27

Example 21

- 4 A tree, whose original foliage was estimated to have a mass of 1500 kg, lost 20% of its foliage in a storm, then lost 15% of what was left in a storm the next day, then lost 40% of what was left in a third storm. Estimate the remaining mass of foliage.
- 5 A new strain of influenza is gradually losing its power. The days lost to sick leave across the city from the new strain fell by 5%, 8%, 23% and 47% in successive months. How many days sick leave would a company be expected to lose this month due to this strain of influenza if the days lost four months ago were:
 - a 56 days b 3500 days c 18 days d 600 days?
- 6 Shares in the Metropolitan Brickworks have been falling by 23% per year for the last five years. Find the present worth of a parcel of shares whose original worth five years ago was:
 - a \$1000 b \$120 000 c \$25 660 d \$3 860 000

Example 22

- 7 a A shirt is discounted by 50% and the resulting price is then increased by 50%. By what percentage is the price increased or decreased from its original value?
 b The price of a shirt is increased by 50% and the resulting price is then decreased by 50%. By what percentage is the price increased or decreased from its original value?
 c Can you explain the relationship between your answers to parts **a** and **b**?

Example 23

- 8 A book shop has a 50% sale on all stock, and has a container of books with the sale price reduced by a further factor of 16%.
 - a What was the total discount on each book in the container?
 - b If a book in the container is now selling for \$10.50, what was its original price?
- 9 Calculate the total increase or decrease in a quantity when:
 - a it is increased by 20% and then decreased by 20%
 - b it is increased by 80% and then decreased by 80%
 - c it is increased by 10% and then decreased by 10%
 - d it is increased by 30% and then decreased by 30%.

Example 24

- 10 The price of gemfish has been rising. The price has risen by 10%, 15% and 35% in three successive years, and they now cost \$24 per kg. Find:
 - a the price one year ago b the price two years ago
 - c the original price three years ago.
- 11 The crime rate in Gotham City has been rising each decade. In the last four decades the number of robberies has risen by 64%, 223%, 75% and 12%. If there are now 958 robberies per year, find how many robberies per year there were:
 - a one decade ago b two decades ago
 - c three decades ago d four decades ago

- 12** A particular strain of bacteria increases its population on a certain prepared Petri dish by 34% every hour. Calculate the size of the original population four hours ago if there are now 56 000 bacteria.
- 13** Flash Jim is desperate to attract customers to his used car yard. He has cut prices recently by 5%, then by 10%, then by 24%. Find, correct to the nearest \$100, the original price of a used car now priced at:
- a** \$10 000 **b** \$35 000 **c** \$4600 **d** \$76 800
- 14** Here is a table of the annual inflation rate in Australia in the years ending 30 June 2001 to 30 June 2006 (from the Reserve Bank of Australia website).

Year	2001	2002	2003	2004	2005	2006
Inflation rate	6.0%	2.8%	2.7%	2.5%	2.5%	4.0%

- Calculate the percentage increase, correct to one decimal place, in prices:
- a** over the whole six years **b** during the first two years
c during the first three years **d** during the last three years.
- 15** The radioactivity of any sample of the element radon-222 decreases by 16.58% every 24 hours. Find, to two decimal places, the percentage reduction in radiation in:
- a** 2 days **b** 3 days **c** 4 days **d** 5 days
- 16** The radioactivity of any sample of the element strontium-90 decreases by 90.75% every century. Find the percentage reduction in radioactivity over each of the periods given below. (Calculate percentages correct to 3 decimal places.)
- a** two centuries **b** three centuries
c four centuries **d** five centuries

3F Compound interest



In all the examples in this section, the interest is compounded annually. This means that at the end of each year, the interest earned is added to the principal invested or borrowed. That increased amount then becomes the amount on which interest is earned in the following year. This is called **compound interest**.

Example 25

Gail has invested \$100 000 for six years with the Mountain Bank. The bank pays her interest at the rate of 7.5% p.a., compounded annually.

- a** How much will the investment be worth at the end of one year?
- b** How much will the investment be worth at the end of two years?
- c** How much will the investment be worth at the end of three years?
- d** How much will the investment be worth at the end of six years?
- e** What is the percentage increase on her original investment at the end of six years?
- f** What is the total interest earned over the six years?
- g** What would the simple interest on the investment have been, assuming the same interest rate of 7.5% p.a.?

Solution

Each year the investment is worth 107.5% of its value the previous year.

- a** Amount at the end of one year = $100\,000 \times 1.075$
= \$107 500
- b** Amount at the end of two years = $100\,000 \times 1.075 \times 1.075$
= $100\,000 \times (1.075)^2$
= \$115 562.50
- c** Amount at the end of three years = $100\,000 \times 1.075 \times 1.075 \times 1.075$
= $100\,000 \times (1.075)^3$
 $\approx$ \$124 229.69
- d** Amount at the end of six years = $100\,000 \times (1.075)^6$
 $\approx$ \$154 330.15
- e** Final amount = original amount $\times (1.075)^6$
 $\approx$ original amount $\times 1.5433$

So the total increase over six years is about 54.33%.

$$\begin{aligned}\mathbf{f} \quad \text{Total interest} &\approx 154\,330.15 - 100\,000 \\ &= \$54\,330.15\end{aligned}$$

$$\begin{aligned}\mathbf{g} \quad \text{Simple interest} &= PRT \\ &= 100\,000 \times 0.075 \times 6 \\ &= \$45\,000\end{aligned}$$

Note: Compound interest for two or more years is always greater than simple interest for two or more years.

Compound interest on a loan

Exactly the same principles apply when someone borrows money from a bank and the bank charges compound interest on the loan. If no repayments are made, the amount owing compounds in the same way, and can grow quite rapidly.

This is shown in the following example.

Example 26

Hussain is setting up a plumbing business and needs to borrow \$150 000 from a bank to buy a truck and other equipment. The bank will charge him interest of 11% p.a., compounded annually. Hussain will pay the whole loan off all at once four years later.

- How much will Hussain owe the bank at the end of one year?
- How much will Hussain owe the bank at the end of two years?
- How much will Hussain owe the bank at the end of four years?
- What is the percentage increase in the money owed at the end of four years?
- What is the total interest that Hussain will pay on the loan?
- What would the simple interest on the loan have been, assuming the same interest rate of 11% p.a.?

Solution

Each year Hussain owes 111% of what he owed the previous year.

- Amount owing at the end of one year $= 150\,000 \times 1.11$
 $= \$166\,500$
- Amount at the end of two years $= 150\,000 \times 1.11 \times 1.11$
 $= 150\,000 \times (1.11)^2$
 $= \$184\,815$

(continued on next page)

$$\begin{aligned}\text{c Amount at the end of four years} &= 150\,000 \times 1.11 \times 1.11 \times 1.11 \times 1.11 \\ &= 150\,000 \times (1.11)^4 \\ &\approx \$227\,710.56\end{aligned}$$

$$\begin{aligned}\text{d Final amount} &= \text{original amount} \times (1.11)^4 \\ &\approx \text{original amount} \times 1.5181\end{aligned}$$

So the total increase over four years is about 51.81%.

$$\begin{aligned}\text{e Total interest} &\approx 227\,710.56 - 150\,000 \\ &= \$77\,710.56\end{aligned}$$

$$\begin{aligned}\text{f Simple interest} &= PRT \\ &= 150\,000 \times 0.11 \times 4 \\ &= 66\,000\end{aligned}$$

Note: Making no repayments on a loan that is accruing compound interest is a risky business practice because, as this example makes clear, the amount owing grows with increasing rapidity as time goes on. This is particularly relevant to credit card debt.



Compound interest

Suppose that an amount P is invested at an interest rate, say 7%, compounded annually. The interest is calculated each year on the new balance. The new balance is obtained by adding on 7% of the balance of the previous year. Thus:

$$\begin{aligned}\text{Amount after four years} &= P \times 1.07 \times 1.07 \times 1.07 \times 1.07 \\ &= P \times (1.07)^4\end{aligned}$$

Reversing the process to find the original amount

As always, division reverses the process to find the original amount, as in the following example.

Example 27

Eleni wants to borrow money for three years to start a business, and then pay all the money back, with interest, at the end of that time. The bank will not allow her final debt, including interest, to exceed \$300 000. Interest is 9% p.a., compounded annually. What is the maximum amount that Eleni can borrow?

Solution

Each year Eleni will owe 109% of what she owed the previous year.

Hence $\text{final debt} = \text{original debt} \times 1.09 \times 1.09 \times 1.09$

$$\text{final debt} = \text{original debt} \times (1.09)^3$$

Reversing this, $\text{original debt} = \text{final debt} \div (1.09)^3$

$$= 300\,000 \div (1.09)^3$$

$$\approx \$231\,655$$

Exercise 3F

Example 25

- 1 a** Christine invested \$100 000 for six years at 5% p.a. interest, compounded annually.

- i** By multiplying by 1.05, find the value of the investment after one year.
- ii** By multiplying by $(1.05)^2$, find the value of the investment after two years.
- iii** By multiplying by $(1.05)^6$, find the value of the investment after six years.
- iv** Find the percentage increase in the investment over the six years.
- v** Find the total interest earned over the six years.

- b** Find the simple interest on the principal of \$100 000 over the six years at the same rate of 5% p.a.

Example 26

- 2 a** Gary has borrowed \$200 000 for six years at 8% p.a. interest, compounded annually, in order to start his indoor decorating business. He intends to pay the whole amount back, plus interest, at the end of the six years.

- i** Find the amount owing after one year.
- ii** Find the amount owing after two years.
- iii** Find the amount owing after six years.
- iv** Find the percentage increase in the loan over the six years.
- v** Find the total interest charged over the six years.

- b** Find the simple interest on the principal of \$200 000 over the six years at the same rate of 8% p.a.

- 3** A couple take out a housing loan of \$320 000 over a period of 20 years. They make no repayments over the 20-year period of the loan. Compound interest is payable at $6\frac{1}{2}\%$ p.a., compounded annually. How much would they owe at the end of the 20-year period, and what is the total percentage increase?

- 4 The population of a city increases annually at a compound rate of 3.2% for five years. If the population is 21 000 initially, what is it at the end of the five-year period, and what is the total percentage increase?
- 5 **a** Find the compound interest on \$1000 at 5% p.a., compounded annually for 200 years.
b Find the simple interest on \$1000 at 5% p.a. for 200 years.
- 6 **a** Find the compound interest on \$300 000 at 7% p.a., compounded annually for 30 years.
b Find the simple interest on \$300 000 at 7% p.a. for 30 years.
- 7 \$10 000 is borrowed for five years and compound interest at 10% p.a. is charged by the lender.
a How much money is owed to the lender after the five-year period?
b How much of this amount is interest?
- 8 Money borrowed at 8% p.a. interest, compounded annually, grew to \$100 000 in four years.
a Find the total percentage increase.
b Hence find the original amount invested.
- 9 Suzette wants to invest a sum of money now so that it will grow to \$180 000 in 10 years time. How much should she invest now, given that the interest rate is 6% compounded annually?
- 10 A bank offers 8% p.a. compound interest. How much needs to be invested if the investment is to be worth \$100 000 in:
a 10 years **b** 20 years **c** 25 years **d** 100 years?
- 11 The population of the coastal town of Coral Bar has been growing at 5.8% every year and has now reached 67 440. Find the population:
a one year ago **b** two years ago
c five years ago **d** ten years ago.
- 12 An investment has been earning compound interest at 6% p.a., compounded annually, for many years, and is now worth \$128 000. Find how much it was worth:
a one year ago **b** two years ago
c five years ago **d** ten years ago.
- 13 A man now owes the bank \$56 000, after having taken out a loan five years ago. Find the original amount that he borrowed if the rate of interest per annum, compounded annually, has been:
a 3% **b** 5.6% **c** 9.25% **d** 15%

Example 27

- 14** Mr Brown has had further difficulties with the virus that attacks his spreadsheet entries. Here is the remains of a spreadsheet that he prepared in answer to questions from business friends. The spreadsheet calculated interest compounded annually, on various amounts, at various interest rates, for various periods of time. Help him reconstruct the missing entries.

	Principal	Rate p.a.	Time in years	Final amount	Total interest
a	\$4000	6%	20		
b	\$10 000	8.2%	15		
c	\$2 000 000	4.8%	10		
d		6%	20	\$4000	
e		8.2%	15	\$10 000	
f		4.8%	10	\$2 000 000	

- 15** Ms Smith invested \$50 000 at 6% p.a. interest, compounded annually, for four years. The tax department wants to know exactly how much interest she earned each year. Calculate these figures for Ms Smith.
- 16** Mrs Robinson has taken out a loan of \$300 000 at 8% p.a. interest, compounded annually, for four years. She wants to know exactly how much interest she will be charged each year so that she can include it as a tax deduction in her income tax return. Calculate these figures for Mrs Robinson.
- 17** **a** Find the total percentage growth in a compound interest investment:
- i** at 15% for two years **ii** at 10% for three years
 - iii** at 6% for five years **iv** at 5% for six years
 - v** at 3% for 10 years **vi** at 2% for 15 years.
- b** What do you observe about these results?
- 18** A couple took out a six-year loan to start a business. For the first three years, compound interest of 8% p.a. was charged. For the second three years, compound interest of 12% p.a. was charged. Find the total percentage increase in the amount owing.
- 19** One six-year loan attracts compound interest calculated at 2%, 4%, 6%, 8%, 10% and 12% in successive years. Another six-year loan attracts compound interest calculated at 12%, 10%, 8%, 6%, 4% and 2% in successive years. Find the total percentage increase in money owing in both cases, compare the two results, and explain what has happened.
- 20** An investment at an interest rate of 10% p.a., compounded annually, returned interest of \$40 000 after five years. Calculate the original amount invested.



Depreciation occurs when the value of an asset reduces as time passes. For example, a company may buy a car for \$40 000, but after four years the car will be worth a lot less, because the motor will be worn, the car will be out of date, the body and interior may have a few scratches, and so forth.

Accountants usually make the assumption that an asset such as a car depreciates at the same rate every year. This rate is called the **depreciation rate**. It works like compound interest. The depreciation is applied each year to the current value. In the following example, the depreciation rate is taken to be 20%.

Example 28

The Medicine Home Delivery Company bought a car four years ago for \$40 000, and assumed that the value of the car would depreciate at 20% p.a.

- What value did the car have at the end of two years?
- What value does the car have now, after four years?
- What is the percentage decrease in value over the four years?
- What is the average depreciation on the car over the four years? (Express your answer in dollars per year)

Solution

The value each year is taken to be $100\% - 20\% = 80\%$ of the value in the previous year.

$$\begin{aligned}\text{a Value at the end of two years} &= 40\,000 \times 0.80 \times 0.80 \\ &= 40\,000 \times (0.80)^2 \\ &= \$25\,600\end{aligned}$$

$$\begin{aligned}\text{b Value at the end of four years} &= 40\,000 \times 0.80 \times 0.80 \times 0.80 \times 0.80 \\ &= 40\,000 \times (0.80)^4 \\ &= \$16\,384\end{aligned}$$

$$\begin{aligned}\text{c Final value} &= \text{original value} \times (0.80)^4 \\ &= \text{original value} \times 0.4096\end{aligned}$$

Hence the percentage decrease over four years is $100\% - 40.96\% = 59.04\%$.

$$\begin{aligned}\text{d Depreciation over four years} &= 40\,000 - 16\,384 \\ &= \$23\,616\end{aligned}$$

$$\begin{aligned}\text{Average depreciation per year} &= 23\,616 \div 4 \\ &= \$5904 \text{ per year}\end{aligned}$$



Depreciation

Suppose that an asset with original value P depreciates at, for example, 7% every year. To find the depreciated value, decrease the current value by 7% each year. Thus:

$$\begin{aligned}\text{Value after four years} &= P \times 0.93 \times 0.93 \times 0.93 \times 0.93 \\ &= P \times (0.93)^4\end{aligned}$$

Reversing the process to find the original amount

If we are given a depreciated value and the rate of depreciation, we can find the original value by division.

Example 29

A school buys new computers every four years. At the end of the four years, it offers them for sale to the students on the assumption that they have depreciated at 35% p.a. (per annum). The school is presently advertising some computers at \$400 each.

- a What did each computer cost the school originally?
- b What is the average depreciation on each computer, in dollars per year?

Solution

- a Each year a computer is worth $100\% - 35\% = 65\%$ of its value the previous year.

$$\text{Hence final value} = \text{original value} \times 0.65 \times 0.65 \times 0.65 \times 0.65$$

$$\text{final value} = \text{original value} \times (0.65)^4$$

$$\text{Reversing this, original value} = \text{final value} \div (0.65)^4$$

$$= 400 \div (0.65)^4$$

$$\approx \$2241$$

- b Depreciation over four years $\approx 2241 - 400$

$$= \$1841$$

$$\text{Average depreciation per year} \approx 1841 \div 4$$

$$\approx \$460$$



Exercise 3G

Note: The depreciation rates in this exercise are taken from the Australian Taxation Office's Schedule of Depreciation. These are intended for income tax purposes. A company may have reasons to use different rates.

Example 28

- 1** The landlord of a large block of home units purchased washing machines for its units four years ago for \$400 000, and is assuming a depreciation rate of 30%.
 - a** By multiplying by 0.70, find the value after one year.
 - b** By multiplying by $(0.70)^2$, find the value after two years.
 - c** By multiplying by $(0.70)^3$, find the value after three years.
 - d** By multiplying by $(0.70)^4$, find the value after four years.
 - e** What is the percentage decrease in value over the four years?
 - f** What is the average depreciation on the washing machines, in dollars p.a. over the four years?
- 2** The Hungry Hour Cafe purchased an air-conditioning system six years ago for \$250 000, and is assuming a depreciation rate of 20%.
 - a** Find the value after one year.
 - b** Find the value after two years.
 - c** Find the value after six years.
 - d** What is the percentage decrease in value over the six years?
 - e** What is the average depreciation, in dollars p.a., on the air-conditioning system over the six years?
- 3**
 - a** A business spent \$560 000 installing alarms at its premises and then depreciated them at 20% p. a.. Find the value after five years, and the percentage depreciation of their value.
 - b** The business borrowed the money to install the alarms, paying 7.2% p.a. compound interest for the five years. How much did it owe at the end of the five years?
- 4** The population of a sea lion colony decreases at a compound rate of 2% p.a. for 10 years. If the population is 8000 initially, what is it at the end of the ten-year period?
- 5** The Northern Start Bus Company bought a bus for \$480 000, depreciated it at 30% p.a., and sold it again seven years later for \$60 000. Was the price that they obtained better or worse than the depreciated value, and by how much?

- 6 The Backyard Rubbish Experts bought a fleet of small trucks for \$1 340 000 and depreciated them at 22.5% p.a. Five years later they sold them for \$310 000. Was the price that they obtained better or worse than the depreciated value, and by how much?
- 7 A company spent \$3 460 000 buying computers for its offices and depreciated them for taxation purposes at 40% p.a. Find their value at the end of each of the first four years, and the amount of the depreciation that the company could claim against its taxable income for each of those four years.
- 8 A landlord spent \$3400 on vacuum cleaners for his block of home units and depreciated them for taxation purposes at 25% p.a. Find their value at the end of each of the first three years, and the amount of the depreciation that the landlord could claim against his taxable income for each of those three years.
- 9 Lara and Kate each received \$100 000 from their parents. Lara invested the money at 6.2% p.a. compounded annually, whereas Kate bought a luxury car that depreciated at a rate of 20% p.a. What were the values of their investments at the end of five years?
- 10 Taxis depreciate at 50% p.a., and other cars depreciate at 22.5% p.a.
 - a What is the total percentage depreciation on each type of vehicle after seven years?
 - b What is the difference in value, to the nearest dollar, after seven years of a fleet of taxis and a fleet of other cars, if both fleets cost \$1 000 000?
- 11 Mr Wong's ten-year-old used car is worth \$4000, and has been depreciating at 22.5% p.a. (Calculate amounts of money in whole dollars.)
 - a Use division by 0.775 to find how much it was worth a year ago.
 - b Find how much it was worth two years ago.
 - c Find how much it was worth ten years ago.
 - d What is the total percentage depreciation on the car over the ten-year period?
 - e What was the average depreciation in dollars per year over the ten-year period?
- 12 St Scholasticus' Grammar School bought photocopying machines six years ago, which it then depreciated at 25% p.a. They are now worth \$72 000.
 - a How much were they worth one year ago?
 - b How much were they worth two years ago?
 - c How much were they worth six years ago?
 - d What is the total percentage depreciation on them over the six-year period?
 - e What was the average depreciation in dollars per year over the six-year period?

Example 29

- 13** Ms Wu's seven-year-old car is worth \$5600, and has been depreciating at 22.5% p.a. Calculate your answers to the nearest dollar.
- a** How much was it worth four years ago?
 - b** How much was it worth seven years ago?
 - c** What is the total percentage depreciation on the car over the seven-year period?
 - d** What was the average depreciation in dollars per year over the seven-year period?
 - e** Ms Wu, however, only bought the car four years ago, at its depreciated value at that time. What has been Ms Wu's average depreciation in dollars over the four years she has owned the car?
 - f** What was the average depreciation in dollars over the first three years of the car's life?
- 14** I take 900 mL of a liquid and dilute it with 100 mL of water. Then I take 900 mL of the mixture and again dilute it with 100 mL of water. I repeat this process twenty times.
- a** What proportion of the original liquid remains in the mixture at the end?
 - b** How much mixture should I take if I want it to contain 20 mL of the original liquid?
- 15** I take a sealed glass container and remove 80% of the air. Then I remove 80% of the remaining air. I do this process six times altogether. What percentage of the original air is left in the container?

Review exercise



- 1** Sarah decides to spend 40% of her weekly earnings on social activities, give 15% to her mother to repay a loan, and save the rest. She earns \$84 a week.
 - a** How much does Sarah spend each week on social activities?
 - b** What percentage of her weekly earnings does she save?
- 2** Nick's share portfolio consists of shares, with value \$10 000, in the banking industry, \$3000 in mining shares and \$15 000 in the gold market.
 - a** What percentage of his share portfolio is made up of shares in the mining sector?
 - b** What percentage of Nick's share portfolio are not banking shares?
- 3** A real estate agent charges a commission of 8.9% on every property sale.
 - a** If a house sells for \$540 000, how much commission will the real estate agent receive, and how much is left for the seller?
 - b** If the real estate agent receives a commission of \$8455 for selling a house, what was the selling price of the house, and what did the seller actually receive?
- 4** It cost the owners of the Corner Newsagency \$3 500 000 to run their business last year. They recorded a profit of 4.5%.
 - a** What was their profit last year?
 - b** What was the total of their sales?
 - c** In the previous year, their costs were \$2 750 000 and their sales were only \$2 635 000. What percentage loss did they make on their costs?
- 5** Grant earned \$1260 interest on money he had invested four years ago at a simple interest rate of 4.5% p.a. How much did Grant originally invest?
- 6** A country is experiencing inflation of 12% p.a.
 - a** If the price of bread is adjusted in line with inflation, what will an annual bread bill of \$2500 become in the next year?
 - b** If Janienne earns \$72 000 in one year and \$78 000 the next, is her salary increase keeping pace with inflation?
- 7** A regional country medical centre has lost the services of one of its 16 doctors due to retirement, and has been unable to replace her.
 - a** What percentage loss is represented by the retiring doctor?
 - b** If the medical centre treated 400 patients per day last year and need to reduce this number by the percentage found in part **a**, how many patients per day will they be able to treat in the coming year?
- 8** A shop made a profit of 6.2% on total costs last year. If the actual profit was \$156 000, what were the total costs and what were the total sales?



- 9** In the January sales, the Best Dress shop has discounted all its prices by 18%.
- a** What is the discounted price of a dress with a marked price of \$240?
 - b** What was the original price of a dress with a discounted price of \$49.20?
- 10** The number of books in a local library varies from year to year. Three years ago, the number fell by 25%, then it rose 41% the following year, and finally rose 8% last year.
- a** What is the percentage increase or decrease over the three years, correct to the nearest 1%?
 - b** If there were 429 000 books in the library three years ago, approximately how many books are in the library now?
- 11** The original asking price for a farm dropped by 30% a year ago, but did not attract a buyer. The price has now been further reduced by 15%.
- a** By what percentage has the original asking price been reduced?
 - b** If the farm is now for sale at \$2 677 500, what was the original asking price of the farm?
- 12** The height of a mature tree is measured on the same day each year. Each year for the last six years, the growth has been 9% less than the previous year's growth.
- a** What is the percentage decrease in growth over the six years, correct to the nearest percent?
 - b** If the growth this year was 320 mm, what was the growth six years ago, correct to the nearest millimetre?
- 13** Sam's investment of \$50 000 for five years earns her interest at the rate of 6.3% p.a., compounded annually.
- a** How much will the investment be worth at the end of one year?
 - b** How much will the investment be worth at the end of two years?
 - c** How much will the investment be worth at the end of six years?
 - d** What is the percentage increase of her original investment at the end of six years?
 - e** What is the total interest earned over the six years?
 - f** What would the simple interest on the investment have been, assuming the same interest rate of 6.3% p.a.?
- 14** A company buys new company cars every three years. At the end of the three years, it offers them for sale to the employees on the assumption that they have depreciated at 30% p.a. The company is presently advertising some cars at \$30 000 each.
- a** What did each car cost the company originally, correct to the nearest thousand dollars?
 - b** What is the average depreciation in dollars p.a., correct to the nearest hundred dollars, on each car over the three-year period?

Challenge exercise



- 1 The population of a town decreases by 15% during 2010. What percentage increase, correct to two decimal places, is necessary during 2011 for the population to be restored to the level it was at immediately before the decrease in 2010?
- 2 The length of a rectangle is increased by 15% and the width is decreased by 11%. What is the exact percentage change in the area?
- 3 The radius of a circular pool of water increases by 8%. What is the exact percentage change in the area of the pool of water?
- 4 The area of a circular pool of oil is increased by 8%. What is the percentage increase of the radius?
- 5 A man earns a salary of \$1440 for a 44-hour week. His weekly salary is increased by 10% and his hours are reduced by 10%. Find his new hourly salary.
- 6 In a particular country in 2010, 15% of the population is unemployed and 85% is employed. In 2011, 10% of the people unemployed became employed and 10% of those employed became unemployed. What percentage of the population is employed now?
- 7 The number of trees on Green Plateau fell by 5% every year for 10 years. Then the numbers rose by 5% every year for 20 years. What was the total percentage gain or loss of trees over the thirty-year period?



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CHAPTER

4

Number and Algebra

Factorisation

The distributive law can be used to rewrite a product involving brackets as an expression without brackets. For instance, the product $3(a + 2)$ can be rewritten as $3a + 6$; this is called the **expanded** form of the expression, and the process is called **expansion**.

The process of writing an algebraic expression as a product of two or more algebraic factors is called **factorisation**. Factorisation is the reverse process to expansion. For example, we can write $3a + 6$ as $3(a + 2)$. This is called the **factorised** form.

In this chapter we will look at how to factorise expressions such as $x^2 + 7x + 12$. We recall that such an expression is called a **quadratic**.

4A

Factorisation using common factors



If each term in the algebraic expression to be factorised contains a **common factor**, then this common factor is a factor of the entire expression. To find the other factor, divide each term by the common factor. The common factor is placed outside brackets. For this reason the procedure is sometimes called ‘taking the common factor outside the brackets’.

Example 1

Factorise $4a + 12$.

Solution

4 is a common factor of $4a$ and 12 thus

$$4a + 12 = 4(a + 3)$$

Notice that the answer can be checked by expanding $4(a + 3)$.

In general, take out as many common factors as possible. The common factors may involve both numbers and pronumerals. It may be easier if you first take out the common factor of the numbers and then the common factor of the pronumerals.

Example 2

Factorise $12x^2 + 3x$.

Solution

$$\begin{aligned} 12x^2 + 3x &= 3(4x^2 + x) \\ &= 3x(4x + 1) \end{aligned}$$

Example 3

Factorise $36ab - 27a$.

Solution

$$\begin{aligned} 36ab - 27a &= 9(4ab - 3a) \\ &= 9a(4b - 3) \end{aligned}$$

Note: It is more efficient to take out all the common factors in one step, as shown in the following examples.

Example 4

Factorise:

a $3x + 9$

b $2x^2 + 4x$

c $-7a^2 - 49$

d $7a^2 + 63ab$

Solution

a $3x + 9 = 3(x + 3)$

b $2x^2 + 4x = 2x(x + 2)$

c $-7a^2 - 49 = -7 \times a^2 + (-7) \times 7$
 $= -7(a^2 + 7)$

d $7a^2 + 63ab = 7a(a + 9b)$

$(7(-a^2 - 7))$ is also correct

Example 5

Factorise:

a $5pq^2 + 10p^2q + 25p^2q^2$

b $16ab + 10b^2 - 2a^2b$

Solution

a $5pq^2 + 10p^2q + 25p^2q^2 = 5pq(q + 2p + 5pq)$

b $16ab + 10b^2 - 2a^2b = 2b(8a + 5b - a^2)$

Exercise 4A

1 Complete each factorisation.

a $12x = 12 \times \square$

b $24a = 12 \times \square$

c $36ab = 9a \times \square$

d $15ac = 5c \times \square$

e $y^2 = y \times \square$

f $6y^2 = 3y \times \square$

g $24a^2 = 6a \times \square$

h $-6b^2 = 2b \times \square$

i $8a^2b = 2ab \times \square$

j $4x^2y = 2x \times \square$

k $12m^2n = 3mn \times \square$

l $25a^2b^2 = 5ab \times \square$

2 Fill in the blanks by finding the missing factors.

a $12ab = 4a \times \square$

b $12ab = 12 \times \square$

c $12ab = 6b \times \square$

d $12ab = 4ab \times \square$

$$\text{e } 12a + 18 = 6 \times \square$$

$$\text{f } 15p - 10 = 5 \times \square$$

$$\text{g } 20mn - 15n = 5 \times \square$$

$$\text{h } 20mn - 15n = 5n \times \square$$

$$\text{i } a^2 + 4a = (a + 4) \times \square$$

$$\text{j } b^2 - 10b = b \times \square$$

$$\text{k } 6yz^2 - 18yz = 3z \times \square$$

$$\text{l } 6yz^2 - 18yz = yz \times \square$$

$$\text{m } 6yz^2 - 18yz = -3yz \times \square$$

$$\text{n } 6yz^2 - 18yz = 6yz \times \square$$

Example 1

3 Factorise:

$$\text{a } 6x + 24$$

$$\text{b } 5a + 15$$

$$\text{c } ac + 5c$$

$$\text{d } a^2 + a$$

$$\text{e } y^2 + xy$$

$$\text{f } 3x + 27$$

$$\text{g } 4x + 24$$

$$\text{h } 7a - 63$$

$$\text{i } 9a + 36$$

$$\text{j } y^2 + 6y$$

$$\text{k } y^2 - 3y$$

$$\text{l } 12x + 18$$

$$\text{m } 24y - 36$$

$$\text{n } -14a - 21$$

$$\text{o } -6y - 9$$

$$\text{p } -4 - 12b$$

Example 2

4 Factorise:

$$\text{a } 4ab + 16a$$

$$\text{b } 12a^2 + 8a$$

$$\text{c } 18m^2n + 9mn^2$$

$$\text{d } 15a^2b^2 + 10ab^2$$

$$\text{e } 4a^2 + 6a$$

$$\text{f } 8a^2 + 12ab$$

Example 3

5 Factorise:

$$\text{a } 3b - 6b^2$$

$$\text{b } 4x^2 - 6xy$$

$$\text{c } 9mn - 12m^2n$$

$$\text{d } 18y - 9y^2$$

$$\text{e } 4a - 6ab^2$$

$$\text{f } 6xy - 4x^2$$

$$\text{g } 14mn^2 - 21m^2n$$

$$\text{h } 6pq^2 - 21qp^2$$

$$\text{i } 10ab^2 - 25a^2b$$

Example 4

6 Factorise:

$$\text{a } -10b^2 + 5b$$

$$\text{b } -16a^2b - 8ab$$

$$\text{c } -x^2y - 3xy$$

$$\text{d } -4pq + 16p^2$$

$$\text{e } -5x^2y + 30x$$

$$\text{f } 18p^2 - 4pq$$

$$\text{g } -8a^2b^2 - 2ab$$

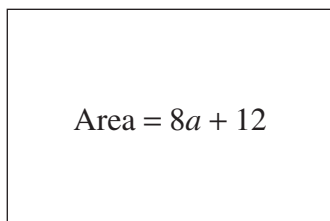
$$\text{h } 12xy^2 - 3x^2y$$

$$\text{i } -25m^2n^2 - 10mn^2$$

7 In each part, an expression for the area of the rectangle has been given. Find an expression for the missing side length.

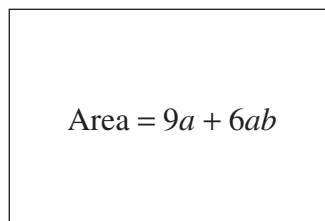
a

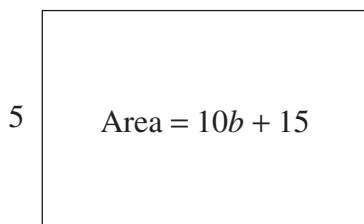
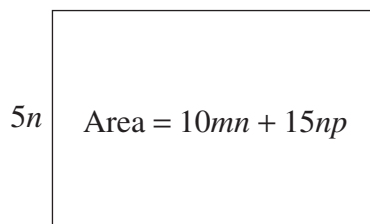
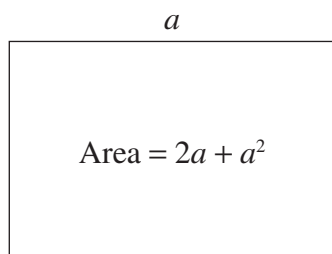
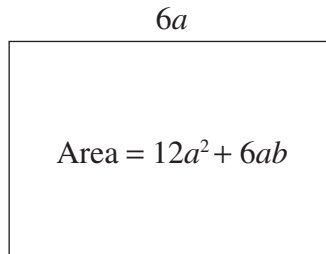
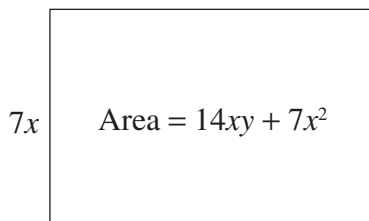
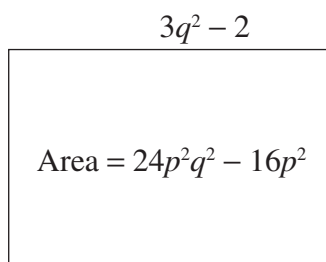
$$2a + 3$$



b

$$3a$$



c**d****e****f****g****h**

Example 5

8 Factorise:

a $4a^2b - 2ab + 8ab^2$

b $4m^2n - 4mn + 16n^2$

c $7ab + 14a^2 + 21b$

d $2m^2 + 4mn + 6n$

e $5a^2b + 3ab + 4ab^2$

f $6a^2 + 8ab + 10ab^2$

g $5p^2q^2 + 10pq^2 + 15p^2q$

h $5\ell^2 - 15\ell m - 20m^2$

4B Factorisation using the difference of two squares



You will recall from Section 1G the important identity

$$(a + b)(a - b) = a^2 - b^2,$$

which is called the difference of two squares. We can now use this result the other way around to factorise an expression that is the difference of two squares.

That is:

$$a^2 - b^2 = (a + b)(a - b)$$

That is, the factors of $a^2 - b^2$ are $a + b$ and $a - b$.

Example 6

Factorise:

a $x^2 - 9$

b $25 - y^2$

c $4x^2 - 9$

Solution

a $x^2 - 9 = x^2 - 3^2$
 $= (x + 3)(x - 3)$

Check the answer by expanding $(x + 3)(x - 3)$ to see that $x^2 - 9$ is obtained.

b $25 - y^2 = 5^2 - y^2$
 $= (5 + y)(5 - y)$

c $4x^2 - 9 = (2x)^2 - 3^2$
 $= (2x + 3)(2x - 3)$

Example 7

Factorise $3a^2 - 27$.

Solution

$$\begin{aligned} 3a^2 - 27 &= 3(a^2 - 9) && \text{(A common factor of 3 is factored out first.)} \\ &= 3(a^2 - 3^2) \\ &= 3(a + 3)(a - 3) \end{aligned}$$

Example 8

Factorise $-16 + 9x^2$.

Solution

$$\begin{aligned} -16 + 9x^2 &= 9x^2 - 16 \\ &= (3x)^2 - 4^2 \\ &= (3x - 4)(3x + 4) \end{aligned}$$



Factorisation using the difference of two squares identity

$$a^2 - b^2 = (a + b)(a - b)$$



Exercise 4B

Example 6

1 Factorise:

a $x^2 - 16$

b $x^2 - 49$

c $x^2 - 100$

d $x^2 - 144$

e $a^2 - 121$

f $m^2 - 64$

g $b^2 - 25$

h $d^2 - 400$

i $(2x)^2 - 25$

j $(3x)^2 - 16$

k $(4x)^2 - 1$

l $(5m)^2 - 9$

m $9x^2 - 4$

n $16y^2 - 49$

o $100a^2 - 49b^2$

p $64m^2 - 81p^2$

q $1 - 4a^2$

r $9 - 16y^2$

s $25a^2 - 100b^2$

t $-9 + x^2$

Example 7

2 Factorise:

a $3x^2 - 48$

b $4x^2 - 100$

c $5x^2 - 45$

d $6x^2 - 24$

e $10x^2 - 1000$

f $7x^2 - 63$

g $8x^2 - 50$

h $12m^2 - 75$

i $3 - 12b^2$

j $20 - 5y^2$

k $27a^2 - 12b^2$

l $16x^2 - 100y^2$

m $45m^2 - 125n^2$

n $27a^2 - 192\ell^2$

o $-8x^2 + 32y^2$

p $-200p^2 + 32q^2$

Example 8

3 Factorise:

a $-9 + x^2$

b $-25 + x^2$

c $-4 + 9x^2$

d $-36 + x^2$

e $-9x^2 + 4$

f $-81x^2 + 16$

g $-100x^2 + 9$

h $-49 + 16x^2$

i $-25x^2 + 121$

j $-18 + 50x^2$

k $-12x^2 + 27$

l $-80 + 125x^2$

m $-36x^2 + 400$

n $-175 + 28x^2$

o $-40x^2 + 90$

- 4 Use the factorisation of the difference of two squares to evaluate the following. One has been done for you.

$$\begin{aligned} 17^2 - 3^2 &= (17 + 3)(17 - 3) \\ &= 20 \times 14 \\ &= 280 \end{aligned}$$

- a** $23^2 - 7^2$ **b** $23^2 - 3^2$ **c** $36^2 - 6^2$
d $94^2 - 6^2$ **e** $1.8^2 - 0.2^2$ **f** $2.8^2 - 2.2^2$
g $11.3^2 - 8.7^2$ **h** $92.6^2 - 7.4^2$ **i** $3.214^2 - 2.214^2$

- 5 **a** Evaluate:

- i** $4^2 - 3^2$ **ii** $5^2 - 4^2$ **iii** $6^2 - 5^2$ **iv** $7^2 - 6^2$
v $8^2 - 7^2$ **vi** $9^2 - 8^2$ **vii** $10^2 - 9^2$ **viii** $101^2 - 100^2$

- b** What do you notice?

- c** Use the factorisation of $(n + 1)^2 - n^2$ to prove the result of part **b**.

- 6 The following leads you through the geometrical proof that $a^2 - b^2 = (a - b)(a + b)$.

In the diagram opposite, square $ECFG$, of side length b , is cut out of square $ABCD$, which has side length a .

- a** What is the area of hexagon $ABEGFD$?

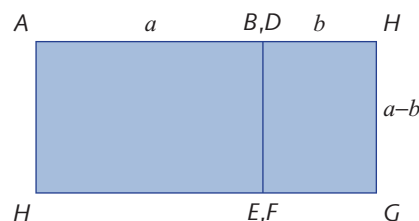
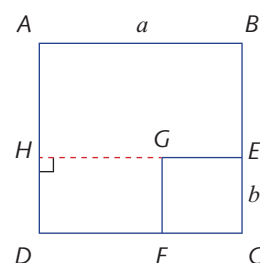
- b** What is the length of BE ?

- c** What is the length of DF ?

- d** The rectangle $HGFD$ is moved so that DF is placed on top of BE (see diagram).

- i** What is the area of the large shaded rectangle?

- ii** What have we proved?





A **simple quadratic** expression is an expression of the form $x^2 + bx + c$, where b and c are given numbers. When we expand $(x + 3)(x + 4)$, we obtain a simple quadratic.

$$\begin{aligned}x(x + 4) + 3(x + 4) &= x^2 + 4x + 3x + 12 \\ &= x^2 + 7x + 12\end{aligned}$$

We want to develop a method for reversing this process.

In the expansion $(x + 3)(x + 4) = x^2 + 7x + 12$, notice that the coefficient of x is $3 + 4 = 7$. The term that is independent of x , the constant term, is $3 \times 4 = 12$. This suggests a method of factorising.

In general, when we expand $(x + p)(x + q)$, we obtain

$$x^2 + px + qx + pq = x^2 + (p + q)x + pq$$

The coefficient of x is the sum of p and q , and the constant term is the product of p and q .



Factorisation of simple quadratics

To factorise a simple quadratic, look for two numbers that add to give the coefficient of x , and that multiply together to give the constant term.

For example, to factorise $x^2 + 8x + 15$, we look for two numbers that multiply to give 15 and add to give 8. Of the pairs that multiply to give 15 (15×1 , 5×3 , $-5 \times (-3)$ and $-15 \times (-1)$), only 5 and 3 add to give 8.

Therefore, $x^2 + 8x + 15 = (x + 3)(x + 5)$.

The result can be checked by expanding $(x + 3)(x + 5)$.

Example 9

Factorise $x^2 - 3x - 18$.

Solution

We are looking for two numbers that multiply to give -18 and add to give -3 . The numbers -6 and 3 satisfy both conditions.

$$x^2 - 3x - 18 = (x - 6)(x + 3)$$

Consider factorising $x^2 - 36$.

Since the constant term is -36 and the coefficient of x is 0 , we are looking for two numbers that multiply to give -36 and add to give 0 . The numbers -6 and 6 satisfy both conditions.

Thus $x^2 - 36 = (x - 6)(x + 6)$.

Note: It is not really sensible to do the example in this way – it is best to recognise $x^2 - 36$ as a difference of squares. However, it does show that the method of factorisation in this section is consistent with the earlier technique.

Example 10

Factorise $x^2 + 8x + 16$.

Solution

We are looking for two numbers that multiply to give 16 and add to give 8 . The numbers 4 and 4 satisfy both conditions.

$$\begin{aligned}\text{Thus } x^2 + 8x + 16 &= (x + 4)(x + 4) \\ &= (x + 4)^2\end{aligned}$$



Exercise 4C

Example 9

1 Factorise these quadratic expressions.

a $x^2 + 5x + 6$

b $x^2 + 11x + 18$

c $x^2 + 7x + 10$

d $x^2 + 11x + 30$

e $x^2 + 9x + 14$

f $x^2 + 19x + 90$

g $x^2 + 9x + 20$

h $x^2 + 13x + 22$

i $x^2 + 7x + 12$

j $x^2 + 12x + 32$

k $x^2 + 13x + 40$

l $x^2 + 20x + 75$

m $x^2 + 12x + 35$

n $x^2 + 28x + 27$

o $x^2 + 15x + 56$

p $x^2 + 18x + 56$

2 Factorise these quadratic expressions.

a $x^2 - 5x + 6$

b $x^2 - 14x + 33$

c $x^2 - 17x + 30$

d $x^2 - 13x + 42$

e $x^2 - 9x + 14$

f $x^2 - 47x + 90$

g $x^2 - 15x + 44$

h $x^2 - 25x + 100$

i $x^2 - 18x + 80$

j $x^2 - 21x + 80$

k $x^2 - 14x + 40$

l $x^2 - 11x + 24$

m $x^2 - 13x + 42$

n $x^2 - 8x + 15$

o $x^2 - 30x + 56$

p $x^2 - 14x + 24$

3 Factorise these quadratic expressions.

a $x^2 + x - 6$

b $x^2 - 8x - 33$

c $x^2 + x - 30$

d $x^2 - 19x - 42$

e $x^2 - 5x - 14$

f $x^2 - 9x - 90$

g $x^2 - 7x - 44$

h $x^2 + 15x - 100$

i $x^2 + 2x - 80$

j $x^2 - 7x - 60$

k $x^2 + 3x - 40$

l $x^2 - 10x - 24$

m $x^2 + 4x - 21$

n $x^2 + 2x - 15$

o $x^2 - x + 56$

p $x^2 + 5x - 24$

4 Factorise these quadratic expressions.

a $x^2 - 3x + 2$

b $x^2 + 8x + 12$

c $x^2 - 3x - 10$

d $x^2 + 11x + 30$

e $x^2 - 5x - 14$

f $x^2 - 9x - 90$

g $x^2 - 5x + 4$

h $x^2 - 7x - 18$

i $x^2 - x - 12$

j $x^2 - 11x + 28$

k $x^2 + 3x - 10$

l $x^2 + x - 90$

5 Factorise these quadratic expressions.

a $x^2 + 6x + 9$

b $x^2 + 14x + 49$

c $x^2 + 20x + 100$

d $x^2 - 10x + 25$

e $x^2 - 18x + 81$

f $x^2 + 16x + 64$

g $x^2 + 10x + 25$

h $x^2 + 12x + 36$

i $x^2 + 30x + 225$

j $x^2 - 16x + 64$

k $x^2 - 20x + 100$

l $x^2 - 8x + 16$

Example 10



Sometimes a common factor can be taken out of a quadratic expression so that the expression inside the brackets becomes a simple quadratic that can be factorised.

Example 11

Factorise:

a $3x^2 + 9x + 6$

b $6x^2 - 54$

c $-x^2 - x + 2$

Solution

$$\begin{aligned}\mathbf{a} \quad 3x^2 + 9x + 6 &= 3(x^2 + 3x + 2) \quad (\text{Take out the common factor.}) \\ &= 3(x + 2)(x + 1)\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad 6x^2 - 54 &= 6(x^2 - 9) \\ &= 6(x + 3)(x - 3)\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad -x^2 - x + 2 &= -(x^2 + x - 2) \quad (\text{Factor } -1 \text{ from each term.}) \\ &= -(x + 2)(x - 1)\end{aligned}$$

**Exercise 4D**

Example 11a

1 Factorise:

a $2x^2 + 14x + 24$

b $3x^2 + 24x + 36$

c $3x^2 - 27x + 24$

d $4x^2 - 24x + 36$

e $7x^2 + 14x + 7$

f $5x^2 - 5x - 30$

g $4x^2 - 4x - 48$

h $2x^2 - 18x + 36$

i $5x^2 + 40x + 35$

j $3x^2 + 9x - 120$

k $3x^2 - 3x - 90$

l $5x^2 + 60x + 180$

m $2x^2 - 4x - 96$

n $5x^2 + 65x + 180$

o $3x^2 + 30x - 72$

p $2x^2 + 40x + 200$

q $3x^2 - 18x + 27$

r $5x^2 - 20x + 20$

s $5x^2 - 40x + 80$

t $2x^2 + 24x + 72$

u $3x^2 - 24x + 36$

Example 11b

2 Factorise:

a $4x^2 - 16$

b $2x^2 - 18$

c $3x^2 - 48$

d $3a^2 - 27$

e $6x^2 - 600$

f $3a^2 - 27b^2$

g $45x^2 - 20$

h $18t^2 - 50s^2$

i $5a^2 - 20$

j $27x^2 - 3y^2$

m $128 - 2x^2$

p $\frac{1}{2}a^2 - 32$

s $\frac{1}{4}a^2 - 9$

k $45 - 5b^2$

n $\frac{1}{2}a^2 - 2b^2$

q $7x^2 - 175$

t $\frac{1}{5}x^2 - 20$

l $12 - 3m^2$

o $27x^2 - \frac{1}{3}y^2$

r $\frac{1}{3}y^2 - 12$

u $\frac{1}{4}x^2 - y^2$

Example 11c

3 Factorise:

a $-x^2 - 8x - 12$

d $9 + 8x - x^2$

g $-x^2 + 47x + 48$

j $-x^2 + 3x + 40$

m $-x^2 - 5x + 24$

p $11x - x^2 - 24$

s $-16x - 63 - x^2$

b $12 - 11x - x^2$

e $-x^2 - 4x - 4$

h $-x^2 - 6x - 9$

k $42 + x - x^2$

n $22x - x^2 - 40$

q $-3x^2 - 30x + 72$

t $-x^2 - 35 + 12x$

c $7 - 6x - x^2$

f $-x^2 - 14x - 45$

i $-2x^2 + 24x - 72$

l $-x^2 + 13x - 36$

o $28 + 3x - x^2$

r $-56 - x^2 - 15x$

u $7x + 18 - x^2$

4E Using factorisation to simplify algebraic expressions



In this section, we show how some quotients and products involving quadratics can be simplified by first factorising and then cancelling. By factorising each quadratic, common factors which can be cancelled can be identified.

Example 12

Simplify $\frac{x+1}{x^2+3x+2}$.

Solution

$$\frac{x+1}{x^2+3x+2} = \frac{x+1}{(x+2)(x+1)} \quad \text{(Factorise the denominator.)}$$

$$= \frac{\cancel{x+1}}{(x+2)\cancel{(x+1)}} \quad \text{(Cancel the common factor.)}$$

$$= \frac{1}{x+2}$$

In the above example, and throughout the rest of the chapter, we will assume that the factor being cancelled does not have a value of zero.

Example 13

Simplify $\frac{x^2+4x-12}{x^2+3x-10}$.

Solution

$$\frac{x^2+4x-12}{x^2+3x-10} = \frac{(x+6)\cancel{(x-2)}}{(x+5)\cancel{(x-2)}}$$

$$= \frac{x+6}{x+5}$$

Example 14

Simplify $\frac{x^2 + 2x}{x^2 - 1} \times \frac{x^2 - 5x + 4}{x^2 - 2x - 8}$.

Solution

$$\begin{aligned}\frac{x^2 + 2x}{x^2 - 1} \times \frac{x^2 - 5x + 4}{x^2 - 2x - 8} &= \frac{x(x+2)}{(x+1)(x-1)} \times \frac{(x-4)(x-1)}{(x-4)(x+2)} \\ &= \frac{\cancel{x}(\cancel{x+2})}{(x+1)(\cancel{x-1})} \times \frac{(\cancel{x-4})(\cancel{x-1})}{(\cancel{x-4})(\cancel{x+2})} \\ &= \frac{x}{x+1}\end{aligned}$$

Example 15

Simplify $\frac{x^2 - x - 6}{x^2 + x - 2} \div \frac{2x^2 - 6x}{x^2 - 1}$.

Solution

Invert and multiply before factorising.

$$\begin{aligned}\frac{x^2 - x - 6}{x^2 + x - 2} \div \frac{2x^2 - 6x}{x^2 - 1} &= \frac{x^2 - x - 6}{x^2 + x - 2} \times \frac{x^2 - 1}{2x^2 - 6x} \\ &= \frac{(\cancel{x+2})(\cancel{x-3})}{(\cancel{x-1})(\cancel{x+2})} \times \frac{(x+1)(\cancel{x-1})}{2x(\cancel{x-3})} \\ &= \frac{x+1}{2x}\end{aligned}$$

Exercise 4E

Example 12

1 Simplify:

a $\frac{x}{x^2 + 3x}$

b $\frac{x^2 + x}{x^2 + 3x + 2}$

c $\frac{2x^2 - x}{6x - 3}$

d $\frac{3x}{x^2 - x}$

e $\frac{x^2 - 3x}{x^2 - 9}$

f $\frac{4x^2 + x}{8x + 2}$

2 Simplify:

a $\frac{x^2 - 1}{x^2 - x - 2}$

b $\frac{x^2 - 4}{x^2 - 3x - 10}$

c $\frac{x^2 - 1}{x^2 + x - 2}$

d $\frac{x^2 + 4x - 5}{x^2 - 2x + 1}$

e $\frac{x^2 + x - 6}{x^2 + 2x - 3}$

f $\frac{x^2 + x - 12}{x^2 - 3x}$

g $\frac{x^2 - 9}{x^2 - 5x - 24}$

h $\frac{x^2 - x - 2}{x^2 - 2x}$

i $\frac{x^2 + 4x + 4}{x^2 + 5x + 6}$

j $\frac{x^2 - x - 20}{x^2 + x - 12}$

k $\frac{x^2 + 6x + 9}{x^2 + x - 6}$

l $\frac{2x^2 - 18}{3x^2 + 3x - 18}$

m $\frac{x^2 - 5x + 6}{x^2 - 4x + 4}$

n $\frac{x^2 + 7x + 12}{x^2 + 8x + 16}$

o $\frac{3x^2 + 3x - 36}{x^2 + 8x + 16}$

3 Simplify:

a $\frac{n - m}{m - n}$

b $\frac{k^2 - k\ell}{\ell - k}$

c $\frac{6x^2 + 6xy}{y^2 - x^2}$

d $\frac{18k^2 - 8\ell^2}{6k - 4\ell}$

e $\frac{12p^2 - 27q^2}{6p^2 - 9pq}$

f $\frac{pq}{p^2q^2 + pq}$

g $\frac{m^2 + mn}{n^2 + nm}$

h $\frac{s^2 - t^2}{t - s}$

i $\frac{3x - 3y}{xy - x^2}$

j $\frac{3p^3 - 3pq^2}{p^2 + pq}$

k $\frac{x^3 - xy^2}{2x + 2y}$

l $\frac{m^3 - m^2n}{n - m}$

4 Simplify:

a $\frac{x}{x-1} \times \frac{x^2 - x}{2}$

b $\frac{x^2 - x - 2}{x^2 + 3x} \times \frac{5x}{x-2}$

c $\frac{2x}{4x-1} \times \frac{8x-2}{3x^2}$

d $\frac{x^2 - 2x - 8}{x^2 + 3x} \times \frac{x+3}{x^2 - 3x - 4}$

e $\frac{x}{x^2 - 1} \times \frac{x+1}{x^2}$

f $\frac{x^2 - 3x - 10}{x^2 + 2x} \times \frac{2x^2 - 2x}{x^2 - 6x + 5}$

g $\frac{3x^2}{2x+1} \times \frac{4x^2 - 1}{x^3 + 5x^2}$

h $\frac{x^2 - x - 2}{x^2 + 3x} \times \frac{5x}{x-2}$

5 Simplify:

$$\mathbf{a} \quad \frac{2x}{4x-1} \div \frac{3x^2}{8x-2}$$

$$\mathbf{c} \quad \frac{3x^2-3x}{2x-2} \div \frac{x-1}{1}$$

$$\mathbf{e} \quad \frac{3x}{x+3} \div \frac{6x^2-3x}{x^2-9}$$

$$\mathbf{g} \quad \frac{3x^2-3x}{x+1} \div \frac{1-x}{x^2+x}$$

$$\mathbf{i} \quad \frac{x^2-49}{x^2-9} \div \frac{x+7}{x+3}$$

$$\mathbf{b} \quad \frac{3x}{x^2-1} \div \frac{6x^2}{(x+1)^2}$$

$$\mathbf{d} \quad \frac{x^2-x-2}{x^2+4x+3} \div \frac{x^2-3x+2}{x^2+3x}$$

$$\mathbf{f} \quad \frac{2x}{x+3} \div \frac{x^2-9}{6x^2-3x}$$

$$\mathbf{h} \quad \frac{x^2-4}{2x^2-6x} \div \frac{6-x-x^2}{9-x^2}$$

$$\mathbf{j} \quad \frac{3m^2+3mn}{2n(m-n)} \div \frac{m+n}{n^2}$$

6 Simplify:

$$\mathbf{a} \quad \frac{x^2-5x+6}{x^2-16} \times \frac{x^2+5x+4}{x^2-4} \div \frac{x-3}{x-4}$$

$$\mathbf{c} \quad \frac{3p}{p+3} \div \frac{6p^2-3p}{p^2-9} \times \frac{2p-1}{p-3}$$

$$\mathbf{e} \quad \frac{p^2-q^2}{2p^2+p} \times \frac{2pq+q}{p^2-pq} \div (p+q)$$

$$\mathbf{f} \quad \frac{2a^2-32}{a^2+7a+12} \div \frac{4a^2-4a-48}{12a^2+15a} \div \frac{(a-1)(4a+5)}{a}$$

$$\mathbf{b} \quad \frac{25a^2-1}{9x^2-4y^2} \times \frac{3x+2y}{5a+1} \div \frac{5a-1}{3x-2y}$$

$$\mathbf{d} \quad \frac{5\ell^2+15\ell}{\ell^2+\ell-2} \div \frac{\ell+3}{\ell^2+3\ell+2} \times \frac{\ell-1}{\ell+1}$$

Review exercise



1 Factorise:

a $4x + 16$

d $10m + 25$

g $5mn - 10n$

j $b^2 - 6b$

m $a^2b - 4ab^2$

b $7x - 21$

e $4ab + 7a$

h $4uv - 8v$

k $3a^2b + 6ab$

n $3pq - 6p^2$

c $6a - 9$

f $6pq - 11p$

i $a^2 + 9a$

l $4m^2n - 12mn$

o $6p^3q - 18pq$

2 Factorise:

a $x^2 - 9$

d $a^2 - 100$

g $9 - 4b^2$

j $9a^2 - b^2$

m $1 - 36b^2$

b $x^2 - 16$

e $9a^2 - 25$

h $100 - 81b^2$

k $2m^2 - 50$

n $4y^2 - \frac{1}{4}$

c $y^2 - 144$

f $16m^2 - 1$

i $16x^2 - y^2$

l $3a^2 - 27$

o $p^2q^2 - 1$

3 Factorise:

a $x^2 + 8x + 12$

d $x^2 + 11x + 28$

g $x^2 - 14x + 24$

j $x^2 - 2x - 48$

m $x^2 - 7x - 8$

p $x^2 + 12x + 35$

b $x^2 + 9x + 18$

e $x^2 - 11x + 24$

h $x^2 - 25x + 24$

k $x^2 - 4x - 12$

n $x^2 - x - 132$

q $a^2 + 19a + 60$

c $x^2 + 11x + 30$

f $x^2 - 10x + 24$

i $x^2 + x - 20$

l $x^2 + 3x - 40$

o $x^2 + 7x + 12$

r $x^2 - 50x + 96$

4 Factorise:

a $a^2 - 22a + 121$

d $a^2 + 24a + 144$

b $m^2 - 14m + 49$

e $a^2 - 12a + 36$

c $s^2 + 8s + 16$

f $z^2 - 40z + 400$

5 Factorise:

a $2x^2 + 18x + 40$

d $2x^2 + 8x - 90$

b $3x^2 - 30x + 63$

e $3x^2 - 6x - 105$

c $5x^2 - 50x + 120$

f $2x^2 - 6x - 260$

6 Simplify:

a $\frac{y^2 - 3y}{y^2 - 6y + 9}$

b $\frac{a^2 - 4}{a^2 + 5a + 6}$

c $\frac{15p^2q - 24pq}{27p^2q - 6pq}$

d $\frac{m^2 - 4m + 3}{m^2 - 9}$

e $\frac{5x^2}{x+3} \times \frac{x^2 - x - 12}{10x}$

f $\frac{y^2 + 7y}{y^2 + 10y + 9} \times \frac{y^2 + 2y + 1}{y^2 - 49}$

g $\frac{5x^2 - 20}{x^2 + 6x} \div \frac{x^2 - 4x + 4}{x^2 + 8x + 12}$

h $\frac{5x^2 - 5x}{x^2 + 4x + 3} \div \frac{1 - x}{10x^2 + 10x}$



7 Factorise:

a $25a^2 - 16b^2$

b $a^2 + 14a + 49$

c $a^2 - a - 20$

d $1 - 36m^2$

e $4 - 9x^2y^2$

f $\frac{1}{9} - \frac{a^2}{25}$

g $m^2 - \frac{1}{4}$

h $x^3 - 49xy^2$

i $3a^2 - 75$

j $b^2 - 20b + 96$

k $n^2 - 31n + 150$

l $m^2 + 20m + 91$

m $a^2 - 7ab - 98b^2$

n $m^2 - 4m - 165$

o $x^2 + 3xy - 4y^2$

p $20m^2n - 5n^3$

q $a^2 - 2a - 63$

r $5q^2 - 5pq - 30p^2$

s $x^2 - 3x - 130$

t $42 - x - x^2$

u $x^2 + 7x - 18$

8 Simplify:

a $\frac{x^2 + 2x - 3}{3(x - 1)}$

b $\frac{x^2 - x - 2}{x^2 - 2x}$

c $\frac{x^2 + x - 6}{x^2 + 2x - 3}$

d $\frac{x^2 + 4x - 5}{x^2 - 2x + 1}$

e $\frac{12p^2 - 27q^2}{6p^2 - 9pq}$

f $\frac{2x^3 - 2xy^2}{3x^3 + 3x^2y}$

g $\frac{3x^2 - 3xy}{2y - 2x}$

h $\frac{x^2 + 4x + 4}{x^2 + 5x + 6}$

i $\frac{x^2 - 2x}{x^2 - 5x + 6}$

9 Simplify:

a $\frac{x + 1}{x^2 - 4x - 5}$

b $\frac{x^2 - 3x - 10}{xy + 2y}$

c $\frac{cx - dx}{dy - cy} \times \frac{y^2}{x^2}$

d $\frac{x^2 - 3x - 4}{x^2 - 3x} \div \frac{x^2 - 4x}{x - 3}$

e $\frac{x^2 + 3x - 10}{x^2 + 2x - 15} \div \frac{x^2 + 3x + 2}{x^2 - 4x + 3}$

f $\frac{3x^2 - 9x - 12}{x^2 + 2x - 3} \times \frac{x^2 - 2x - 15}{2x^2 - 4x - 6}$

g $\frac{8x^2 - 8}{x^2 + 4x + 4} \times \frac{x^2 + x - 2}{2x - 2}$

h $\frac{x^2 - 3x + 2}{5x^2 - 5x} \times \frac{x^2 - 3x}{x^2 - 9}$

i $\frac{4x + 4}{5x^2 - 30x + 25} \div \frac{6x + 6}{5x^2 + 20x - 25}$

Challenge exercise



1 Factorise:

a $x^4 - 1$

b $x^4 - 16$

c $x^2 - 3$

d $x^2 - 5$

e $x^2 - 2\sqrt{2}x + 2$

f $x^2 + 2\sqrt{2}x + 2$

g $x^2 + x + \frac{1}{4}$

h $x^2 + 3x + \frac{9}{4}$

i $x^2 - x + \frac{1}{4}$

j $x^2 - 3x + \frac{9}{4}$

k $x^2 - xy - 2y^2$

l $x^2 + xy - 2y^2$

2 Simplify:

a $\frac{x^2 + x - 2}{x^2 - x - 20} \times \frac{x^2 + 5x + 4}{x^2 - x} \div \left(\frac{x^2 + 3x + 2}{x^2 - 2x - 15} \times \frac{x + 3}{x^2} \right)$

b $\frac{x^2 - 64}{x^2 + 24x + 128} \times \frac{x^2 + 12x - 64}{x^2 - 16} \div \frac{x^2 - 16x + 64}{x^2 - 10x + 16}$

c $\frac{x^2 - 18x + 80}{x^2 - 5x - 50} \times \frac{x^2 - 6x - 7}{x^2 - 15x + 56} \div \frac{x - 1}{x + 5}$

3 Factorise:

a $(x + 2)^2 - 8x$

b $(x - 3)^2 + 12x$

c $(x + a)^2 - 4ax$

d $(x - a)^2 + 4ax$

4 Simplify $\left(\frac{a}{b} + \frac{c}{d}\right) \div \left(\frac{a}{b} - \frac{c}{d}\right)$.

5 Factorise:

a $(x^2 + 1)^2 - 4$

b $(x^2 - 2)^2 - 4$

c $x^2 + 4x + 4 - y^2$

d $x^2 + 8x + 16 - a^2$

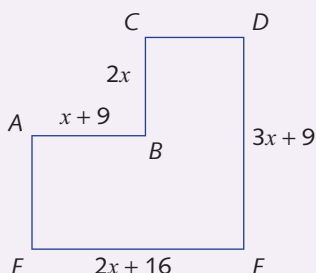
e $m^2 - 2m + 1 - n^2$

f $p^2 - 5p + \frac{25}{4} - q^2$

6 **a** By adding and subtracting $4x^2$, factorise $x^4 + 4$.

b Factorise $x^4 + 4a^4$.

- 7 A swimming pool is designed in an L-shape with dimensions in metres as shown.



The pool is enlarged or reduced depending on the value of x .

- a** Find, in terms of x , the length of:

i AF **ii** CD

- b** Show that the perimeter is equal to $(10x + 50)$ m.

- c** What is the perimeter if $x = 3$?

- d** Find the area of the swimming pool in terms of x . Expand and simplify your answer.

- e** It is decided that a square swimming pool would be a better use of space.

i By factorising your answer to part **d**, find the dimensions, in terms of x , of a square swimming pool with the same area as the L-shaped swimming pool.

ii What is the perimeter, in terms of x , of this square swimming pool?

- 8 For positive whole numbers a and b , prove that:

a if $\frac{a}{b} < 1$ then $\frac{a+1}{b+1} > \frac{a}{b}$

b if $\frac{a}{b} > 1$ then $\frac{a+1}{b+1} < \frac{a}{b}$

- 9 A right-angled triangle has a hypotenuse of length b cm and one other side of length a cm. If $b - a = 1$, find the length of the third side in terms of a and b .

10 Factorise $\left(1 + \frac{y^2 + z^2 - x^2}{2yz}\right) \div \left(1 - \frac{x^2 + y^2 - z^2}{2xy}\right)$.



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CHAPTER

5

Number and Algebra

Linear equations and inequalities

We have seen that many real-world problems can be converted to equations. We have also seen how to solve simple equations. In this chapter, we consider a wider variety of equations and use them to solve practical problems.

Equations arise naturally when solving problems. In fact, a lot of problem-solving relies on us being able to translate a given word or real world problem into an equation, or equations, solve the equation(s) and relate the solution to the original problem. Turning a complicated problem into an equation enables us to understand and solve difficult problems. Before we reach this stage, we need to have at hand a collection of techniques for solving equations.

In earlier times, people used a number of ad hoc methods for solving equations. Only since the development of modern algebra have standard procedures and notations been introduced that enable us to solve equations quickly and efficiently.

5A Expressions



Two examples of **linear expressions** are $2x + 3$ and $x + 7$. The expression $2x + 3$ is called **linear** because the graph of $y = 2x + 3$ is a straight line, as we saw in Year 8. We begin with some revision examples and exercises.

Example 1

Brian is 6 cm taller than Geoff. Represent this information algebraically.

Solution

Since we do not know how tall Geoff is, we call his height x cm.
Then Brian is $(x + 6)$ cm tall.

Example 2

The length of a rectangle is 3 m more than its width. The width of the rectangle is w m.

- a** Express the length of the rectangle in terms of w .
- b** Express the perimeter of the rectangle in terms of w .

Solution

- a** Length of rectangle = $(w + 3)$ m
- b** Perimeter = $w + w + (w + 3) + (w + 3)$
 $= (4w + 6)$ m

Exercise 5A

Example 1

- 1 Fiona is 5 cm taller than Tristan. Tristan's height is h cm. What is Fiona's height?
- 2 Seuret weighs 5 kg more than his younger sister Vivian. If Vivian weighs w kg, what is Seuret's weight?
- 3 When Andriana's age is doubled, the number is 3 more than Helen's age. If Andriana's age is x years, what is Helen's age?

Example 2

- 4 The length of a rectangle is 5 metres more than its width.
 - a** If w metres is the width of the rectangle, express the length of the rectangle in terms of w .
 - b** If ℓ metres is the length of the rectangle, express the width of the rectangle in terms of ℓ .

- 5** In a competition, Deeksha scored 18 points more than Greta and Deirdre scored 5 points less than twice the number of points Greta scored. If a is the number of points Greta scored:
- express the number of points Deeksha scored in terms of a
 - express the number of points Deirdre scored in terms of a .
- 6** The length of a rectangular paddock is 20 m less than three times its width. If the width of the paddock is x m, express the length of the paddock in terms of x .
- 7** In a triathlon race, Luca ran at an average speed 5 times his average swimming speed. Also, when his average running speed was multiplied by 4, this number was 3 less than his average speed for the cycling leg.
- If x km/h is Luca's average swimming speed, find expressions in terms of x for his:
- average running speed
 - average cycling speed.
- 8** Match each of the following mathematical expressions with its corresponding English expression.
- | | |
|----------------------------|---|
| a $4 + 2x$ | i Six less than four times x |
| b $x - 5$ | ii Three times one more than x |
| c $2x - 4$ | iii Two less than one quarter of x |
| d $\frac{x + 2}{4}$ | iv Three times one less than x |
| e $3(x + 1)$ | v One quarter of two less than x |
| f $4x - 6$ | vi One quarter of two more than x |
| g $\frac{x}{4} - 2$ | vii Four less than twice x |
| h $\frac{x - 2}{4}$ | viii Six more than half of x |
| i $x + 6$ | ix One more than three times x |
| j $3x + 1$ | x Five less than x |
| k $\frac{x}{2} + 6$ | xi Six more than x |
| l $3(x - 1)$ | xii Four more than twice x |
- 9** Gemma is 6 cm shorter than Gavin and 4 cm taller than Brent. If x cm represents Gemma's height, express:
- Gavin's height in terms of x
 - Brent's height in terms of x .

5B Solving simple linear equations



A statement such as $x + 7 = 11$ is called an **equation** and we may **solve** the equation to find a value for x that makes the statement true. In this case, the **solution** is $x = 4$.

Reading equations

The equation $2x + 4 = 10$ can be read as ‘two times a number plus 4 is equal to 10’.

The instruction ‘Solve the equation $2x + 4 = 10$ ’ can also be read as ‘A number is multiplied by 2 and 4 is then added. The result is 10. Find the number.’

In this case, the number is 3. The solution is $x = 3$.

Equivalent equations

Consider these equations:

$$2x + 7 = 11 \quad (1)$$

$$2x + 9 = 13 \quad (2)$$

Equation (2) is obtained from equation (1) by adding 2 to each side of the equation.

So equation (1) is obtained from equation (2) by subtracting 2 from each side of the equation.

Equations (1) and (2) are said to be **equivalent equations**.

Equation (3), below, is obtained from equation (1) by subtracting 7 from each side of the equation.

$$2x = 4 \quad (3)$$

$$x = 2 \quad (4)$$

Equation (4) is obtained from equation (3) by dividing each side of the equation by 2. You can obtain equation (3) from equation (4) by multiplying each side of the equation by 2.

All of the above equations are **equivalent**.



Equivalent equations

- If we add the same number to, or subtract the same number from, both sides of an equation, the new equation is **equivalent** to the original equation.
- If we multiply or divide both sides of an equation by the same non-zero number, the new equation is **equivalent** to the original equation.
- Equivalent equations have exactly the same solutions.

In the following examples, equivalent equations are formed to solve the equations.

Example 3

Solve:

a $3x + 5 = 20$ **b** $5x - 7 = 18$ **c** $3 - 2x = 15$ **d** $-3p = \frac{2}{5}$

Solution

a $3x + 5 = 20$
 $3x = 15$ (Subtract 5 from both sides.)
 $x = 5$ (Divide both sides by 3.)

b $5x - 7 = 18$
 $5x = 25$ (Add 7 to both sides.)
 $x = 5$ (Divide both sides by 5.)

c $3 - 2x = 15$
 $-2x = 12$ (Subtract 3 from both sides.)
 $x = -6$ (Divide both sides by -2 .)

d $-3p = \frac{2}{5}$
 $p = -\frac{2}{15}$ (Divide both sides by -3 .)

Example 4

Solve:

a $3x + 7 = 2x + 13$ **b** $5a - 21 = 14 - 2a$

Solution

a $3x + 7 = 2x + 13$
 $3x + 7 - 2x = 2x + 13 - 2x$ (Subtract $2x$ from both sides.)
 $x + 7 = 13$
 $x = 6$ (Subtract 7 from both sides.)

b $5a - 21 = 14 - 2a$
 $7a - 21 = 14$ (Add $2a$ to both sides.)
 $7a = 35$ (Add 21 to both sides.)
 $a = 5$ (Divide both sides by 7.)



Exercise 5B

1 Solve these equations.

a $a + 2 = 5$

b $b + 7 = 19$

c $c - 6 = 11$

d $d - 15 = 3$

e $2a = 6$

f $3b = 9$

g $5c = 35$

h $6d = 42$

i $-3m = 6$

j $-2n = 8$

k $8p = -24$

l $9q = -27$

m $b + 7 = 29\frac{1}{2}$

n $a + 3 = 15\frac{1}{4}$

o $3a = \frac{2}{3}$

p $x - 6 = 5\frac{1}{3}$

q $-4y = \frac{8}{9}$

r $2x = -\frac{5}{6}$

Example 3

2 Solve these equations.

a $2a + 5 = 7$

b $3b + 4 = 19$

c $3c - 1 = 20$

d $5d - 7 = 23$

e $3e - 2 = 16$

f $4f - 3 = 13$

g $3g + 17 = 5$

h $5h + 21 = 11$

i $6a + 17 = -1$

j $4a + 23 = -9$

k $3a - 16 = -31$

l $7b - 17 = -66$

m $2b + 5 = 7\frac{1}{2}$

n $2x + 11 = 7\frac{1}{4}$

o $2x - 13 = 5\frac{2}{3}$

p $3a + 4 = -8\frac{1}{2}$

q $4m - 9 = -13\frac{4}{5}$

r $-2b + 4 = 9\frac{1}{4}$

3 Solve these equations.

a $2 - 3a = 8$

b $3 - 4b = 15$

c $5 - 2c = -13$

d $3 - 5d = -22$

e $-6 - 7e = 15$

f $-4 - 3f = 14$

Example 4

4 Solve these equations for x and check your answers.

a $5x + 5 = 3x + 1$

b $7x + 15 = 2x + 20$

c $8x + 5 = 4x + 21$

d $9x - 7 = 7x + 3$

e $5x - 6 = x - 2$

f $4x + 7 = x - 2$

g $3x + 1 = 9 - x$

h $4x - 3 = 18 - 3x$

i $2x - 3 = 7 - x$

j $8 - 3x = 2x - 7$

k $3x - 1 = 4 - x$

l $2x + 9 = 11 - 3x$



In this section, we look at solving equations in which brackets are involved. In previous work in this area, you have always expanded the brackets first. In some examples, we do not do this.

Example 5

Solve:

a $3(x + 2) = 21$

b $2(3 - x) = 12$

Solution

a $3(x + 2) = 21$

$$x + 2 = 7$$

(Divide both sides by 3.)

$$x = 5$$

b $2(3 - x) = 12$

$$3 - x = 6$$

(Divide both sides by 2.)

$$-x = 3$$

$$x = -3$$

Example 6

Solve $3(x + 5) = 31$.

Solution

Method 1

$$3(x + 5) = 31$$

$$3x + 15 = 31$$

$$3x = 16$$

$$x = \frac{16}{3}$$

$$x = 5\frac{1}{3}$$

Method 2

$$3(x + 5) = 31$$

$$x + 5 = \frac{31}{3}$$

$$x = \frac{16}{3}$$

When brackets are involved, the equation is usually solved by expanding the brackets first. In the above example, Method 1 is preferable to Method 2.

Example 7

Solve:

a $2(x + 1) + 4(x + 3) = 26$

b $3(a + 5) = 2(a + 6)$

c $3(y - 3) - 2(y - 4) = 4$

Solution

a $2(x + 1) + 4(x + 3) = 26$

$$2x + 2 + 4x + 12 = 26$$

(Expand the brackets.)

$$6x + 14 = 26$$

$$6x = 12$$

$$x = 2$$

b $3(a + 5) = 2(a + 6)$

$$3a + 15 = 2a + 12$$

$$a + 15 = 12$$

(Subtract $2a$ from both sides.)

$$a = -3$$

c $3(y - 3) - 2(y - 4) = 4$

$$3y - 9 - 2y + 8 = 4$$

$$y - 1 = 4$$

$$y = 5$$

(Expand both sets of brackets,
being careful with the signs.)

Exercise 5C

Example 5

1 Solve for x :

a $2(x + 3) = 8$

b $3(x - 2) = 15$

c $4(x - 1) = 12$

d $5(x + 1) = 10$

e $3(x - 3) = 18$

f $2(x + 4) = 14$

g $2(3 - x) = 4$

h $3(5 - x) = 9$

i $4(8 - x) = 36$

j $-3(2x - 6) = 12$

k $-4(3x - 7) = -8$

l $-2(7x + 6) = 86$

Example 6

2 Solve for x :

a $2(x + 3) = 15$

b $5(x - 2) = 16$

c $7(3 - x) = 20$

d $4(7 - x) = 7$

e $-2(x - 5) = 7$

f $-3(2x - 1) = 2$

g $5(x - 3) = \frac{2}{3}$

h $2(2x - 6) = \frac{2}{5}$

i $-3(5x + 2) = \frac{1}{4}$

3 Solve:

a $2(a + 1) + 4(a + 2) = 22$

c $5(c + 2) - 2(c + 1) = 17$

e $2(x + 3) - 3(x - 4) = 20$

g $3(a + 1) = 2(a + 2)$

i $2(a + 4) = 5(a + 7)$

k $-2(x + 4) + 3(x - 2) = 16$

m $\frac{1}{2}(2x + 5) + 6(x - 2) = 4\frac{1}{2}$

b $4(b - 1) + 3(b + 2) = 30$

d $4(2d + 1) - 5(d - 2) = 17$

f $5(2y - 3) - 3(y - 5) = 21$

h $5(a + 3) = 3(2a + 1)$

j $5(2a - 1) = 2(3a + 2)$

l $-5(x + 3) - 4(x + 1) = 17$

n $\frac{1}{2}(4x + 1) + 2(x - 2) = 13$

5D Linear equations involving fractions



When there are fractions in equations, the standard procedure is to remove the fractions by multiplying both sides of the equation by an appropriate whole number. The next step is to remove the brackets.

Example 8

Solve:

a $2x + \frac{1}{2} = \frac{2}{3}$

b $3x - \frac{1}{4} = \frac{4}{5}$

Solution

a $2x + \frac{1}{2} = \frac{2}{3}$

$$6\left(2x + \frac{1}{2}\right) = 6 \times \frac{2}{3} \quad \text{(Multiply each side by 6, the lowest common denominator.)}$$

$$12x + 3 = 4$$

$$12x = 1$$

$$x = \frac{1}{12}$$

b $3x - \frac{1}{4} = \frac{4}{5}$

$$20\left(3x - \frac{1}{4}\right) = 20 \times \frac{4}{5}$$

$$60x - 5 = 16$$

$$60x = 21$$

$$x = \frac{21}{60}$$

$$= \frac{7}{20}$$

Example 9

Solve $\frac{2x}{3} + \frac{1}{5} = 4$.

Solution

$$\frac{2x}{3} + \frac{1}{5} = 4$$

$$15\left(\frac{2x}{3} + \frac{1}{5}\right) = 15 \times 4 \quad (\text{Multiply both sides by 15.})$$

$$15 \times \frac{2x}{3} + 15 \times \frac{1}{5} = 15 \times 4$$

$$10x + 3 = 60$$

$$10x = 57$$

$$x = \frac{57}{10} \text{ or } x = 5\frac{7}{10}$$

Example 10

Solve $10 - \frac{a+3}{4} = 6$.

Solution

$$10 - \frac{a+3}{4} = 6$$

$$4 \times 10 - 4 \times \frac{a+3}{4} = 4 \times 6 \quad (\text{Multiply both sides by 4.})$$

$$40 - (a+3) = 24$$

$$37 - a = 24$$

$$-a = -13$$

$$a = 13$$

Example 11

Solve $2.1x + 3.5 = 9.4$

Solution

$$2.1x + 3.5 = 9.4$$

$$21x + 35 = 94 \quad (\text{Multiply both sides by 10.})$$

$$21x = 59$$

$$x = \frac{59}{21}$$

Example 12

Solve $\frac{2x}{3} - 3 = x + \frac{3}{4}$.

Solution

$$\begin{aligned}\frac{2x}{3} - 3 &= x + \frac{3}{4} \\ 8x - 36 &= 12x + 9 && \text{(Multiply both sides by 12.)} \\ -4x &= 45 \\ x &= -\frac{45}{4}\end{aligned}$$

Example 13

Solve $\frac{a+5}{4} = \frac{a+3}{3}$.

Solution

$$\begin{aligned}\frac{a+5}{4} &= \frac{a+3}{3} \\ 3(a+5) &= 4(a+3) && \text{(Multiply both sides by 12.)} \\ 3a + 15 &= 4a + 12 \\ 15 &= a + 12 \\ 3 &= a \\ a &= 3\end{aligned}$$



Solving linear equations

To solve linear equations:

- Remove all fractions by multiplying both sides of the equation by the lowest common denominator.
- Remove all brackets.
- Collect like terms and solve the equation.



Exercise 5D

Example 8

1 Solve:

a $2x - \frac{1}{2} = \frac{1}{4}$

b $3x + \frac{3}{2} = \frac{5}{3}$

c $\frac{4}{3} - 2y = \frac{1}{2}$

d $\frac{7}{2} - 3y = \frac{2}{3}$

e $\frac{2}{3} + 4x = \frac{3}{5}$

f $-\frac{3}{4} - 5y = \frac{2}{5}$

g $-2x + \frac{1}{3} = \frac{1}{5}$

h $-\frac{2}{3} + 3x = \frac{1}{5}$

i $-3y - \frac{2}{5} = \frac{1}{4}$

Example 9

2 Solve:

a $\frac{a}{3} + 5 = 3$

b $\frac{3a}{4} - \frac{4}{5} = \frac{2}{3}$

c $\frac{b}{3} - 5 = 3$

d $\frac{3b}{7} + 6 = 2$

e $2 - \frac{x}{3} = 4$

f $3 - \frac{3x}{4} = 6$

g $\frac{2}{3}(m - 3) = 1$

h $3\left(\frac{m}{5} + 2\right) = 2$

i $-3\left(\frac{x}{6} + 1\right) = 4$

Example 10

3 Solve:

a $\frac{2y + 1}{3} + 4 = 7$

b $\frac{2x + 5}{3} = 9$

c $\frac{5p - 2}{4} - 1 = 6$

d $\frac{3y - 1}{2} + 2 = 9$

e $\frac{2x - 3}{5} = 3$

f $\frac{4a + 3}{5} - 2 = 1$

g $2 + \frac{2x - 1}{5} = 5$

h $18 - \frac{7x + 2}{3} = 8$

i $1 - \frac{5y - 3}{4} = -2$

Example 11

4 Solve:

a $e + 1.8 = 2.9$

b $f + 3.6 = 7.5$

c $g - 2.8 = 3.8$

d $h - 1.6 = 8.7$

e $1.2u = 15.6$

f $3.6r = 9$

g $-4.7x = 49.35$

h $-6.2y = 52.08$

i $1.2x + 4 = 10$

j $3.8x - 7 = 8.2$

k $4.6x - 2.6 = 25$

l $8.2x - 52.6 = 45.8$

Example 12

5 Solve these equations for x and check your answers.

a $\frac{3x}{4} - 1 = \frac{x}{2} + 3$

b $\frac{x}{3} + 2 = \frac{4x}{3} + 3$

c $\frac{2x}{5} + 4 = \frac{3x}{10} - 5$

d $\frac{5x}{8} - 7 = \frac{3x}{4} + 2$

e $\frac{5x}{6} - 3 = 7 - \frac{x}{3}$

f $\frac{x}{3} - 4 = 6 - \frac{2x}{5}$

g $\frac{5}{3} - \frac{x}{2} = \frac{3x}{4} + \frac{7}{6}$

h $\frac{11}{12} - \frac{5x}{6} = \frac{3x}{4} - \frac{2}{3}$

6 Solve for x :

a $1.6x + 10 = 0.9x + 12$

c $8.3x + 12 = 36 - 1.7x$

e $1.5x + 3.9 = 6.7 - 0.5x$

b $5.9x - 7 = 2.4x + 35$

d $4.8 - 1.3x = 23 + 1.3x$

f $11.8x + 7.6 = 61.6 - 1.7x$

Example 13

7 Solve for a :

a $\frac{a+3}{2} = \frac{a+1}{5}$

c $\frac{2a+1}{3} = \frac{3a+1}{4}$

e $\frac{3a+2}{4} + 2 = a$

g $\frac{3a-2}{4} = \frac{a-5}{2}$

i $\frac{4a-1}{3} + a = 2$

b $\frac{a+1}{3} = \frac{2a-1}{7}$

d $\frac{a+1}{2} + 1 = \frac{a-1}{5}$

f $\frac{2a+1}{2} + \frac{a}{3} = 4$

h $\frac{2a-1}{3} - 2 = \frac{a+3}{4}$

j $\frac{a}{2} + \frac{a-1}{3} = \frac{a+1}{4}$

8 Solve:

a $5a + 9 = 24$

c $\frac{c}{3} - 2 = 4$

e $\frac{4e}{3} + \frac{1}{2} = 2$

g $2g + 5 = 7g - 6$

i $2(i-1) = 5(i+6)$

k $2(k+1) - 3(k-2) = 7$

m $\frac{m+1}{3} = \frac{m-2}{5}$

o $\frac{2q-2}{5} + 1 = 4q$

b $3b - 7 = 32$

d $\frac{d}{2} + 6 = 3$

f $\frac{2f}{3} - 1 = \frac{3}{7}$

h $4h - 2 = 5 - 3h$

j $3(j+2) = 2(2j-1)$

l $4(\ell-1) + 3(\ell+2) = 8$

n $\frac{2n-1}{3} = \frac{4n+1}{5}$

p $\frac{2r+1}{3} + 2 = \frac{3r-1}{4}$

5E Using linear equations to solve problems



Problem solving often involves introducing algebra, translating the problem into an equation and then solving the equation. An important first step is to introduce an appropriate pronumeral for one of the unknown quantities.

Example 14

Three children earn weekly pocket money. Andrew earns \$2 more than Gina, and Katya earns twice the amount Gina earns. The total of the weekly pocket money is \$22.

- a How much money does Gina earn?
- b How much money do Andrew and Katya earn?

Solution

- a Let $\$m$ be the amount of pocket money Gina earns in a week.
Then Andrew earns $\$(m + 2)$ and Katya earns $\$(2m)$,
so $m + (m + 2) + 2m = 22$ (The total weekly pocket money is \$22.)
$$4m + 2 = 22$$
$$4m = 20$$
$$m = 5$$

So Gina earns \$5 per week.
- b Andrew earns \$7 and Katya earns \$10 per week.

Example 15

Ali and Jasmine each have a number of swap cards. Jasmine has 25 more cards than Ali, and in total the two children have 149 cards.

- a How many cards does Ali have?
- b How many cards does Jasmine have?

Solution

- a Let x be the number of cards Ali has.
Jasmine has $(x + 25)$ cards.
Total number of cards is 149,
so $x + (x + 25) = 149$
$$2x + 25 = 149$$
$$2x = 124$$
$$x = 62$$

So Ali has 62 cards.
- b Jasmine has $62 + 25 = 87$ cards.

Harder examples involving rates

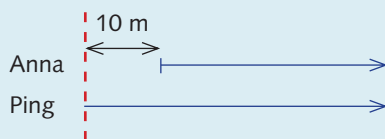
Example 16

Ping and Anna compete in a handicap sprint race. Anna starts the race 10 m ahead of Ping. Ping runs at an average speed that is 20% faster than Anna's average speed. The two sprinters will be level in the race after 9 seconds. Find the average speed of:

- a** Anna **b** Ping.

Solution

- a** The diagram below represents the situation.



Let x be Anna's average speed, measured in m/s.

$$\begin{aligned}\text{Ping's average speed is } 120\% \text{ of } x &= \frac{120x}{100} \\ &= \frac{6x}{5} \text{ m/s}\end{aligned}$$

We use distance = average speed $\times$ time taken

After 9 seconds, Anna has run a distance of $9x$ m and Ping has run

$$\text{a distance of } \frac{6x}{5} \times 9 = \frac{54x}{5} \text{ m.}$$

Anna started 10 m in front of Ping. So when they are level,

$$\begin{aligned}9x + 10 &= \frac{54x}{5} \\ 45x + 50 &= 54x \\ 50 &= 9x \\ x &= \frac{50}{9} \\ x &= 5\frac{5}{9}\end{aligned}$$

So Anna runs at an average speed of $5\frac{5}{9}$ m/s.

b Ping runs at $\frac{6x}{5} = \frac{6}{5} \times \frac{50}{9} = 6\frac{2}{3}$ m/s.

Example 17

For a training run, a triathlete covers 50 km in $4\frac{1}{4}$ hours. She runs part of the way at a speed of 10 km/h, cycles part of the way at a speed of 40 km/h and swims the remaining distance at a speed of $2\frac{1}{2}$ km/h. The athlete runs for twice the time it takes to complete the cycle leg. How long did she take to complete the cycle leg?

Solution

In problems like this, we use the relationship

$$\text{average speed} = \frac{\text{distance}}{\text{time taken}}$$

or distance travelled = average speed $\times$ time taken.

Let t hours be the time for the cycle leg.

Then $2t$ hours is the time for the running leg and $(4\frac{1}{4} - t - 2t)$ hours is the time for the swim leg.

Now, distance of run + distance of cycle + distance of swim = total distance,

$$\text{so } 10 \times 2t + 40 \times t + 2\frac{1}{2} \times \left(4\frac{1}{4} - t - 2t\right) = 50$$

$$20t + 40t + \frac{5}{2} \left(\frac{17}{4} - 3t\right) = 50$$

$$60t + \frac{85}{8} - \frac{15t}{2} = 50$$

$$480t + 85 - 60t = 400$$

$$420t = 315$$

$$t = \frac{315}{420}$$

$$t = \frac{3}{4}$$

The athlete takes $\frac{3}{4}$ hour, or 45 minutes, to complete the cycle leg.



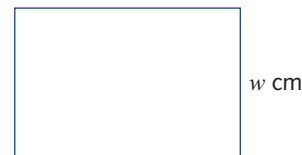
Exercise 5E

In each of the following, form an equation and solve.

- 1 Jacques thinks of a number x . When he adds 17 to his number, the result is 32. What is the value of x ?

- 2 When 16 is added to twice Simone's age, the answer is 44. How old is Simone?
- 3 When 14 is added to half of Suzette's weight in kilograms, the result is 42. How much does Suzette weigh?
- 4 Yolán buys 8 pens and receives 80 cents change from \$20.00. How much does a pen cost, assuming each pen costs the same amount?
- 5 If the sum of $2p$ and 19 is the same as the sum of $4p$ and 11, find the value of p .
- 6 If the sum of half of q and 6 is equal to the sum of one third of q and 2, find the value of q .
- 7 Derek is presently 20 years older than his daughter, Alana.
 - a If x represents Alana's present age, express each of the following in terms of x .
 - i Derek's present age
 - ii Alana's age in 12 years time
 - iii Derek's age in 12 years time
 - b If Derek's age 12 years from now is twice Alana's age 12 years from now, find their present ages.
- 8 Alan, Brendan and Calum each have a number of plastic toys from a fast food store. Brendan has 5 more toys than Alan, and Calum has twice as many toys as Alan.
 - a If x represents the number of toys Alan has, express each of the following in terms of x :
 - i the number of toys Brendan has
 - ii the number of toys Calum has
 - iii the total number of toys the three boys have
 - b If the boys have 37 toys in total, determine how many toys each boy has.
- 9 The length of a swimming pool is 2 m more than four times its width.
 - a If x metres represents the width of the pool, express the length of the pool in terms of x .
 - b If the perimeter of the pool is 124 m, find the length and width of the pool.
- 10 Ms Minas earns \$3600 more than Mr Brown, and Ms Lee earns \$2000 less than Mr Brown.
 - a If $\$x$ represents Mr Brown's salary, express the salary of:
 - i Ms Minas in terms of x
 - ii Ms Lee in terms of x .
 - b If the total of the three incomes is \$151 600, find the income of each person.
- 11 Dudley has a number of 10-cent and 20-cent coins. If their total value is \$6, how many are there of each coin if there are:
 - a equal numbers of each coin
 - b twice as many 10-cents coins as there are 20-cents coins
 - c twice as many 20-cents coins as there are 10-cents coins?

- 12** The distance between two towns, A and B , is 350 km. Find x if:
- a** the trip takes x hours at an average speed of 80 km/h
 - b** the trip takes $3\frac{1}{2}$ hours at an average speed of x km/h
 - c** the trip takes 3 hours, travelling x hours at 120 km/h and the remaining time at 60 km/h.
- 13** A classroom has a length 6 m shorter than its width and a perimeter of 30 m. Find the length and width of the classroom.
- 14** Of the total amount of money in his wallet, a man gave one-third to his daughter and one-quarter to his son. After this he only had \$250 left. How much was originally in his wallet?
- 15** A student has an average mark of 68 from 10 tests. What mark must be gained in the next test to raise his average to 70?
- 16** Luisa travels 25 km in $3\frac{3}{4}$ hours. She walks part of the way at 4 km/h and cycles the rest at 12 km/h. How far did she walk?
- 17** A man is four times as old as his son. In five years he will be only three times as old as his son. What is the man's present age?
- 18** A bottle of cordial contains 5 litres, of which 10% is pure fruit juice. How many litres of water must be added to dilute it to 4% fruit juice?
- 19** A coffee blender mixes two types of coffee. Type A costs \$13 per kg and type B costs \$18 per kg. How many grams of each type of coffee should be blended so that 1 kg costs \$15?
- 20** A collection of coins consisting of 10-cent, 20-cent and 50-cent pieces has a value of \$4.50. The number of 20-cent pieces is twice the number of 10-cent pieces and the number of 50-cent pieces is 3 less than twice the number of 10-cent pieces. How many 10-cent, 20-cent and 50-cent pieces are there in the collection?
- 21** If 4 litres of laboratory alcohol is 80% pure (80% alcohol, 20% water), how many litres of water must be added so that the resulting mixture will contain 30% alcohol?
- 22** A rectangular garden pond has a length 60 cm more than its width. Let the width of the pond be w cm.
- a** Find the length of the pond in terms of w .
 - b** If the length of the pond is 150 cm, what are the dimensions of the pond?
 - c** If the perimeter of the pond is 520 cm, what are the dimensions of the pond?



The pond has a 50 cm wide path around its perimeter.

d Express, in terms of w , the:

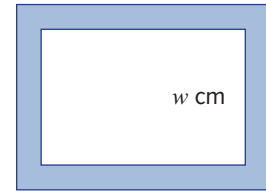
- i** length **ii** width

of the outer rectangle formed by the path.

e Express, in terms of w , the area of the path.

f If the width of the pond is 120 cm, find the area of the path.

g If the area of the path is 5.6 m^2 , find the width of the pond.



- 23** A father is concerned about his daughter's progress in mathematics. In order to encourage her, he agrees to give her 10 cents for every problem she solves correctly and to penalise her 15 cents for every problem she gets wrong. The girl completed 22 problems for homework.
- a** If the girl got 10 problems correct, how many problems did she get incorrect?
- b** If the girl got x problems correct, how many problems did she get incorrect?
- c** How much money does the father have to give his daughter for getting x questions correct?
- d** How much does the daughter have to pay in penalties for the incorrect questions?
- e** Using the fact that the girl made a profit of 20 cents, write down an equation involving x .
- f** How many questions did the girl get correct?
- 24** During a power failure, Sonia lights two identical candles of length 10 cm at 7 p.m. One candle burns out by 11 p.m. and the other candle burns out at 12 midnight. Assume that the length of each candle goes down by a constant rate.
- a** How high is each candle at 8 p.m.?
- b** How high is each candle at 9 p.m.?
- c** How high is each candle t hours after 7 p.m.?
- d** At what time is one candle twice as high as the other candle?

5F Literal equations

In this chapter, we have discussed methods for solving linear equations.

Sometimes equations arise in which some of the coefficients are pronumerals. These are called **literal equations**. We need to be told which pronumeral we are solving for.

Example 18

Solve:

a $ax + b = c$ for x

b $a(x + b) = c$ for x

Solution

a $ax + b = c$

$$ax = c - b \quad (\text{Subtract } b \text{ from both sides.})$$

$$x = \frac{c - b}{a} \quad (\text{Divide both sides by } a.)$$

b $a(x + b) = c$

$$ax + ab = c$$

$$ax = c - ab$$

$$x = \frac{c - ab}{a}$$

Example 19

Solve $mx - n = nx + m$ for x .

Solution

$$mx - n = nx + m$$

$$mx - nx - n = m \quad (\text{Subtract } nx \text{ from both sides.})$$

$$mx - nx = m + n \quad (\text{Add } n \text{ to both sides.})$$

$$x(m - n) = m + n \quad (\text{Factorise the left-hand side.})$$

$$x = \frac{m + n}{m - n} \quad (\text{Divide both sides by } m - n.)$$



Exercise 5F

Example 18

1 Solve each of these equations for x .

a $x + b = c$

c $p - x = q$

e $cx = b$

g $a(x + b) = c$

i $\frac{x}{a} = b$

k $\frac{x}{a} + b = c$

m $\frac{ax}{b} + c = d$

o $\frac{mx - n}{n} = m$

q $\frac{x}{a} - \frac{a}{b} = b$

b $x - d = e$

d $-x + m = n$

f $c - bx = e$

h $m(nx + p) = n$

j $\frac{x + a}{b} = c$

l $\frac{mx}{n} = p$

n $\frac{ax + b}{c} = d$

p $\frac{x}{f} + \frac{g}{h} = k$

r $\frac{x}{b} - b = \frac{a}{b}$

Example 19

2 Solve each of these equations for x .

a $ax + b = cx + d$

c $a(x + b) = cx + d$

b $mx + n = nx - m$

d $a(x - b) = c(x - d)$

5G Inequalities



The numbers -6 , -2 , 3 and 5 are shown on a number line.



Using the 'greater than' sign, we write $5 > 3$ because 5 is to the right of 3 . Similarly, $3 > -2$ and $-2 > -6$.

We can also use the 'less than' sign and write $3 < 5$, $-2 < 3$ and $-6 < -2$. Notice that every negative number is less than 0 . For example, $-6 < 0$.

Statements such as $-6 < -2$ and $5 > 3$ are called **inequalities**.

The symbols $\leq$ and $\geq$

The symbol $\leq$ means 'is less than or equal to'.

The symbol $\geq$ means 'is greater than or equal to'.

The statement ' $4 \leq 6$ ' means that either ' $4 < 6$ ' or ' $4 = 6$ ' is true. It is correct because the statement ' $4 < 6$ ' is true. It does not matter that the second statement, ' $4 = 6$ ', is false.

It is also correct to say that $3 \geq 3$, because the statement ' $3 = 3$ ' is true. Once again, it does not matter that the statement ' $3 > 3$ ' is false.

Statements such as ' $4 \leq 6$ ' and ' $3 \geq 3$ ' are also called **inequalities**.

Sets and intervals

We will need ways of describing and graphing various collections of numbers.

An example is the collection of all numbers that are less than 2 .

On the number line, we graph this in the following way.



The open dot, $\circ$, is used to indicate that 2 is not in the collection. The collection we are interested in is the interval that begins at 2 (but does not include 2) and extends indefinitely to the left, as indicated. The arrow head means 'goes on indefinitely'.

Another example is the collection of all numbers that are greater than or equal to -1 .

On the number line, we graph this in the following way.



Here, the filled dot indicates that -1 is included in the interval. The interval begins at and includes -1 , and extends indefinitely to the right.

The word ‘set’ means the same as ‘collection’. Instead of spelling out the long description ‘the set of numbers x that are less than 2’, we write

$$\{x: x < 2\}.$$

In a similar way, instead of the long description ‘the set of all numbers x that are greater than or equal to -1 ’, we write

$$\{x: x \geq -1\}.$$

Example 20

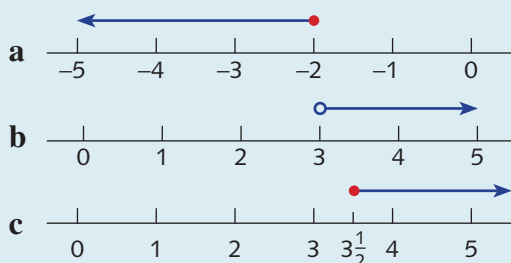
Graph each set on the number line.

a $\{x: x \leq -2\}$

b $\{x: x > 3\}$

c $\{x: x \geq 3\frac{1}{2}\}$

Solution



Exercise 5G

1 Copy and insert $>$ or $<$ to make each statement true.

a $7 \dots 2$

b $3 \dots -4$

c $-4 \dots -2$

d $-54 \dots -500$

e $-6 \dots 0$

f $-13 \dots -45$

g $21 \dots 40$

h $-2 \dots 5$

i $99 \dots -100$

2 Copy and insert $\leq$ or $\geq$ to make each statement true.

a $-7 \dots -2$

b $5 \dots -7$

c $-5 \dots -5$

d $-10 \dots -50$

e $0 \dots 0$

f $-23 \dots -45$

g $12 \dots 26$

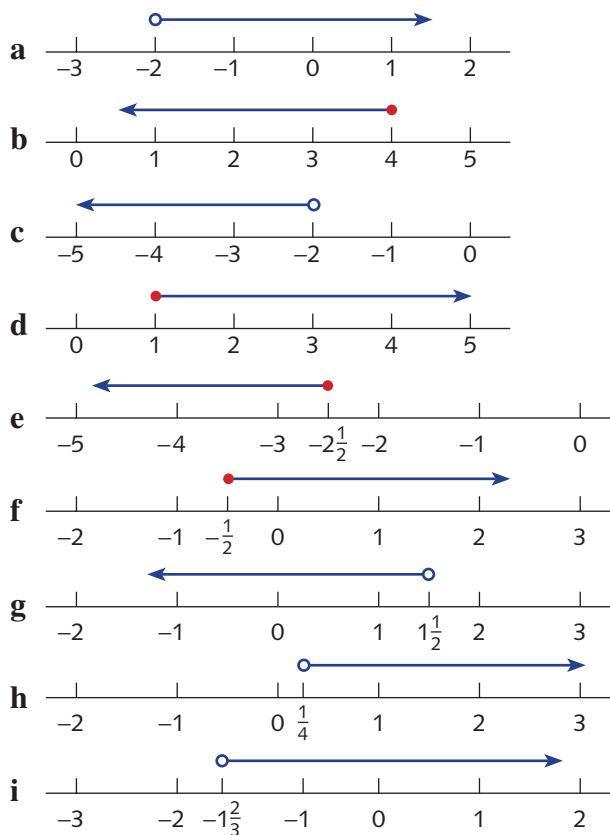
h $9 \dots 9$

i $98 \dots 89$

3 Graph each set on the number line.

- a** $\{x: x > 10\}$ **b** $\{x: x \leq 0\}$ **c** $\{x: x < -4\}$
d $\{x: x > -3\}$ **e** $\{x: x \leq -2\}$ **f** $\{x: x \geq -2\}$
g $\{x: x < 2\frac{1}{2}\}$ **h** $\{x: x \geq -4\frac{1}{2}\}$ **i** $\{x: x \leq -1\frac{1}{2}\}$

4 Use set notation to describe each interval.



5H Solving linear inequalities

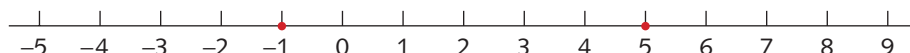


We first look at the effects of addition, subtraction, multiplication and division on inequalities.

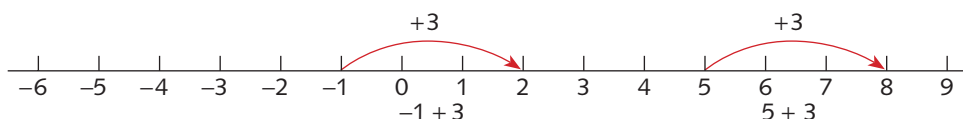
Addition and subtraction

We know that $-1 < 5$.

On the number line:

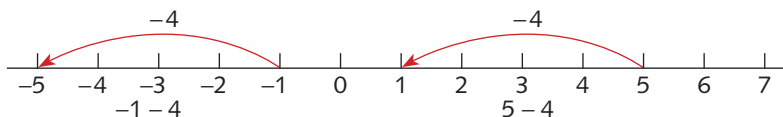


If we add 3 to each number, each moves 3 to the right.



$$-1 + 3 < 5 + 3$$

If we subtract 4 from each number, each moves 4 to the left.



$$-1 - 4 < 5 - 4$$

So we see that in each case the inequality still holds.



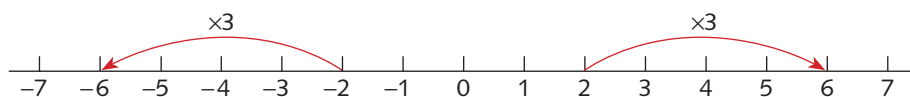
Addition and subtraction

- If we add the same number to both sides of an inequality, then the resulting inequality is true.
- If we subtract the same number from both sides of an inequality, then the resulting inequality is true.

Multiplication and division

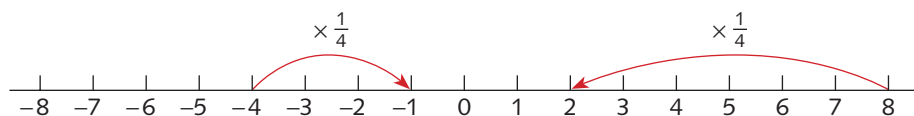
Multiplication by a positive number

If we multiply any number by 3, it moves to 3 times the distance it was from 0 and stays on the same side of 0. The effect of multiplying both sides of an inequality by 3 is shown in a diagram.



$$-2 < 2 \quad \text{so} \quad 3 \times -2 < 3 \times 2$$

If we multiply by $\frac{1}{4}$, then the new number is $\frac{1}{4}$ of the distance from 0, but still on the same side of 0. The effect on an inequality can be seen in a diagram.

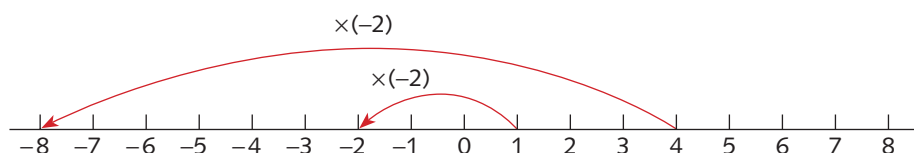


$$-4 < 8 \quad \text{so} \quad \frac{1}{4} \times -4 < \frac{1}{4} \times 8$$

So the inequality remains true if we multiply both sides by a positive number.

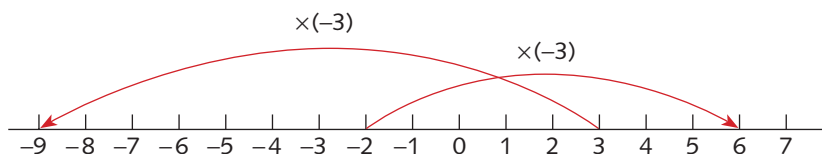
Multiplication by a negative number

We know that $1 < 4$. If we multiply both numbers by -2 , they move to the opposite side of 0, and are twice as far from 0 as they were before.



$$1 < 4 \quad \text{but} \quad -2 \times 1 > -2 \times 4$$

Similarly,



$$3 > -2 \quad \text{but} \quad -3 \times 3 < -3 \times (-2)$$

That is, in both cases the inequality sign is *reversed*.

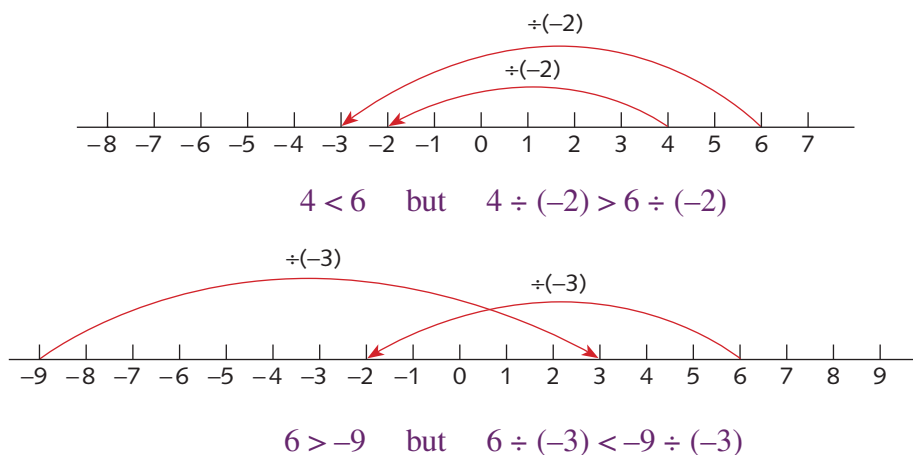


Multiplication

- If we multiply both sides of an inequality by a positive number, then the resulting inequality is true.
- If we multiply both sides of an inequality by a negative number, then we must reverse the inequality sign to make the resulting inequality true.

Division by a negative number

These diagrams explain what happens to an inequality when we divide both sides by a negative number.



Division

- If we divide both sides of an inequality by a positive number, then the resulting inequality is true.
- If we divide both sides of an inequality by a negative number, then we must reverse the inequality sign to make the resulting inequality true.

Linear inequalities

Solving the linear inequality $4x - 5 < 3$ means finding all of the values for x that satisfy that inequality.

It is easy to find some numbers that satisfy the inequality. For example, if $x = -2$,

$$4x - 5 = -13 < 3.$$

Similarly, if $x = 0$,

$$4x - 5 = -5 < 3.$$

However, $x = 4$ does not satisfy the inequality, because when $x = 4$,

$$4x - 5 = 11 > 3.$$

So $x = -2$ and $x = 0$ satisfy the inequality, but $x = 4$ does not.

There is a systematic way of solving linear inequalities, as shown in the following examples. We often graph the solution set on the number line.

The method for solving linear inequalities is the same as that for solving linear equations, except for multiplying and dividing both sides by a negative number.

Example 21

- a** Solve the inequality $4x - 5 < 3$.
b Graph the solution set on the number line.

Solution

- a** If x satisfies the inequality

$$4x - 5 < 3$$

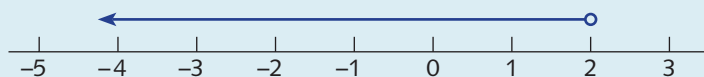
then $4x < 8$ (Add 5 to both sides.)

so $x < 2$ (Divide both sides by 4.)

Conversely, if $x < 2$, then $4x < 8$ and $4x - 5 < 8 - 5 = 3$.

The solution is $\{x: x < 2\}$.

- b** The solution set is indicated on the number line.



Example 22

Solve each of the following inequalities:

a $-2x \leq 6$

b $-\frac{x}{3} > 4$

Solution

a $-2x \leq 6$

$x \geq -3$ (Divide both sides by -2 and *reverse* the inequality sign.)

b $-\frac{x}{3} > 4$

$x < -12$ (Multiply both sides by -3 and *reverse* the inequality sign.)

Just as in solving linear equations, all of the steps we do in solving linear inequalities are reversible.

This means that any number x that satisfies the original inequality also satisfies the subsequent ones; and conversely, any number that satisfies a subsequent inequality will also satisfy the original one. We say that the inequalities are **equivalent**.

Equivalent inequalities have exactly the same solutions.

Two standard methods of setting out are shown in Examples 23 and 24.

Example 23

- a** Solve the inequality $-2x + 3 \leq 4$.
b Graph the solution set on the number line.

Solution

a Method 1

$$-2x + 3 \leq 4$$

$$-2x \leq 1 \quad (\text{Subtract 3 from both sides.})$$

$$x \geq -\frac{1}{2} \quad (\text{Divide by } -2 \text{ and reverse the inequality sign.})$$

Method 2

$$-2x + 3 \leq 4$$

$$3 \leq 4 + 2x \quad (\text{Add } 2x \text{ to both sides.})$$

$$3 - 4 \leq 2x \quad (\text{Subtract 4 from both sides.})$$

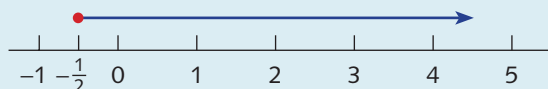
$$-1 \leq 2x$$

$$-\frac{1}{2} \leq x \quad (\text{Divide both sides by 2.})$$

$$\text{so } x \geq -\frac{1}{2}$$

The solution set is $\left\{x: x \geq -\frac{1}{2}\right\}$.

- b** The solution set is as shown.



Example 24

- a** Solve the inequality $2x + 3 < 3x - 4$.
b Graph the solution set on the number line.

Solution

a Method 1

$$2x + 3 < 3x - 4$$

$$3 < x - 4 \quad (\text{Subtract } 2x \text{ from both sides.})$$

$$7 < x \quad (\text{Add 4 to both sides.})$$

$$\text{so } x > 7$$

Method 2

$$2x + 3 < 3x - 4$$

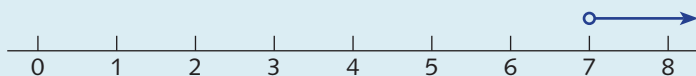
$$2x < 3x - 7 \quad (\text{Subtract 3 from both sides.})$$

$$-x < -7 \quad (\text{Subtract } 3x \text{ from both sides.})$$

$$x > 7 \quad (\text{Multiply by } -1 \text{ and reverse the inequality sign.})$$

The solution set is $\{x : x > 7\}$.

b The solution set is as shown:



Solving linear inequalities

- Multiply out brackets and multiply through by the lowest common denominator of fractions.
- Move all of the terms involving x to one side and all of the constant terms to the other side of the inequality sign.
- Simplify.
- Divide by the coefficient of x , remembering to reverse the inequality sign if the coefficient is negative.



Exercise 5H

- 1**
- If $x < 6$, how many different values can x take?
 - If $x > 9$, how many different values can x take?
 - Give three values of x that satisfy $x > -1$.
 - Give three non-whole numbers that satisfy $x > 2$.
- 2** Solve each of these inequalities. Graph each solution set to parts **a–f** on a number line.

a $x + 3 \geq 7$

b $x - 7 \leq 4$

c $x + 12 > 9$

d $x - 2 < 3$

e $x + 4 \geq -8$

f $x - 10 > -6$

g $x - 5 > -12$

h $2x \geq 6$

i $3x > -15$

j $\frac{x}{3} < 7$

k $\frac{x}{5} \geq 4$

l $\frac{x}{2} \leq -3$

Example 21

3 Solve:

a $2x + 1 \geq 5$

b $3x - 4 \leq 8$

c $4x - 6 \leq -2$

d $\frac{2x}{3} > 5$

e $\frac{3x}{5} \geq 2$

f $\frac{4x}{7} \leq -2$

g $3(x + 5) \geq 9$

h $2(5x - 2) > 5$

i $\frac{2x}{7} \leq 3$

j $\frac{x}{3} - \frac{1}{2} \geq 1$

k $\frac{2x}{5} + \frac{1}{4} > 4$

l $2(x - 3) \leq 5$

Example 22

4 Solve:

a $-4x \leq 20$

b $-10x \geq 130$

c $-12x > -42$

d $-\frac{x}{2} \leq 5$

e $-\frac{x}{7} \geq 4$

f $-\frac{x}{5} > 4$

g $-\frac{x}{12} < 4$

h $-\frac{x}{2} \geq -8$

i $2 - \frac{x}{9} > 5$

Example 23

5 Solve:

a $3 - 2x > 5$

b $2 - 5x \leq -8$

c $4(7 - x) < 5$

d $6 - \frac{2x}{3} < 4$

e $4 - \frac{2x}{5} \geq 6$

f $8 - \frac{3x}{7} \geq 2$

Example 24

6 Solve:

a $\frac{x + 3}{2} \leq \frac{3 - x}{2}$

b $\frac{2x - 1}{3} - \frac{3x + 2}{4} > 3$

7 Solve:

a $1.2x + 6.8 \leq 15.2$

b $3.6x - 1.8 > 28.8$

c $5.6 - 2.7x > 21.8$

d $2.4 - 0.7x \leq 12.9$

e $1.6(x + 7) \leq 1.5(x - 3)$

f $2.8(x - 4) > 1.3(x + 3.5)$

8 Solve:

a $2x - 14 \leq 8$

b $-5x + 3 \geq 78$

c $\frac{2x + 1}{6} > -3$

d $-\frac{x + 2}{3} \leq 7$

e $\frac{x - 8}{2} - \frac{2x}{3} \geq 3$

f $\frac{x}{4} > -\frac{x + 12}{5}$

- 9** When 5 is added to twice p , the result is greater than 17. What values can p take?
- 10** When 16 is subtracted from half of q , the result is less than 18. What values can q take?
- 11** When $2p$ is subtracted from 10, the result is greater than or equal to 4. What values can p take?
- 12** The sum of $4d$ and 6 is greater than the sum of $2d$ and 18. What values can d take?
- 13** A number a is increased by 3 and this amount is then doubled. If the result of this is greater than a , what values can a take?
- 14** Two car hire firms offer the following deals.
Movit: \$25 plus 6 cents per km
Rentlow: \$20 plus 8 cents per km
- a** If a client is to drive x km, express the cost of hiring a car from:
- i** Movit, in terms of x **ii** Rentlow, in terms of x .
- b** For what distances does Movit offer the better deal?



Review exercise

- 1 David weighs 5 kg less than Christos. If Christos weighs w kg, express David's weight in terms of w .
- 2 When John's age is doubled, the number is 5 more than Kayla's age. If John is x years old, what is Kayla's age?
- 3 The length of a rectangle is 10 m greater than the width of the rectangle.
 - a If w m is the width of the rectangle, express the length of the rectangle in terms of w .
 - b If the length of the rectangle is ℓ m, express the width of the rectangle in terms of ℓ .
- 4 Solve these equations.

a $a + 7 = 5$	b $b + 11 = 8$	c $-3m = 18$
d $-2n = 16$	e $3p = -24$	f $4q = -27$
g $6a + 21 = -1$	h $5a + 22 = -9$	i $7a - 26 = -35$
j $7b - 27 = -16$	k $-6 - 8e = 24$	l $-5 - 7f = 24$
m $5x - 4 = 11 - x$	n $18 - 7x = 3x - 15$	o $4x - 2 = 7 - 2x$
p $5x + 13 = 14 - 6x$	q $3 - 2x = 7 - 4x$	r $-5p + 8 = 6 - 7p$
- 5 Solve these equations.

a $5(x + 3) = 18$	b $5(x - 2) = 15$
c $12(x - 1) = 96$	d $7(x + 1) = 10$
e $7(x + 3) = 15$	f $5(x - 22) = 26$
g $7(3 - x) = 20$	h $6(11 - x) = 17$
i $6(a + 1) - 4(a + 2) = 24$	j $5(b - 1) - 8(b - 2) = 30$
k $5(a + 3) = 7(a + 11)$	l $15(3a - 1) = 2(6a + 7)$
m $-3(x - 2) = 2(3x - 1)$	n $-4(5m + 6) = -3(4 - 5m)$
- 6 Solve:

a $5 - 1.4x = 13.68$	b $9 - 2.6a = 29.8$	c $11 - 3.6c = 3.8$
d $12.6 - 4.5\ell = -5.4$	e $1.6(x + 7) = 17.6$	f $2.8(x - 4) = 16.8$
g $3.4(x - 5) = 8.5$	h $4.3(x + 11) = 53.75$	i $3.5(8 - x) = 29.5$



7 Solve these equations.

a $5x + 9 = 35$

b $3x - 11 = 41$

c $\frac{x}{3} - 2 = -7$

d $\frac{x}{2} + 6 = -3$

e $\frac{5p-2}{4} + \frac{1}{2} = -2$

f $\frac{5p}{4} - 1 = \frac{3}{7}$

g $3x + 11 = 7x - 23$

h $4x - 2 = 15 - 3x$

i $-2(z - 1) = 5(z + 6)$

j $5(y + 2) = 2(2y - 1)$

k $12(k + 1) - 3(k - 2) = -7$

l $4(\ell - 1) - 3(\ell + 2) = 18$

m $\frac{m+1}{7} = \frac{2m-2}{7}$

n $\frac{2n-1}{3} = \frac{5n+1}{7}$

o $\frac{2y+1}{3} + 7 = 11$

p $\frac{7-4x}{4} - 1 = 6$

q $\frac{3s-2}{5} + 1 = 6s$

r $\frac{2x+1}{3} + 7 = \frac{3x-1}{4}$

s $\frac{2a+5}{6} + a = 4$

t $\frac{x+2}{5} + \frac{x-1}{2} = \frac{x+1}{3}$

8 The length of a rectangular lawn is 15 m more than four times its width.

a If x m is the width of the lawn, express the length of the lawn in terms of x .

b If the perimeter of the lawn is 265 m, find the length and width of the lawn.

9 Mr Guernsey earns \$14 800 more than Mr Jersey, and Mrs Mann earns \$22 000 less than Mr Jersey.

a If $\$x$ represents Mr Jersey's salary, express the salary of:

i Mr Guernsey in terms of x

ii Mrs Mann in terms of x .

b If the total of the three incomes is \$351 600, find the income of each person.

10 A person has a number of 10-cent and 20-cent coins. If their total value is \$18, how many are there of each coin if there are:

a equal numbers of each coin

b twice as many 10-cent coins as there are 20-cent coins

c twice as many 20-cent coins as there are 10-cent coins?

11 The distance between two towns A and B is 450 km. Find x if:

a the trip takes x hours at an average speed of 90 km/h

b the trip takes $5\frac{1}{4}$ hours at an average speed of x km/h

c the trip takes 5 hours, travelling x hours at 110 km/h and the remaining time at 60 km/h.



12 A room has a length 8 m shorter than its width and a perimeter of 80 m. Find the length and width of the room.

13 Solve these equations for x .

a $\frac{mx+n}{a} = b$ b $m(x-n) = p$ c $a(bx-c) = d$

d $mx-n = px+q$ e $\frac{x}{a} + \frac{b}{c} = d$ f $\frac{m(x+n)}{a} = b$

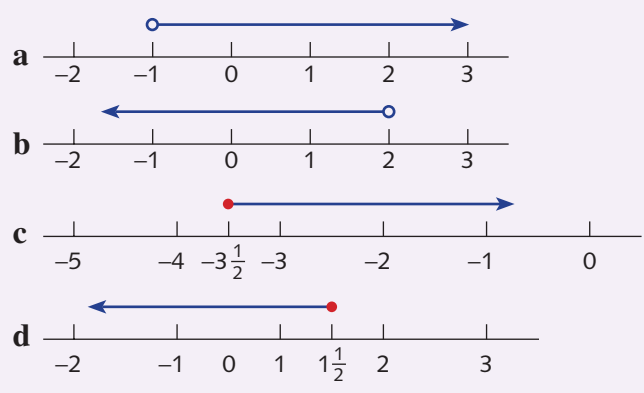
14 Copy each statement and insert the correct symbol, $\geq$ or $\leq$.

- a $11 \dots 5$ b $-8 \dots -12$ c $4 \dots 15$
d $-14 \dots -16$ e $-2 \dots -6$ f $-8 \dots -7$
g $11 \dots -5$ h $-11 \dots 5$ i $-10 \dots -100$

15 Graph each set on the number line.

- a $\{x: x > 2\}$ b $\{x: x < 3\}$ c $\{x: x < -1\frac{1}{2}\}$
d $\{x: x \geq 4\}$ e $\{x: x \geq \frac{1}{3}\}$ f $\{x: x \leq 2\frac{1}{4}\}$

16 Use set notation to describe each interval.



17 Solve these inequalities.

- a $x+7 \geq 17$ b $x-7 \leq -4$
c $-2x \geq 16$ d $\frac{x}{3} < -5$
e $5x+3 \geq 13$ f $3x-11 \leq 8$
g $\frac{x}{5} - \frac{1}{2} \geq 2$ h $\frac{2x}{11} + \frac{3}{5} > 4$
i $6(x-3) \leq 15$ j $3(x+11) \geq 27$
k $7(15x-12) > 25$ l $4(7-x) < 36$
m $7-4x \leq -12$ n $11-3x > 18$



o $\frac{2x}{18} \leq 6$

q $10(x+7) \leq 15(x-3)$

s $3(5-x) > 8(x-3.2)$

p $15 - \frac{x}{18} < 14$

r $2(x-4) > 8(x+3.5)$

t $6(1-x) \leq 2.4(15-12x)$

18 Solve these inequalities.

a $\frac{x-3}{3} \leq \frac{3+x}{2}$

c $\frac{3x+1}{4} \geq \frac{x+5}{2}$

e $\frac{3-x}{5} < \frac{6-x}{4}$

g $\frac{7+x}{2} + \frac{x-7}{3} \leq 4$

i $\frac{2x-1}{7} \geq \frac{2x+3}{4}$

k $\frac{x+5}{4} + \frac{2x-3}{5} > 1$

b $\frac{2x+3}{2} - \frac{x-4}{3} > 2$

d $\frac{4-x}{2} + \frac{3-x}{4} < 1$

f $\frac{3-2x}{2} \geq \frac{7-10x}{5}$

h $\frac{5-2x}{3} - \frac{5+2x}{4} \geq -1$

j $\frac{6x+1}{3} > \frac{3x-1}{2} + 3$

l $\frac{6-3x}{2} \geq \frac{3x-6}{5}$



Challenge exercise

- 1 At 2 p.m., two aeroplanes leave airports 2880 km apart and fly towards one another. The average speed of one plane is twice that of the other. If they pass each other at 5 p.m., what is the average speed of each plane?
- 2 When a mathematics teacher was asked her age she replied, 'One-fifth of my age three years ago, when added to half my age last year, gives my age eleven years ago.' How old is she?
- 3 \$420 is divided between A , B and C . B receives \$20 less than A , and C receives half as much as A and B together. How much does each receive?
- 4 In a printing works, 75 000 leaflets are run off by two printing presses in $18\frac{3}{4}$ hours. One press delivers 200 more leaflets per hour than the other. Find the number of leaflets produced per hour by each of the machines.
- 5 Ten cm^3 of silver and 5 cm^3 of copper weigh 150 g. One cubic centimetre of silver weighs $1\frac{1}{6}$ times as much as 1 cm^3 of copper. Find the weight of 1 cm^3 of each of the metals.
- 6 A river flows at 5 km/h. A boat goes upstream half as fast as downstream. What is the speed of the boat in still water?
- 7 Twenty litres of milk with unknown butter-fat content from Vat A is mixed with 10 litres of milk with 3% butter-fat from Vat B to produce milk with $3\frac{3}{4}\%$ butter-fat content. Find the percentage of butter-fat in Vat A.
- 8 The ends of a bar have to be machined down until the bar is 3 m long. After 5% of the length had been removed, it was found that 0.5% of the new length still had to come off. What length was removed at each machining and what was the original length of the bar?
- 9 A salesman makes a trip to visit a client. The traffic he encounters keeps his average speed to 40 km/h. On the return trip, he takes a route 6 km longer, but he averages 50 km/h. If he takes the same time each way, how long was the total journey?



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CHAPTER

6

Number and Algebra

Formulas

The area, A square units, of a circle with radius r units is given by $A = \pi r^2$.

The volume, V cubic units, of a cylinder with radius r units and height h units is given by $V = \pi r^2 h$.

Einstein discovered the remarkable formula $E = mc^2$, where:

E = energy

m = mass

and c = the speed of light.

These are examples of **formulas**.



A formula relates different quantities. For instance, the formula $A = \pi r^2$ relates the radius, r units, of a circle to the area, A square units, of the circle. That is, it describes how the area of a circle depends on its radius.

Formulas with a subject

In many formulas that you see there is a single pronumeral on the left-hand side of the equal sign. This pronumeral is called the **subject** of the formula.

For instance, in the formula $E = mc^2$, E is the subject.

Substitution

If the values of all the pronumerals except the subject of a formula are known, then we can find the value of the subject by substitution.

Example 1

Find the value of the subject when the pronumerals in the formula have the values indicated.

a $F = ma$, where $a = 10$, $m = 3.5$

b $m = \frac{a+b}{2}$, where $a = 12$, $b = 26$

Solution

a $F = ma$
 $= 10 \times 3.5$
 $= 35$

b $m = \frac{a+b}{2}$
 $= \frac{12+26}{2}$
 $= 19$

Example 2

The formula for the circumference C of a circle of radius r is $C = 2\pi r$. Find the value of C when $r = 20$:

a in terms of π

b correct to 2 decimal places.

Solution

$$\begin{aligned}\mathbf{a} \quad C &= 2\pi r \\ &= 40\pi\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad C &= 40\pi \\ &= 125.663 \dots && \text{(using a calculator)} \\ &\approx 125.66 && \text{(correct to 2 decimal places)}\end{aligned}$$

Example 3

- a** The area of a triangle $A \text{ cm}^2$ is given by $A = \frac{1}{2}bh$, where $b \text{ cm}$ is the base length and $h \text{ cm}$ is the height. Calculate the area of a triangle with base length 16 cm and height 11 cm.
- b** The simple interest payable when $\$P$ is invested at a rate of $r\%$ per year for t years is given by $I = \frac{Prt}{100}$. Calculate the simple interest payable when $\$1000$ is invested at 3.5% per year for 6 years.

Solution

$$\begin{aligned}\mathbf{a} \quad A &= \frac{1}{2}bh \\ &= \frac{1}{2} \times 16 \times 11 \\ &= 88\end{aligned}$$

The area of the triangle is 88 cm^2 .

$$\begin{aligned}\mathbf{b} \quad I &= \frac{Prt}{100} \\ &= \frac{1000 \times 3.5 \times 6}{100} \\ &= 210\end{aligned}$$

The interest payable is $\$210$.

Substitution into a formula

When the pronumeral whose value is to be found is not the subject of the formula, it is necessary to solve an equation to find the value of this pronumeral. This is shown in the following examples.

Example 4

For a car travelling in a straight line with initial velocity u m/s and acceleration a m/s², the formula for the velocity v m/s at time t seconds is $v = u + at$.

- a** Find u if $a = 2$, $v = 15$ and $t = 7$.
- b** Find a if $v = 10$, $u = 6$ and $t = 3$.
- c** Find t if $v = 20$, $u = 5$ and $a = 3$.

Solution

a $v = u + at$

When $a = 2$, $v = 15$ and $t = 7$.

$$15 = u + 2 \times 7$$

$$15 = u + 14$$

$$u = 1$$

The initial velocity is 1 m/s.

b $v = u + at$

When $v = 10$, $u = 6$ and $t = 3$.

$$10 = 6 + 3a$$

$$4 = 3a$$

$$a = \frac{4}{3}$$

The acceleration is $\frac{4}{3}$ m/s².

c $v = u + at$

When $v = 20$, $u = 5$ and $a = 3$.

$$20 = 5 + 3t$$

$$15 = 3t$$

$$t = 5$$

The time is 5 seconds.

Example 5

The thin lens formula states that

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v},$$

where u is the distance from the object to the lens, v is the distance of the image from the lens and f is the focal length of the lens.

- a** Find f if $u = 2$ and $v = 5$.
- b** Find u if $f = 2$ and $v = 6$.

Solution

a
$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

When $u = 2$ and $v = 5$,

$$\begin{aligned}\frac{1}{f} &= \frac{1}{2} + \frac{1}{5} \\ &= \frac{7}{10}\end{aligned}$$

Taking reciprocals of both sides of the equation gives $f = \frac{10}{7}$.

b
$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

If $f = 2$ and $v = 6$,

$$\begin{aligned}\frac{1}{2} &= \frac{1}{u} + \frac{1}{6} \\ \frac{1}{u} &= \frac{1}{2} - \frac{1}{6} \\ &= \frac{1}{3}.\end{aligned}$$

Taking reciprocals of both sides of the equation gives $u = 3$.

Example 6

The area of a circle $A \text{ cm}^2$ is given by $A = \pi r^2$, where $r \text{ cm}$ is the radius of the circle.
If $A = 20$, find r :

- a** in terms of π
- b** correct to 2 decimal places.

Solution

a $A = \pi r^2$

When $A = 20$,

$$20 = \pi r^2$$

$$\frac{20}{\pi} = r^2 \quad (\text{Divide both sides of equation by } \pi.)$$

$$r = \sqrt{\frac{20}{\pi}} \quad (r \text{ is positive.})$$

b $r \approx 2.52$ (correct to 2 decimal places)

Exercise 6A

Example 1

- 1** For each part, find the value of the subject when the other pronumerals have the value indicated.

a $A = \ell w$, where $\ell = 5$, $w = 8$

b $s = \frac{d}{t}$, where $d = 120$, $t = 6$

c $A = \frac{1}{2}xy$, where $x = 10$, $y = 7$

d $A = \frac{1}{2}(a + b)h$, where $a = 4$, $b = 6$, $h = 10$

e $v = u + at$, where $u = 12$, $a = 10$, $t = 3$

f $E = \frac{1}{2}mv^2$, where $m = 8$, $v = 4$

Examples 2, 3

- 2** For each part, find the value of the subject when the other pronumerals have the value indicated.

a $x = \sqrt{ab}$, where $a = 40$, $b = 50$ (Calculate x correct to 3 decimal places.)

b $V = \pi r^2 h$, where $r = 12$, $h = 20$ (Calculate V correct to 3 decimal places.)

c $t = a + (n - 1)d$, where $a = 30$, $n = 8$, $d = 4$

d $T = 2\pi\sqrt{\frac{\ell}{g}}$, where $\ell = 88.2$, $g = 9.8$ (Calculate T correct to 3 decimal places.)

e $s = ut + \frac{1}{2}at^2$, where $u = 15$, $t = 10$, $a = 5.6$

f $A = P(1 + R)^n$, where $P = 10\,000$, $R = 0.065$, $n = 10$
(Calculate your answer correct to 2 decimal places.)

Example 4

3 For the formula $v = u + at$, find:

a v if $u = 6$, $a = 3$ and $t = 5$

b u if $v = 40$, $a = 5$ and $t = 2$

c a if $v = 60$, $u = 0$ and $t = 5$

d t if $v = 100$, $u = 20$ and $a = 6$

4 a For the formula $S = 2(a - b)$, find a if $S = 60$ and $b = 10$.

b For the formula $I = \frac{180n - 360}{n}$, find n if $I = 120$.

c For the formula $a = \frac{m+n}{2}$, find m if $a = 20$ and $n = 6$.

d For the formula $A = \frac{PRT}{100}$, find P if $A = 1600$, $R = 4$ and $T = 10$.

e For the formula $S = 2(\ell w + \ell h + hw)$, find h if $S = 592$, $\ell = 10$ and $w = 8$.

f For the formula $P = 2(\ell + w)$, find w if $P = 30$ and $\ell = 6$.

g For the formula $s = ut + \frac{1}{2}at^2$, find a if $s = 1000$, $u = 20$ and $t = 5$.

h For the formula $t = a + (n - 1)d$, find n if $t = 58$, $d = 3$ and $a = 7$.

5 Given $v^2 = u^2 + 2ax$ and $v > 0$, find the value of v (correct to 1 decimal place) when:

a $u = 0$, $a = 5$ and $x = 10$

b $u = 6$ and $a = 4$ and $x = 15$

c $u = 2$, $a = 9.8$ and $x = 22$

d $u = 2.3$, $a = 4.9$ and $x = 10.6$

Example 5

6 Given $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$, find the value of f when:

a $u = 2$ and $v = 4$

b $u = 6$ and $v = 9$

c $u = 12$ and $v = 11$

d $u = 2$ and $v = 15$

7 For the formula $s = ut + \frac{1}{2}at^2$, find the value of:

a u , when $s = 10$, $t = 20$ and $a = 2$

b a , when $s = 20$, $u = 5$ and $t = 2$

- 8 Given $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$, find the value of:

a u when $f = 2$ and $v = 4$

b u when $f = 3$ and $v = 4$.

- 9 Given that $P = \frac{M+m}{M-m}$, find the value of P when:

a $M = 8$ and $m = 4$

b $M = 26$ and $m = 17$

c $M = 19.2$ and $m = 5.9$

d $M = \frac{3}{4}$ and $m = \frac{2}{5}$

Example 6

- 10 The area $A \text{ cm}^2$ of a square with side length $x \text{ cm}$ is given by $A = x^2$.
If $A = 20$, find:

a the value of x

b the value of x correct to 2 decimal places.

- 11 For a rectangle of length $\ell \text{ cm}$ and width $w \text{ cm}$, the perimeter $P \text{ cm}$ is given by $P = 2(\ell + w)$.

Use this formula to calculate the length of a rectangle which has width 15 cm and perimeter 57 cm .

- 12 The sum of the first n whole numbers is given by $S = \frac{n}{2}(n + 1)$.

Use this formula to calculate the sum of the first 20 whole numbers.

- 13 The formula for finding the number of degrees Fahrenheit (F) for a temperature given as a number of degrees Celsius (C) is $F = \frac{9}{5}C + 32$.

Fahrenheit temperatures are still used in the USA, but in Australia we commonly use Celsius. Calculate the Fahrenheit temperatures which people in the USA would recognise for:

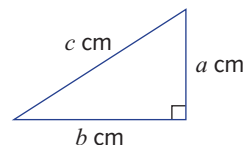
a the freezing point of water, 0°C

b the boiling point of water, 100°C

c a nice summer temperature of 25°C .

- 14 The length of the hypotenuse $c \text{ cm}$ in a right-angled triangle is given by $c^2 = a^2 + b^2$, where $a \text{ cm}$ and $b \text{ cm}$ are the lengths of the perpendicular sides, as in the diagram.

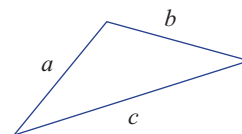
Calculate, correct to the nearest 0.1 cm , the length of the hypotenuse in the right-angled triangle whose perpendicular sides have lengths 16 cm and 20 cm .



- 15** The area $A \text{ cm}^2$ of a triangle with side lengths $a \text{ cm}$, $b \text{ cm}$ and $c \text{ cm}$ is given by Heron's formula

$$A^2 = s(s-a)(s-b)(s-c),$$

where $s = \frac{a+b+c}{2}$ = half the perimeter.



Find the exact areas of the triangles whose side lengths are given below.

- a** 6 cm, 8 cm and 10 cm **b** 5 cm, 12 cm and 13 cm
c 8 cm, 10 cm and 14 cm **d** 13 cm, 14 cm and 15 cm
- 16** Sam throws a stone down to the ground from the top of a cliff s metres high, with an initial speed of $u \text{ m/s}$. It accelerates at $a \text{ m/s}^2$. The stone hits the ground with a speed of $v \text{ m/s}$ given by the formula $v^2 = u^2 + 2as$. Find the speed at which the stone hits the ground, correct to two decimal places, if:
- a** $u = 0$, $a = 9.8$ and $s = 50$ **b** $u = 2$, $a = 10$ and $s = 40$
c $u = 5$, $a = 9.8$ and $s = 35$ **d** $u = 10$, $a = 9.8$ and $s = 50$
- 17** The distance d metres Jim's car takes to stop once the brakes are applied is given by the formula $d = 0.2v + 0.005v^2$, where $v \text{ km/h}$ is the speed of the car when the brakes are applied.

Find the distance the car takes to stop if the brakes are applied when it is travelling at each of the speeds given below. Calculate your answers correct to 3 decimal places where appropriate.

- a** 60 km/h **b** 65 km/h **c** 70 km/h
d 80 km/h **e** 100 km/h **f** 120 km/h

6B Changing the subject of a formula



Sometimes the pronumeral whose value is to be determined is not the subject of the formula. In this situation you have a choice. You can either:

- **rearrange** the formula to make another pronumeral whose value is to be determined the subject and then substitute (Method 1), or
- **substitute** the values for the known pronumerals and then solve the resulting equation for the unknown pronumeral (Method 2).

In either case, an equation has to be solved. In Method 2, the equation involves numbers. In Method 1, the equation involves pronumerals.

It is preferable to use Method 1 when you are asked to find several values of a pronumeral that is not the subject of the formula.

Example 7

The cost \$ C of hiring Scott's car is given by the formula $C = \frac{1}{4}x + 40$, where x is the number of kilometres driven. Find the number of kilometres driven by a person who is charged \$130 for hiring the car.

Solution

Method 1

Rearrange the formula to make x the subject and then substitute the given value of C , as follows.

$$C = \frac{1}{4}x + 40$$

$$C - 40 = \frac{1}{4}x \quad (\text{Subtract 40 from both sides of the formula.})$$

$$x = 4C - 160 \quad (\text{Multiply both sides of the formula by 4.})$$

$$\begin{aligned} \text{When } C = 130, x &= 4 \times 130 - 160 \\ &= 360 \end{aligned}$$

Thus the person drove 360 km.

Method 2

Substitute the numbers and then solve the resulting equation, giving

$$C = \frac{1}{4}x + 40.$$

When $C = 130$,

$$130 = \frac{1}{4}x + 40$$

$$90 = \frac{1}{4}x \quad (\text{Subtract 40 from both sides of the equation.})$$

$$x = 360 \quad (\text{Multiply both sides of the equation by 4.})$$

Thus the person drove 360 km.

Example 8

The manager of a bed-and-breakfast guest house finds that the weekly profit \$ P is given by the formula

$$P = 40G - 600,$$

where G is the number of guests who stay during the week. Make G the subject of the formula and use the result to find the number of guests needed to make a profit of \$800.

Solution

$$P = 40G - 600$$

$$P + 600 = 40G$$

$$G = \frac{P + 600}{40}$$

When $P = 800$,

$$G = \frac{800 + 600}{40}$$

$$= \frac{1400}{40}$$

$$= 35$$

Thirty-five guests are required to make a profit of \$800.

Example 9

Given the formula $v^2 = u^2 + 2as$:

- rearrange the formula to make s the subject
- find the value of s when $u = 4$, $v = 10$ and $a = 2$
- find the value of s when $u = 4$, $v = 12$ and $a = 3$.

(continued on next page)

Solution

a $v^2 = u^2 + 2as$

$v^2 - u^2 = 2as$ (Subtract u^2 from both sides of the formula.)

$s = \frac{v^2 - u^2}{2a}$ (Divide both sides of the equation by $2a$.)

b When $u = 4$, $v = 10$ and $a = 2$.

$$\begin{aligned} s &= \frac{10^2 - 4^2}{2 \times 2} \\ &= \frac{100 - 16}{4} \\ &= \frac{84}{4} \\ &= 21 \end{aligned}$$

c When $u = 4$, $v = 12$ and $a = 3$.

$$\begin{aligned} s &= \frac{12^2 - 4^2}{2 \times 3} \\ &= \frac{144 - 16}{6} \\ &= \frac{128}{6} \\ &= 21\frac{1}{3} \end{aligned}$$

Example 10

Rearrange each of these formulas to make the pronumeral in brackets the subject.

a $E = \frac{p^2}{2m}$ (m)

b $T = 2\pi \sqrt{\frac{\ell}{g}}$ (ℓ)

c $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$ (u)

d $P = \sqrt{h+c} - a$ (h)

Solution

a $E = \frac{p^2}{2m}$

$mE = \frac{p^2}{2}$ (Multiply both sides of the equation by m .)

$m = \frac{p^2}{2E}$ (Divide both sides by E .)

$$\mathbf{b} \quad T = 2\pi \sqrt{\frac{\ell}{g}}$$

$$\frac{T}{2\pi} = \sqrt{\frac{\ell}{g}}$$

(Divide both sides of the equation by 2π .)

$$\frac{\ell}{g} = \left(\frac{T}{2\pi}\right)^2$$

(Square both sides of the equation.)

$$= \frac{T^2}{4\pi^2}$$

$$\ell = \frac{T^2}{4\pi^2} \times g$$

(Multiply both sides of the equation by g .)

$$\text{That is, } \ell = \frac{T^2 g}{4\pi^2}$$

$$\mathbf{c} \quad \frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

$$\frac{1}{f} - \frac{1}{v} = \frac{1}{u}$$

(Subtract $\frac{1}{v}$ from both sides.)

$$\frac{v-f}{fv} = \frac{1}{u}$$

(common denominator on LHS of equation)

$$u = \frac{fv}{v-f}$$

(Take reciprocals of both sides.)

Note: $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$ does not imply $f = u + v$

$$\mathbf{d} \quad P = \sqrt{h+c} - a$$

$$P + a = \sqrt{h+c}$$

(Add a to both sides of the equation.)

$$(P+a)^2 = h+c$$

(Square both sides.)

$$h = (P+a)^2 - c$$

The previous example shows some of the techniques that can be used to rearrange a formula.

Example 11

Make q the subject of the formula $\frac{3p}{4} - \frac{5}{q} = \frac{p^2}{3q}$.

(continued on next page)

Solution

$$\frac{3p}{4} - \frac{5}{q} = \frac{p^2}{3q}$$

$$12q \left(\frac{3p}{4} - \frac{5}{q} \right) = 12q \times \frac{p^2}{3q}$$

$$3p \times 3q - 5 \times 12 = p^2 \times 4$$

(Multiply both sides by the lowest common denominator, $12q$.)

$$9pq - 60 = 4p^2$$

$$9pq = 4p^2 + 60$$

$$q = \frac{4p^2 + 60}{9p}$$



Changing the subject of a formula

When rearranging a formula, the basic strategy is to move all terms involving the new subject to one side, and all the other terms to the other side. To do this:

- fractions can be eliminated by multiplying both sides of the formula by a common denominator
- all like terms should be collected
- the same operation(s) must be performed on both sides of the formula.



Exercise 6B

Example 7

- The profit $\$P$ made each day by a store owner who sells CDs is given by the formula $P = 5n - 150$, where n is the number of CDs sold.
 - What profit is made if the store owner sells 60 CDs?
 - Make n the subject of the formula.
 - How many CDs were sold if the store made:
 - a profit of \$275
 - a profit of \$400
 - a loss of \$100
 - no profit?

Example 8

- The cost $\$C$ of hiring a reception room for a function is given by the formula $C = 12n + 250$, where n is the number of people attending the function.
 - Rearrange the formula to make n the subject.
 - How many people attended the function if the cost of hiring the reception room was:
 - \$730
 - \$1090
 - \$1210
 - \$1690?

- 3** The cost \$ C of hiring an electric floor sander is given by the formula $C = 1.2t + 50$, where t is the number of hours for which the sander is hired.
- Rearrange the formula to make t the subject.
 - For how long was the sander hired if it cost:
 - \$56.00
 - \$59.60
 - \$74.00
 - \$110.00?
- 4** Given the formula $v = u + at$:
- rearrange the formula to make u the subject
 - find the value of u when:
 - $v = 20$, $a = 2$ and $t = 5$
 - $v = 40$, $a = -6$ and $t = 4$
 - $v = -19.6$, $a = 4.9$ and $t = 2.5$
 - rearrange the formula to make a the subject
 - find the value of a when:
 - $v = 20$, $u = 15$ and $t = 2$
 - $v = -26.8$, $u = -14.4$ and $t = 2$
 - $v = \frac{1}{2}$, $u = \frac{2}{3}$ and $t = \frac{5}{6}$
 - rearrange the formula to make t the subject and find t when $v = 6$, $u = 7$ and $a = -3$.
- 5** Given the formula $t = a + (n - 1)d$:
- rearrange the formula to make a the subject
 - find the value of a when:
 - $t = 11$, $n = 4$ and $d = 3$
 - $t = 8$, $n = 5$ and $d = -3$
 - $t = 17.6$, $n = 11$ and $d = 1.4$
 - rearrange the formula to make d the subject
 - find the value of d when:
 - $t = 30$, $a = 5$ and $n = 6$
 - $t = 48$, $a = 3$ and $n = 16$
 - $t = 120$, $a = -30$ and $n = 101$
 - rearrange the formula to make n the subject and find the value of n when $t = 150$, $a = 5$ and $d = 5$.

- 6 Rearrange each of these formulas to make the pronumeral in brackets the subject.

a $y = mx + c$

(c)

b $y = mx + c$

(x)

c $A = \frac{1}{2}bh$

(b)

d $C = 2\pi r$

(r)

e $A = \frac{h}{2}(a + b)$

(h)

f $P = A + 2\ell h$

(ℓ)

g $s = ut + \frac{1}{2}at^2$

(u)

h $s = ut + \frac{1}{2}at^2$

(a)

i $A = 2\pi r^2 + 2\pi rh$

(h)

j $V = \frac{1}{3}\pi r^2 h$

(h)

k $S = \frac{n}{2}(a + \ell)$

(a)

l $S = \frac{n}{2}(a + \ell)$

(n)

m $V = \pi r^2 + \pi rs$

(s)

n $E = mgh + \frac{1}{2}mv^2$

(h)

- 7 The formula for the sum S of the interior angles in a convex n -sided polygon is $S = 180(n - 2)$. Rearrange the formula to make n the subject and use this to find the number of sides in the polygon if the sum of the interior angles is:

a 1080°

b 1800°

c 3240°

- 8 The kinetic energy E joules of a moving object is given by $E = \frac{1}{2}mv^2$, where m kg is the mass of the object and v m/s is its speed.

Rearrange the formula to make m the subject and use this to find the mass of the object when its energy and speed are, respectively:

a 400 joules, 10 m/s **b** 28 joules, 4 m/s **c** 57.6 joules, 2.4 m/s

- 9 The formula for finding the number of degrees Fahrenheit (F) for a temperature given in degrees Celsius (C) is $F = \frac{9}{5}C + 32$.

a Rearrange the formula to make C the subject.

b Use this formula to convert these Fahrenheit temperatures to Celsius temperatures.

i 68°F

ii 23°F

iii 212°F

iv 356°F

v 98.4°F

vi -14.8°F

- 10 When an object is shot up into the air with a speed of u metres per second, its height above the ground h metres and time of flight t seconds are related (ignoring air resistance) by $h = ut - 4.9t^2$.

Find the speed at which an object was fired if it reached a height of 27.5 metres after 5 seconds.

- 11** Rearrange each of these formulas to make the pronumeral in brackets the subject. (All pronumerals represent positive numbers.)

a $c = a^2 + b^2$ (a)

b $K = 3\ell^2 + 4m$ (ℓ)

c $x = \sqrt{ab}$ (b)

d $d = 8\sqrt{\frac{h}{5}}$ (h)

e $T = \frac{2\pi}{n}$ (n)

f $D = \frac{m}{v}$ (v)

g $E = \frac{m}{2r^2}$ (r)

h $T = \frac{a}{4r^2}$ (r)

- 12** Temperatures can be measured in either degrees Fahrenheit or degrees Celsius. To convert from one scale to the other, the following formula is used: $F = \frac{9}{5}C + 32$.

- a** On a particular day in Melbourne, the temperature was 28°C . What is this temperature measured in Fahrenheit?
- b** In Boston, USA, the minimum overnight temperature was 4°F . What is this temperature measured in Celsius?
- c** What number represents the same temperature in $^\circ\text{C}$ and $^\circ\text{F}$?
- d** An approximate conversion formula, used frequently when converting oven temperatures, is $F = 2C + 30$. Use this to convert these temperatures:
- i** an oven temperature of 200°C
 - ii** an oven temperature of 400°F
 - iii** an oven temperature of 180°C
 - iv** an oven temperature of 530°F

In this section we shall learn how to create formulas from given information.

Example 12

Find a formula for n , the number of cents in x dollars and y cents.

Solution

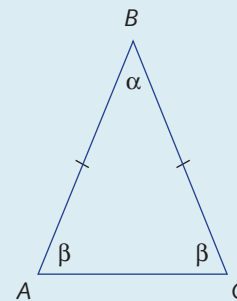
In $\$x$ there are $100x$ cents.

In $\$x$ and y cents there are $(100x + y)$ cents.

The formula is $n = 100x + y$.

Example 13

Here is an isosceles triangle with equal base angles marked.
Find a formula for β in terms of α .



Solution

$$\alpha + 2\beta = 180 \quad (\text{angle sum of triangle})$$

$$2\beta = 180 - \alpha$$

$$\beta = \frac{180 - \alpha}{2}$$

Exercise 6C

Example 12

1 Construct a formula for:

- S in terms of C and P , where $\$S$ is selling price, $\$C$ is cost price and $\$P$ is profit
- R in terms of W and L , where $\$R$ is retail price, $\$W$ is wholesale price and $\$L$ is loss
- D in terms of n , where D is the number of degrees in n right angles
- c in terms of D , where c is the number of cents in $\$D$

- e m in terms of h , where m is the number of minutes in h hours
 - f d in terms of m , where d is the number of days in m weeks.
- 2** Construct a formula for:
- a the number of centimetres n in p metres
 - b the number of grams x in y kilograms
 - c the number of millilitres s in t litres
 - d the number of centimetres q in $5p$ metres
 - e the number of grams x in $\frac{y}{2}$ kilograms.
- 3** Find a formula for:
- a the number of cents z in x dollars and y cents
 - b the number of hours x in y minutes and z seconds
 - c the number of minutes x in y minutes and z seconds
 - d the cost $\$m$ of 1 book if 20 books cost $\$c$
 - e the cost $\$n$ of 1 suit if 5 suits cost $\$m$
 - f the cost $\$m$ of 1 tyre if x tyres cost $\$y$
 - g the cost $\$p$ of n suits if 4 suits cost $\$k$
 - h the cost $\$q$ of x cars if 8 cars cost $\$b$.
- 4** Find a formula relating x and y for each of these statements, making y the subject.
- a y is three less than x .
 - b y is four more than the square of x .
 - c y is eight times the square root of one-fifth of x .
 - d x and y are supplementary angles.
 - e A car travelled 80 km in x hours at an average speed of y km/h.
 - f A car used x litres of petrol on a trip of 80 km and the fuel consumption was y litres/100 km.
- 5** Find a formula relating the given pronumerals for each of these statements.
- a The number of square cm x in y square metres.
 - b The selling price $\$S$ of an article with an original price of $\$m$ when a discount of 20% is given.
 - c The area of a circle A cm² with a radius r cm.
 - d The circumference C metres of a circle with radius r metres.
 - e The sum of the interior angles S° of a polygon of n sides.

Example 13

- f** The length c cm of the hypotenuse and the lengths a cm and b cm of the other two sides in a right-angled triangle.
 - g** The area A cm² of a sector of a circle with a radius of length r cm and angle θ at the centre of the circle.
 - h** The distance d km travelled by a car in t hours at an average speed of 75 km/h.
 - i** The number of hectares h in a rectangular paddock of length 400 m and width w m.
- 6** In each part, find a formula from the information given.
- a** The total cost $\$C$ for a tour of x days if the cost is given as:
 - i** \$18 per day **ii** \$154 plus \$7 per day.
 - b** A hire car firm charges \$20 per day plus 40 cents per km. What is the total cost $\$C$ for a day in which x km was travelled?
 - c** If there are 50 litres of petrol in the tank of a car and petrol is used at the rate of 4 litres per day, what is the number of litres y that remains after x days?
 - d** In a sequence of numbers the first number is 2, the second number is 4, the third is 8, the fourth is 16, etc. Assuming the doubling pattern continues, what is the formula you would use to calculate t , the n th number?
 - e** Cooking instructions for a forequarter of lamb are as follows: Preheat oven to 220°C and cook for 45 min per kg plus an additional 20 min. What is the formula relating the cooking time T minutes and weight w kg?
 - f** A piece of wire of length x cm is bent into a circle of area A cm². What is the formula relating A and x ?
 - g** A right-angled triangle has shorter side lengths $9x$ cm and $40x$ cm, and a perimeter of p cm. What is the formula relating p and x ?
- 7** Gareth the gardener has a large rectangular vegetable patch and he wishes to put in a path around it using concrete pavers that measure 50 cm $\times$ 50 cm. The path is to be 1 paver wide. Let n be the number of pavers required. If the vegetable patch measures x metres by y metres, find a formula for n in terms of x and y .

Review exercise



- 1 If $s = \frac{n}{2}(2a + (n - 1)d)$:
 - a find the value of s when $n = 10$, $a = 6$ and $d = 3$
 - b find the value of a when $s = 350$, $n = 20$ and $d = 4$
 - c find the value of d when $s = 460$, $n = 10$ and $a = 10$.
- 2 The formula for the geometric mean m of two positive numbers a and b is $m = \sqrt{ab}$.
 - a Find m if $a = 16$ and $b = 25$.
 - b Find a if $m = 7$ and $b = 16$.
- 3 If $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$:
 - a find x if $b = 4$, $a = 1$ and $c = -24$
 - b find c if $a = 1$, $x = 6$ and $b = 2$.
- 4 A pillar is in the shape of a cylinder with a hemispherical top. If r metres is the radius of the base and h metres is the total height, the volume V cubic metres is given by the formula $V = \frac{1}{3}\pi r^2(3h - r)$.
 - a Rearrange the formula to make h the subject.
 - b Find the height of the pillar, correct to the nearest centimetre, if the radius of the pillar is 0.5 m and the volume is 10 m^3 .
- 5 Rearrange each of these formulas to make the pronumeral in brackets the subject. (All of the pronumerals represent positive numbers.)

a $A = \ell \times w$ (ℓ)	b $C = 2\pi r$ (r)
c $V = \pi r^2 h$ (h)	d $A = 2h(\ell + b)$ (b)
e $A = 4\pi r^2$ (r)	f $w = 10\sqrt{\frac{x}{a}}$ (x)
g $w = \sqrt{\frac{3V}{\pi h}}$ (V)	h $\frac{1}{x} + \frac{1}{y} = \frac{2}{z}$ (y)
- 6 If a stone is dropped off a cliff, the number of metres it has fallen after a certain number of seconds is found by multiplying the square of the number of seconds by 4.9.
 - a Find the formula for the distance d metres fallen by the stone in t seconds.
 - b Find the distance fallen in 1.5 seconds.



7 If $t = \sqrt{\frac{M}{M-m}}$:

a express the formula with m as the subject

b express the formula with M as the subject

c find the value of M if $m = 3$ and $t = \sqrt{2}$.

- 8 The total surface area $S \text{ cm}^2$ of a cylinder is given in terms of its radius $r \text{ cm}$ and height $h \text{ cm}$ by the formula $S = 2\pi r(r + h)$.

a Express this formula with h as the subject.

b What is the height of such a cylinder if the radius is 7 cm and the total surface area is 500 cm^2 ? Calculate your answer in centimetres, correct to two decimal places.

- 9 The sum S of the squares of the first n whole numbers is given by the formula

$$S = \frac{n(n+1)(2n+1)}{6}.$$

Find the sum of the squares of:

a the first 20 whole numbers

b all the numbers from 5 to 21 inclusive.

- 10 a For the formula $T = 2\pi \sqrt{\frac{W}{gF}}$, make F the subject.

b For the formula $P = \frac{\pi r x}{180} + 2r$, make x the subject.

c For the formula $D = \sqrt{\frac{f+x}{f-x}}$, make x the subject.

- 11 Cans in a supermarket are displayed in a triangular stack with one can at the top, two cans in the second row from the top, three cans in the third row from the top, and so on. What is the number of cans in the display if the number of rows is:

a 4

b 5

c n

d 35?

- 12 Rearrange each of these formulas to make the pronumeral in brackets the subject. Assume all pronumerals take non-negative values.

a $M = \sqrt{\frac{ab}{t} + a^2}$ (t)

b $V = \frac{1}{3}a^2\sqrt{\ell^2 - \frac{a^2}{2}}$ (ℓ)

c $E = \frac{w^2a}{(w^2+m)b^3}$ (a)

d $T = 2\pi\sqrt{\frac{\ell+r}{g}}$ (r)

e $x = a\sqrt{\frac{k}{ph+y}}$ (h)

f $v^2 = u^2 + 2as$ (u)



- 13** The volume $V \text{ cm}^3$ of metal in a tube is given by the formula $V = \pi \ell (r^2 - (r - t)^2)$ where $\ell \text{ cm}$ is the length, $r \text{ cm}$ is the radius of the outside surface and $t \text{ cm}$ the thickness of the material.
- a** Find V if $\ell = 50$, $r = 5$ and $t = 0.25$.
 - b** Make r the subject of the formula.
- 14** When a stone is thrown upward at 25 m/s , the height $h \text{ metres}$ it reaches after $t \text{ seconds}$ is given by the formula $h = 25t - 4.9t^2$. Find the height of the stone after:
- a** 1 second
 - b** 2 seconds.
- 15** The cost $\$C$ of carpeting a room is given by $C = P\ell w$ where $\$P$ is the price of the carpeting per square metre and ℓ and w are the length and width of the room in metres.
- a** Find the cost of carpeting a room 3 metres by 7 metres if the cost of carpeting per square metre is $\$50$.
 - b** Make P the subject of the formula.
- 16** For the formula $I = \frac{180n - 360}{n}$:
- a** find I when $n = 6$
 - b** make n the subject of the formula and find n when $I = 108$.
- 17** Find the formula connecting x and y for each of these statements making y the subject.
- a** y is four more than twice the square of x .
 - b** x and y are complementary angles.
 - c** A car travelled $x \text{ km}$ in $y \text{ hours}$ at a speed of 100 km/h .
 - d** A car travelled 100 km in $y \text{ hours}$ at a speed of $x \text{ km/h}$.



Challenge exercise

- 1 Rearrange each of these formulas to make the pronumeral in brackets the subject.
(All of the pronumerals represent positive numbers.)

a $P = \frac{M+m}{M-m}$ (M)

b $F = \frac{MP}{M+\ell}$ (M)

c $\frac{1}{S} = \frac{1}{R} + \frac{1}{T}$ (R)

d $E = \frac{R}{i} + P$ (i)

e $T = t - \frac{h}{\ell}$ (ℓ)

f $I = \frac{M}{4} \left(R^2 + \frac{L^2}{3} \right)$ (L)

g $L = \ell \sqrt{1 + \frac{v^2}{c^2}}$ (v)

h $T = 2\pi \left(\frac{1+e}{1-e} \right)$ (e)

i $v = c \left(\frac{1}{r} - \frac{1}{s} \right)$ (r)

j $P = 2\pi \sqrt{\frac{h+k}{g}}$ (k)

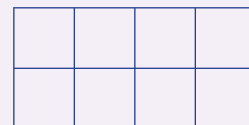
k $P = p \sqrt{1 + \frac{1}{\ell}}$ (ℓ)

l $A = \pi(R^2 - r^2)$ (R)

m $(a+b)^2 + c^2 = (a-d)^2$ (a)

n $(M-m)^2 + p^2 = M^2$ (M)

- 2 Sophie is playing with a box of connecting rods which can be joined together to produce a rectangular grid. The grid shown opposite is of size 4×2 (i.e. length = 4, width = 2) and uses 22 rods.



- a** How many rods are needed to make these grids?

i 1×2

ii 3×2

iii 10×2

iv $n \times 2$

- b** How many rods are needed to make these grids?

i 1×3

ii 2×3

iii 10×3

iv $n \times 3$

- c** How many rods are needed to make a grid of size $n \times m$?

- d** In the 4×2 grid shown above, how many rods have been placed:

i vertically

ii horizontally?

- e** In an $n \times m$ grid, how many rods have been placed:

i vertically

ii horizontally

iii in total?

- f** Does your answer to part **e** **iii** agree with your answer to part **c**?

- g** Sophie makes a 4×7 grid. How many rods did she use?

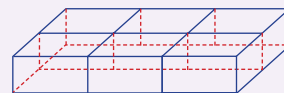


h If she used 97 rods to make a grid of width 6, what was the length of the grid?

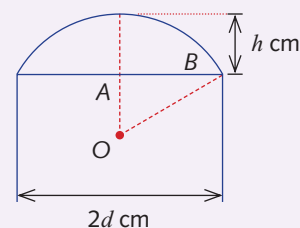
i If she has 100 rods, can she make a grid that uses all 100 rods?

If so, what size grid can she make?

j If the rods can be joined to make a three-dimensional grid, find the formula for the number of rods required to make a grid as shown of size $m \times n \times p$ (that is, length = m , width = n , height = p). The diagram shows a rectangular prism measuring 3 rods by 2 rods by 1 rod.



- 3** A builder wishes to place a circular cap of a given height above an existing window. To do this he needs to know the location of the centre of the circle (the cap is not necessarily a semi-circle) and the radius of the circle. O is the centre of the required circle, the radius of the required circle is r cm, the width of the window is $2d$ cm and the height of the circular cap is h cm.



a Express each of these in terms of r , d and h .

i AB

ii OA

b Show that $r = \frac{h^2 + d^2}{2h}$.

c If the window is 120 cm wide and the cap is 40 cm high, find:

i the radius of the circle

ii how far below the top of the window the centre of the circle must be placed.

d If the builder used a circle of radius 50 cm and this produced a cap of height 20 cm, what was the width of the window?

- 4** A group of n people attend a club meeting. Before the meeting begins, they all shake hands with each other. Write a formula to find H , the number of handshakes exchanged.



- 5** A cyclic quadrilateral has all its vertices on a circle. Its area A is given by Brahmagupta's formula

$$A^2 = (s - a)(s - b)(s - c)(s - d),$$

where a , b , c and d are the side lengths of the quadrilateral and

$s = \frac{a+b+c+d}{2}$ is the 'semi-perimeter'. Find the exact area of a cyclic quadrilateral

with side lengths:

- a** 4, 5, 6, 7
 - b** 7, 4, 4, 3
 - c** 8, 9, 10, 13
 - d** 39, 52, 25, 60
 - e** 51, 40, 68, 75
- 6 a** The population of a town decreases by 2% each year. The population was initially P , and is Q after n years. What is the formula relating Q , P and n ?
- b** The population of a town decreases by 5% each year. The percentage decrease over a period of n years is $a\%$. What is the formula relating a and n ?



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- ACMMG201
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CHAPTER

7

Measurement and Geometry

Congruence and special quadrilaterals

This chapter reviews work on angles, congruence tests and their applications. Congruence is an extremely useful tool in geometrical arguments. It is used here to prove properties of various types of quadrilaterals, and to develop tests for special quadrilaterals.

This chapter is a revision of all the geometry presented in Years 7 and 8.



Most geometrical arguments rely on the relationship between different angles in the one figure. This section reviews the methods introduced so far, and the next section uses these methods to prove some interesting general results.

Describing angles by their sizes

Angles of three particular sizes have special names:

- An angle of 90° is a **right angle**.
- An angle of 180° is a **straight angle**.
- An angle of 360° is a **revolution**.

Other angles are described by being within a range of angle sizes:

- An **acute angle** is an angle between 0° and 90° .
- An **obtuse angle** is an angle between 90° and 180° .
- A **reflex angle** is an angle between 180° and 360° .

For example, 30° is an acute angle, 140° is obtuse, and 220° is reflex.

Particular pairs of angles also have special names:

- Two angles whose sum is 90° are called **complementary angles**.
- Two angles whose sum is 180° are called **supplementary angles**.

For example, 35° and 55° are complementary angles because $35^\circ + 55^\circ = 90^\circ$, and 70° and 110° are supplementary angles because $70^\circ + 110^\circ = 180^\circ$.

Angles at a point

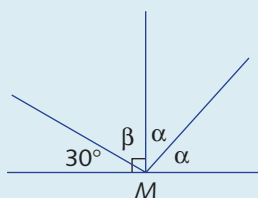
The first group of results concern angles at a point.

- Adjacent angles can be added and subtracted.
- Adjacent angles on a straight line are supplementary.
- Angles in a revolution add to 360° .
- Vertically opposite angles are equal.

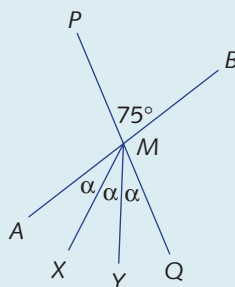
Example 1

Find the value of the pronumeral in each figure.

a



b



Solution

- a** $\beta = 60^\circ$ (complementary angles)
 $2\alpha = 90^\circ$ (supplementary angles)
 $\alpha = 45^\circ$
- b** $3\alpha = 75^\circ$ (vertically opposite angles at M)
 $\alpha = 25^\circ$

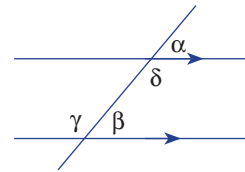
Angles across transversals

Suppose that a transversal crosses two lines.

- If the lines are parallel, then the alternate angles are equal.
- If the lines are parallel, then the corresponding angles are equal.
- If the lines are parallel, then the co-interior angles are supplementary.

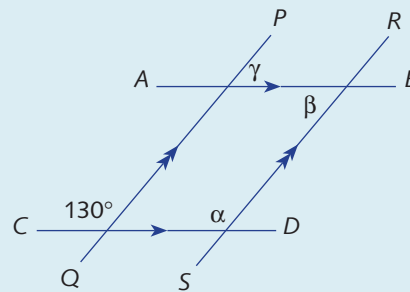
The diagram to the right shows a transversal crossing two parallel lines.

- α and β are equal corresponding angles.
- δ and γ are equal alternate angles.
- δ and β are co-interior angles.



Example 2

Find α , β and γ in the figure to the right.



Solution

- $\alpha = 130^\circ$ (corresponding angles, $PQ \parallel RS$)
 $\beta = 50^\circ$ (co-interior angles, $AB \parallel CD$)
 $\gamma = 50^\circ$ (alternate angles, $PQ \parallel RS$)

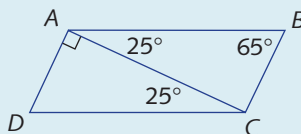
Proving that two lines are parallel

The **converses** of the previous three results are tests to prove that two lines are parallel. Suppose that a transversal crosses two lines.

- If two alternate angles are equal, then the lines are parallel.
- If two corresponding angles are equal, then the lines are parallel.
- If two co-interior angles are supplementary, then the lines are parallel.

Example 3

Identify every pair of parallel lines in the figure to the right.



Solution

First, $AB \parallel DC$ (alternate angles are equal)

Secondly, $\angle BAD = 115^\circ$ (adjacent angles at A)

so $AD \parallel BC$ (co-interior angles are supplementary).

Angle sums of triangles and quadrilaterals

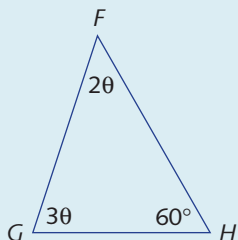
- The sum of the interior angles of a triangle is 180° .
- An exterior angle of a triangle equals the sum of the opposite interior angles.
- The sum of the interior angles of a quadrilateral is 360° .

Proofs of these results are reviewed in Exercise 7B.

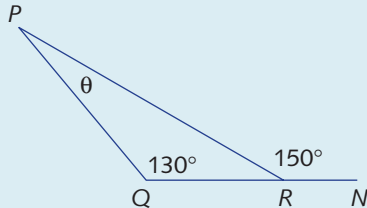
Example 4

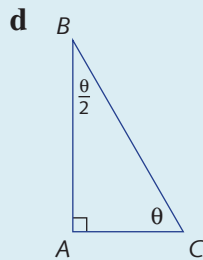
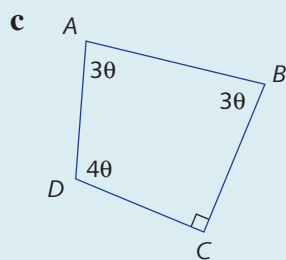
Find θ in each diagram below.

a



b





Solution

a $3\theta + 2\theta + 60^\circ = 180^\circ$ (angle sum of $\triangle FGH$)
 $5\theta = 120^\circ$
 $\theta = 24^\circ$

b $\theta + 130^\circ = 150^\circ$ (exterior angle of $\triangle PQR$)
 $\theta = 20^\circ$

c $3\theta + 3\theta + 4\theta + 90^\circ = 360^\circ$ (angle sum of quadrilateral $ABCD$)
 $10\theta = 270^\circ$
 $\theta = 27^\circ$

d $\theta + \frac{\theta}{2} + 90^\circ = 180^\circ$ (angle sum of $\triangle ABC$)
 $\frac{3\theta}{2} = 90^\circ$
 $\theta = 60^\circ$

Isosceles and equilateral triangles

A triangle is called **isosceles** if it has two equal sides; it is called **equilateral** if all three sides are equal. Thus all equilateral triangles are also isosceles triangles.

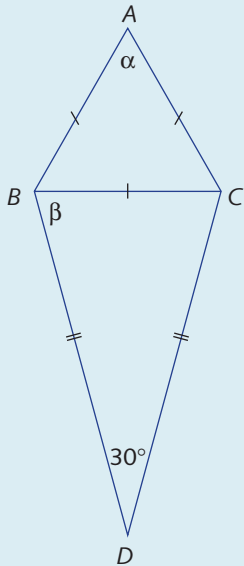
- The base angles of an isosceles triangle are equal.
- Conversely, if two angles of a triangle are equal, then the sides opposite those angles are equal.
- Each interior angle of an equilateral triangle is 60° .
- Conversely, if the three angles of a triangle are equal, then the triangle is equilateral.

Proofs of these results are reviewed in Exercise 7C.

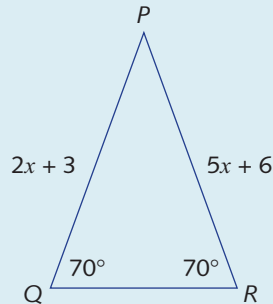
Example 5

Find α , β , γ and x in the diagrams below.

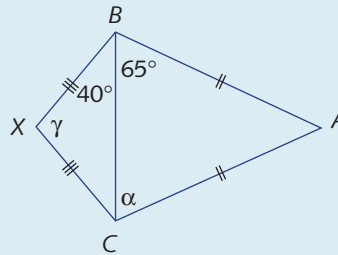
a



b



c



Solution

a First, $\alpha = 60^\circ$

($\triangle ABC$ is equilateral)

Secondly, $\angle BCD = \beta$

(base angles of isosceles $\triangle BCD$)

so $2\beta + 30^\circ = 180^\circ$

(angle sum of $\triangle BCD$)

$\beta = 75^\circ$

b $2x + 3 = 5x + 6$

(opposite angles of $\triangle PQR$ are equal)

$-3 = 3x$

$x = -1$

Thus PQ and PR each have length of 1.

c $\alpha = 65^\circ$

(base angles of isosceles $\triangle ABC$)

$\angle BCX = 40^\circ$

(base angles of isosceles $\triangle BXC$)

$\gamma = 100^\circ$

(angle sum of $\triangle XBC$)



Exercise 7A

1 a Classify these angles using the standard terms.

- i 140° ii 360° iii 33° iv 180° v 90° vi 350°

b Write down the complements of:

- i 20° ii 72° iii 45°

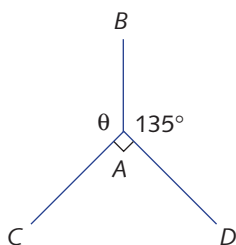
c Write down the supplements of:

- i 20° ii 172° iii 90°

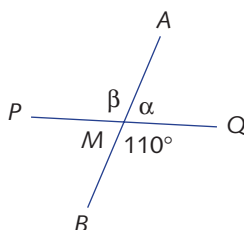
Example 1

2 This question reviews angles at a point. Find the angles marked with pronumerals, giving reasons.

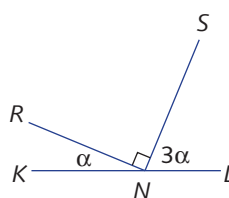
a



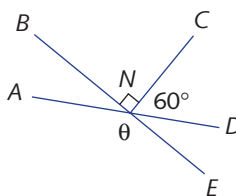
b



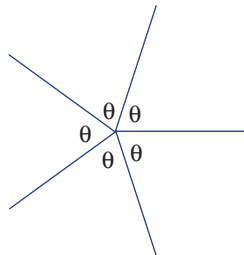
c



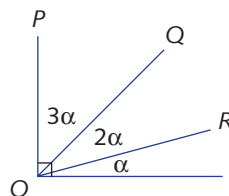
d



e



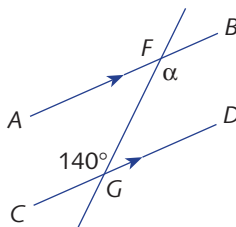
f



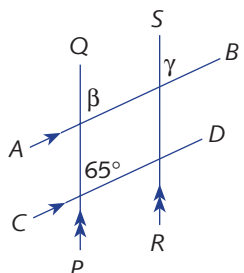
Example 2

3 This question reviews angles across transversals. Find the angles marked with pronumerals, giving reasons.

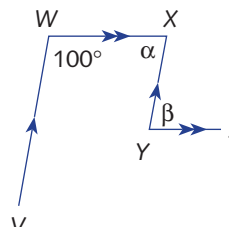
a

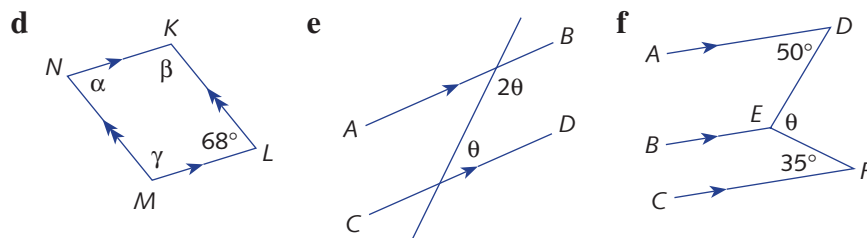


b



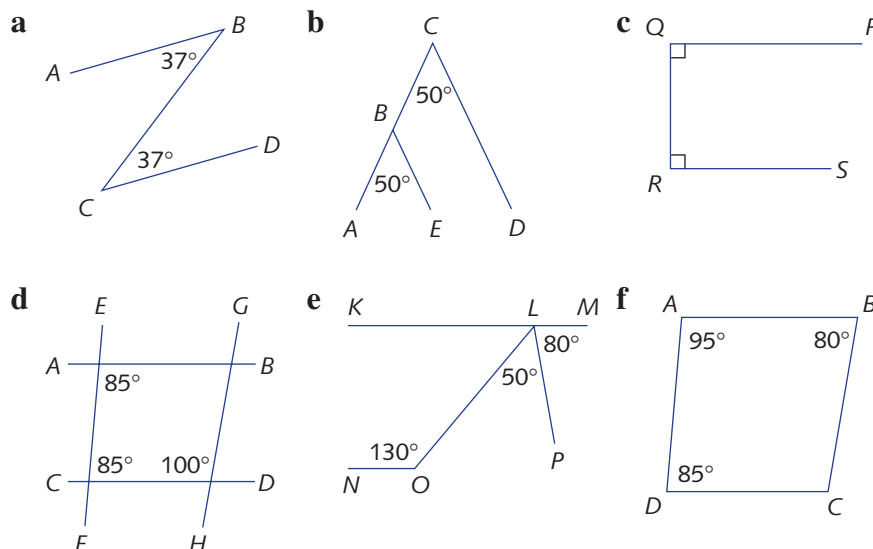
c





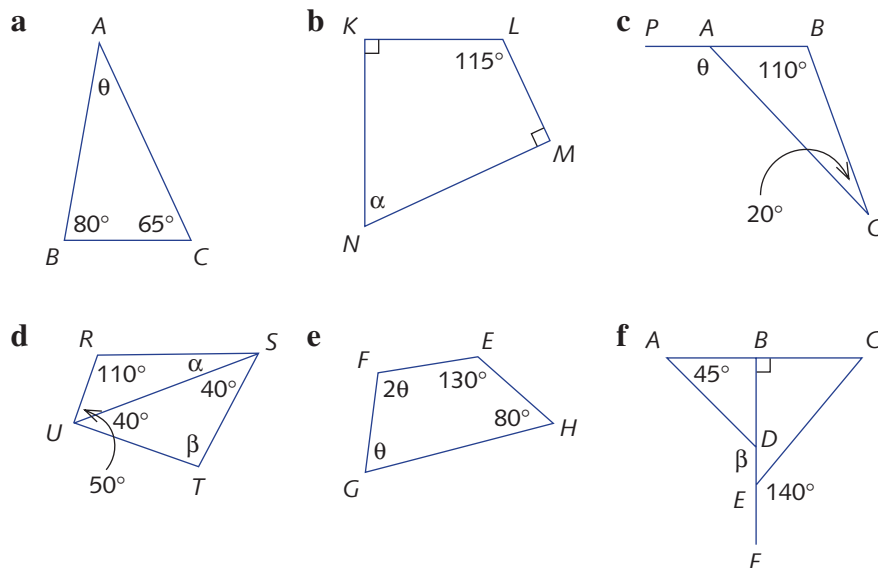
Example 3

- 4** This question reviews tests for two lines to be parallel. In each part, name all pairs of parallel lines, giving reasons.

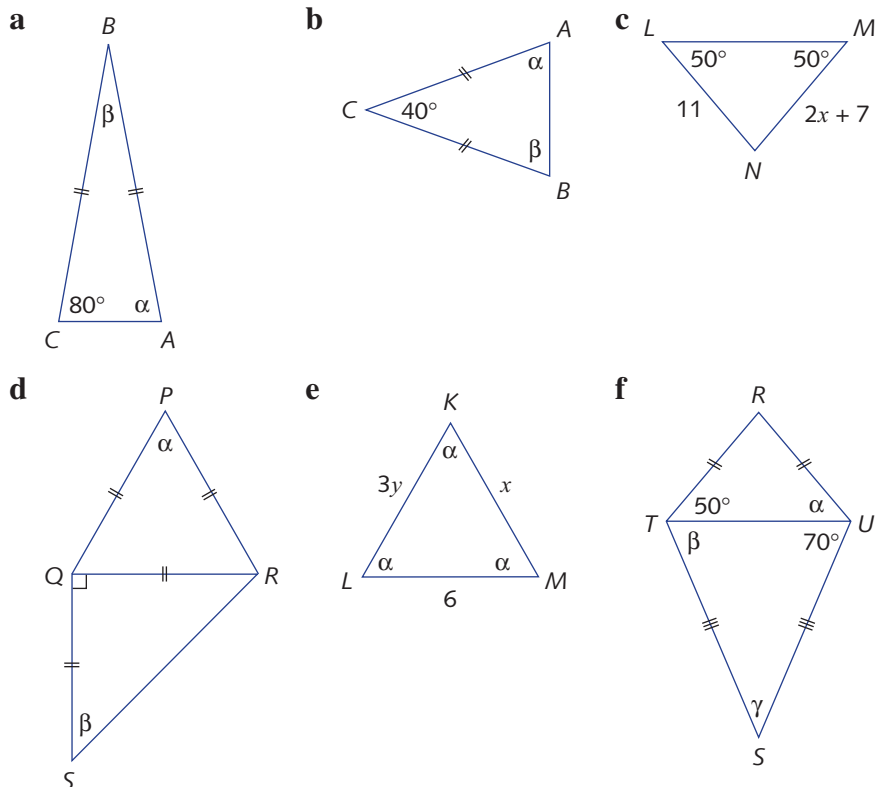


Example 4

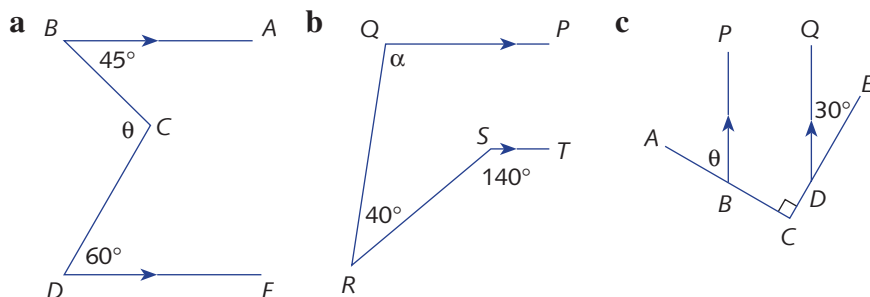
- 5** This question reviews angle sums of triangles and quadrilaterals. Find the angles marked with pronumerals, giving reasons.



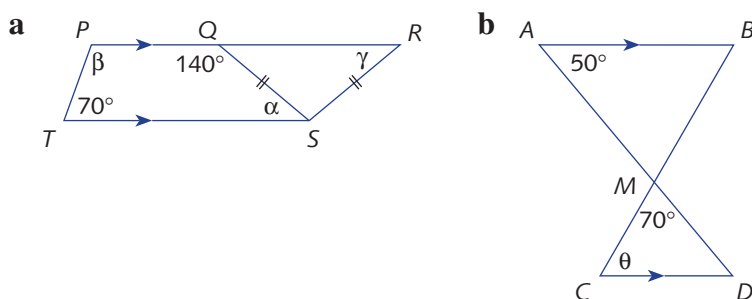
- 6 This question reviews isosceles and equilateral triangles. Find the angles and sides marked with pronumerals, giving reasons.

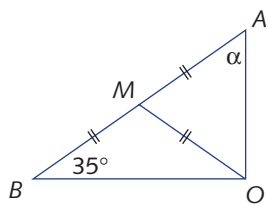
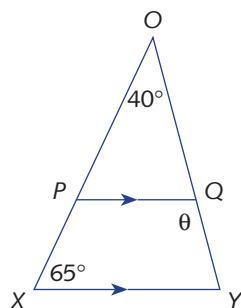
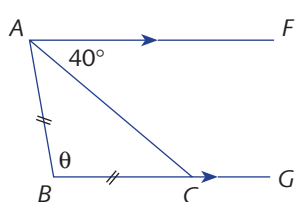
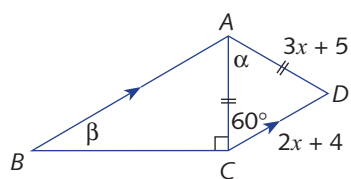
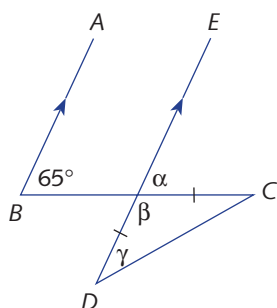
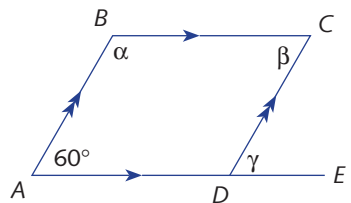


- 7 Use a construction to find the angles marked with pronumerals.



- 8 Find the values of the pronumerals α , β , γ , θ and x . Different methods are combined in each part of this question.



c**d****e****f****g****h**

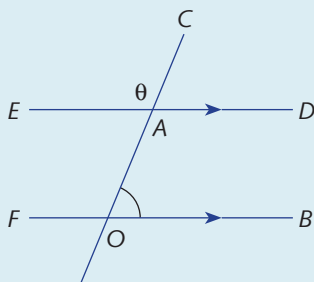


This section uses the methods of Section 7A in more general situations. Construction lines are often needed.

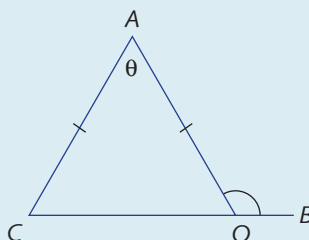
Example 6

Find the angle $\angle AOB$ in terms of θ , giving reasons.

a



b



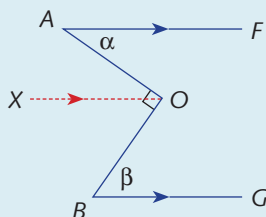
Solution

$$\begin{aligned} \text{a } \angle FOA &= \theta && \text{(corresponding angles, } ED \parallel FB) \\ \angle AOB &= 180^\circ - \theta && \text{(supplementary angle)} \end{aligned}$$

$$\begin{aligned} \text{b } \angle AOC &= \frac{180^\circ - \theta}{2} && (\triangle ACO \text{ is isosceles}) \\ &= 90^\circ - \frac{\theta}{2} \\ \angle AOB &= 180^\circ - \left(90^\circ - \frac{\theta}{2}\right) && \text{(supplementary angles)} \\ &= 90^\circ + \frac{\theta}{2} \end{aligned}$$

Example 7

Show that $\alpha + \beta = 90^\circ$.



(continued on next page)

Solution

Draw XO parallel to AF .

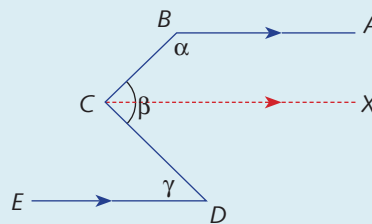
Then $\angle AOX = \alpha$ (alternate angles, $OX \parallel FA$)

and $\angle BOX = \beta$ (alternate angles, $OX \parallel GB$)

so $\alpha + \beta = 90^\circ$ (adjacent angles)

Example 8

Prove that $\alpha + \beta - \gamma = 180^\circ$.



Solution

Draw CX parallel to BA .

Then $\angle BCX = 180^\circ - \alpha$ (co-interior angles, $BA \parallel CX$)

and $\angle DCX = \gamma$ (alternate angles, $CX \parallel ED$)

Hence $\beta = 180^\circ - \alpha + \gamma$ ($\angle BCD = \angle BCX + \angle DCX$)

so $\alpha + \beta - \gamma = 180^\circ$

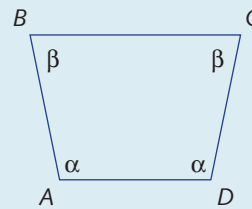
Proving theorems

The results of the previous section can be applied to prove many interesting general results, called **theorems**. For the rest of this chapter, each theorem to be proven is stated in *italics*. Usually the question includes a diagram, introduces the necessary pronumerals and breaks the proof down into a number of steps. Sometimes a construction is required.

Example 9

Prove that: *A quadrilateral with two pairs of equal adjacent angles is a trapezium.*

In the quadrilateral $ABCD$ to the right,
 $\angle A = \angle D = \alpha$ and $\angle B = \angle C = \beta$.



- a Prove that $\alpha + \beta = 180^\circ$.
- b Hence prove that $ABCD$ is a trapezium.

Solution

- a $\alpha + \alpha + \beta + \beta = 360^\circ$ (angle sum of quadrilateral $ABCD$)
 so $\alpha + \beta = 180^\circ$
- b Hence $AD \parallel BC$ (co-interior angles are supplementary)
 Therefore $ABCD$ is a trapezium.

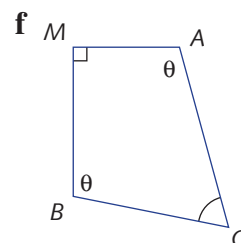
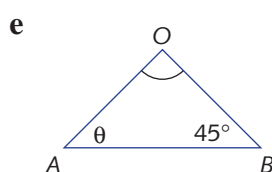
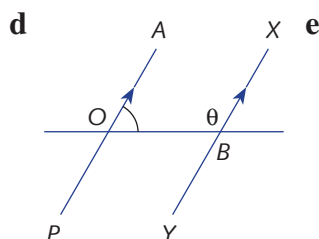
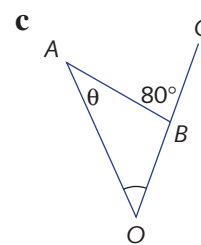
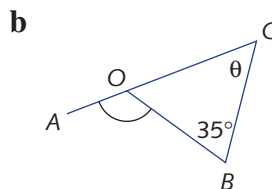
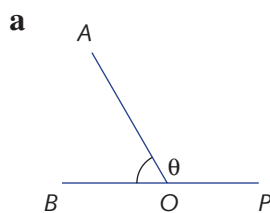
Angle sums of polygons

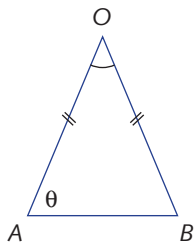
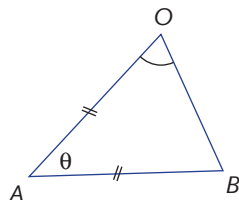
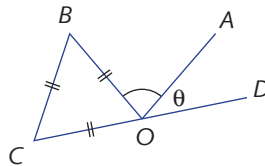
The last four questions of the following exercise deal with interior and exterior angle sums of polygons. These are interesting general results that you may want to remember as part of your known geometric facts.

Exercise 7B

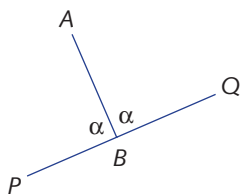
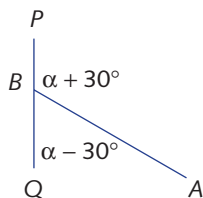
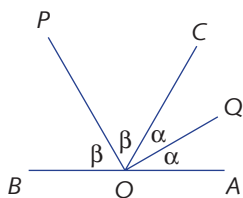
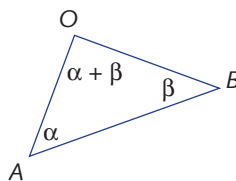
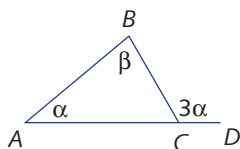
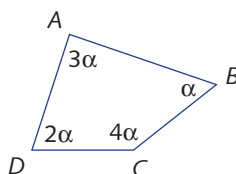
Example 6

- Find the marked angle $\angle AOB$ in terms of θ , giving reasons.

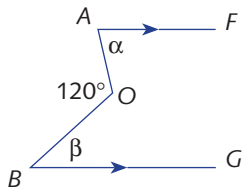
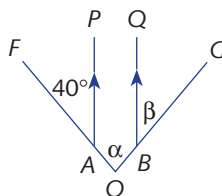
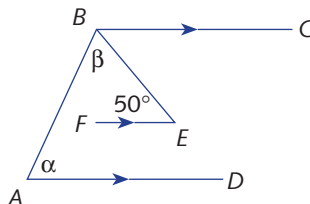


g**h****i**

Example 7

2 a Prove that $AB \perp PQ$.**b** Prove that $\angle ABP = 2 \times \angle ABQ$.**c** Prove that $PO \perp QO$.**d** Prove that $\angle AOB = 90^\circ$.**e** Prove that $\beta = 2\alpha$.**f** Prove that $AB \parallel DC$.

Example 8

3 These questions may require construction lines to be drawn.**a** Prove that $\alpha + \beta = 120^\circ$.**b** Prove that $\alpha - \beta = 40^\circ$.**c** Prove that $\alpha + \beta = 130^\circ$.

4 This question reviews proofs of theorems about angle sums of triangles.

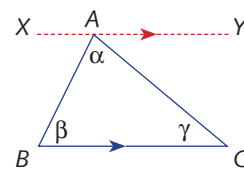
a Prove that: *The sum of the interior angles of a triangle is 180° .*

In $\triangle ABC$ to the right, $\angle BAC = \alpha$, $\angle B = \beta$ and $\angle C = \gamma$.

The line XAY has been constructed parallel to BC .

i Explain why $\angle XAB = \beta$ and $\angle YAC = \gamma$.

ii Hence explain why $\alpha + \beta + \gamma = 180^\circ$.



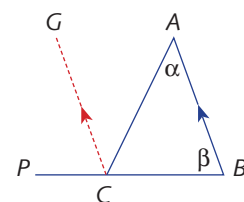
b Prove that: *An exterior angle of a triangle equals the sum of the opposite interior angles.*

In $\triangle ABC$ to the right, $\angle A = \alpha$ and $\angle B = \beta$.

The line CG has been constructed parallel to BA , and the side BC has been produced to P .

i Explain why $\angle ACG = \alpha$ and $\angle PCG = \beta$.

ii Hence explain why $\angle ACP = \alpha + \beta$.



5 Prove that: *An angle in a semicircle is a right angle.*

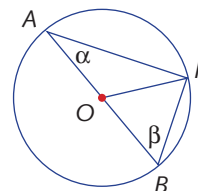
In the circle to the right, AOB is a diameter, and P is any other point on the circle.

Let $\angle A = \alpha$ and $\angle B = \beta$.

a Prove that $\angle APO = \alpha$.

b Prove that $\angle BPO = \beta$.

c Hence prove that $\angle APB = 90^\circ$.



6 **a** Prove that: *The sum of the exterior angles of a triangle is 360° .*

In the diagram to the right, α , β and γ are the sizes of the three exterior angles of $\triangle ABC$.

i Find the sizes of the three interior angles of $\triangle ABC$.

ii Hence show that $\alpha + \beta + \gamma = 360^\circ$.

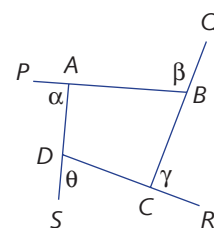
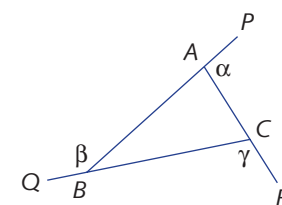
b Draw a quadrilateral $ABCD$. Explain why the sum of the interior angles of $ABCD$ is 360° .

c Prove that: *The sum of the four exterior angles of a convex quadrilateral is 360° .*

In the diagram to the right, α , β , γ and θ are the sizes of the four exterior angles of the convex quadrilateral $ABCD$. (All the interior angles of a convex quadrilateral are less than 180° .)

i Find the sizes of the four interior angles of $ABCD$.

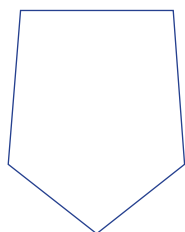
ii Hence show that $\alpha + \beta + \gamma + \theta = 360^\circ$.



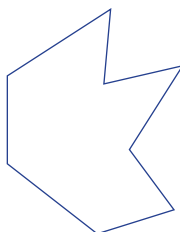
Angle sums of polygons

- 7 a** The diagrams below show a pentagon, an octagon and a dodecagon. By dividing each figure into triangles, find the sum of its interior angles.

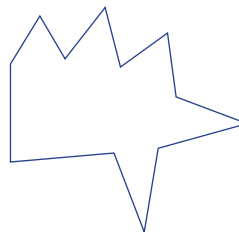
i



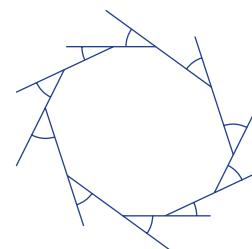
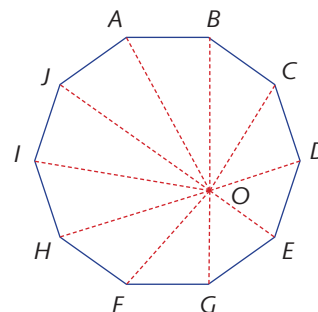
ii



iii

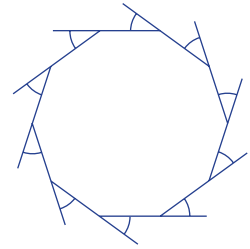


- b** Using a similar technique, find the sum of the interior angles of:
- i** a hexagon **ii** a heptagon **iii** a nonagon **iv** a decagon.
- c** Using these examples as a guide, find a formula for the angle sum of an n -sided polygon.
- 8** In a convex polygon, all the interior angles are less than 180° .
- a** The diagram to the right shows a convex decagon, and any point O inside the decagon. Intervals are drawn from O to each of the ten vertices, dissecting the decagon into ten triangles.
- i** What is the sum of the interior angles of all ten triangles so formed?
- ii** Hence find the sum of the interior angles of the decagon.
- b** Draw a convex hexagon and repeat the process.
- c** Using the same technique, find a formula for the sum of the interior angles of a convex polygon with n sides.
- 9 a** In the convex decagon to the right, each side has been produced to form an exterior angle.
- i** Explain why the sum of the interior angles plus the sum of the exterior angles is 1800° .
- ii** Hence find the sum of the ten exterior angles.
- b** Draw a convex hexagon and repeat the process.
- c** Using the same technique, find the sum of the exterior angles of a convex polygon with n sides.



- 10** In the previous questions, you established that the sum of the interior angles of a decagon is 1440° , and that the sum of the exterior angles of a convex decagon is 360° .

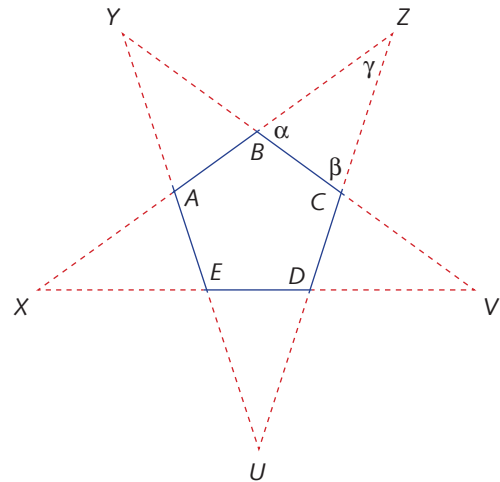
a The diagram to the right shows a regular decagon, with each side produced to form an exterior angle.



- i** Find the size of each interior angle.
 - ii** Find the size of each exterior angle.
- b** Draw a regular hexagon and repeat the process.
- c** Using the same technique, find a formula for the size of each interior angle, and of each exterior angle, of a regular polygon with n sides.
- 11** The figure shown is a regular pentagram. It is formed by producing the sides of a regular pentagon.

a Find the size of each interior angle of the pentagon $ABCDE$.

b Find the values of α , β and γ .





In *ICE-EM Mathematics Year 8 Book 2* we introduced the idea of **congruent figures**.



Congruent figures

- Two plane figures are called **congruent** if one figure can be moved on top of the other figure, by a sequence of translations, rotations and reflections, so that they coincide exactly.
- Congruent figures have exactly the same shape and size.
- When two figures are congruent, we can match every part of one figure with the corresponding part of the other, so that:
 - matching angles have the same size
 - matching paired intervals have the same length
 - matching paired regions have the same area.

The congruence arguments used in this chapter involve only congruent triangles. In Year 8 we developed four tests for two triangles to be congruent.



The four standard congruence tests for triangles

Two triangles are **congruent** if:

SSS: the three sides of one triangle are respectively equal to the three sides of the other triangle, **or**

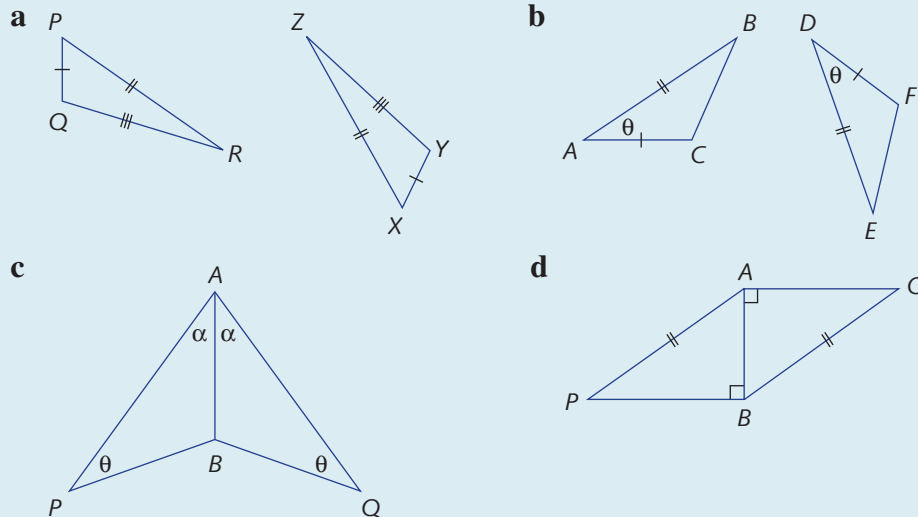
AAS: two angles and one side of one triangle are respectively equal to two angles and the matching side of the other triangle, **or**

SAS: two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other triangle, **or**

RHS: the hypotenuse and one side of one right-angled triangle are respectively equal to the hypotenuse and one side of the other right-angled triangle.

Example 10

In each part, write a congruence statement, giving a test as the reason. Make sure that you write the vertices of the two triangles in matching order.



Solution

- a $\triangle PQR \equiv \triangle XYZ$ (SSS) b $\triangle BAC \equiv \triangle EDF$ (SAS)
 c $\triangle ABP \equiv \triangle ABQ$ (AAS) d $\triangle ABQ \equiv \triangle BAP$ (RHS)

Notice that in parts **c** and **d**, the side AB is **common** to both triangles, and thus provides one of the pairs of equal sides.

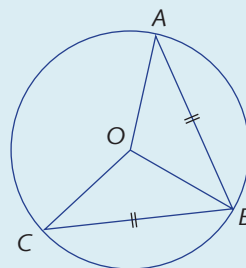
Using congruence to prove that two matching sides or angles are equal

Once we have established that two triangles are congruent, we know that the remaining matching sides and angles are equal, as in the following example.

Example 11

In the diagram to the right, AB and BC are equal chords of a circle with centre O .

- a Prove that $\triangle AOB \equiv \triangle COB$.
 b Hence prove that $\angle AOB = \angle COB$.



(continued on next page)

Solution

a In the triangles AOB and COB :

$$OA = OC \quad (\text{radii})$$

$$OB = OB \quad (\text{common})$$

$$AB = CB \quad (\text{given})$$

$$\text{so } \triangle AOB \equiv \triangle COB \quad (\text{SSS})$$

b Hence $\angle AOB = \angle COB$ (matching angles of congruent triangles)

Using congruence to prove that two lines are parallel

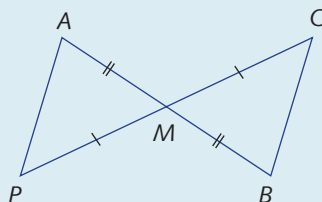
In many situations, congruence is used to prove that two alternate angles, or two corresponding angles, are equal. This allows us to prove that two lines are parallel.

Example 12

In the diagram to the right,
 $AM = BM$ and $PM = QM$

a Prove that $\triangle AMP \equiv \triangle BMQ$.

b Prove that $AP \parallel BQ$.



Solution

a In the triangles AMP and BMQ :

$$AM = BM \quad (\text{given})$$

$$PM = QM \quad (\text{given})$$

$$\angle AMP = \angle BMQ \quad (\text{vertically opposite angles at } M)$$

$$\text{so } \triangle AMP \equiv \triangle BMQ \quad (\text{SAS})$$

b Hence $\angle APM = \angle BQM$ (matching angles of congruent triangles)

so $AP \parallel BQ$ (alternate angles are equal)

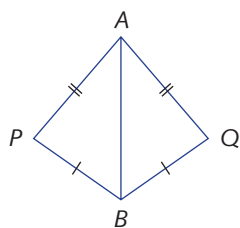


Exercise 7C

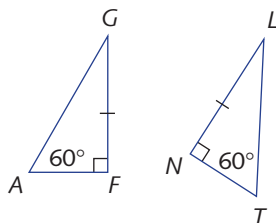
Example 10

- 1 In each part, write a congruence statement, giving a congruence test as the reason. Make sure that you name the vertices in matching order.

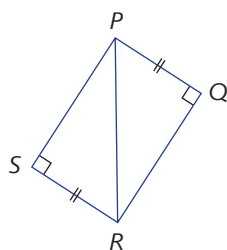
a



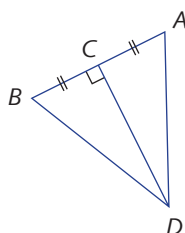
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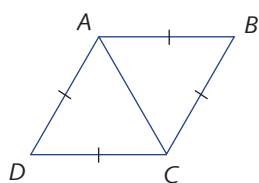
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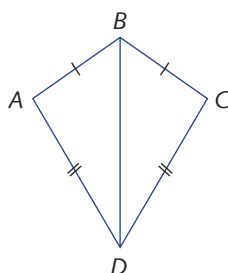
d



e

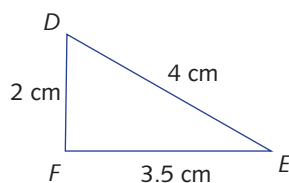
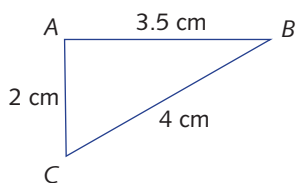


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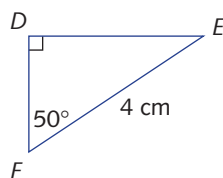
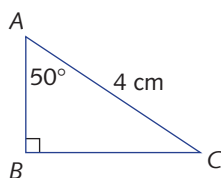


- 2 In each part, write a congruence statement, giving a congruence test as the reason.

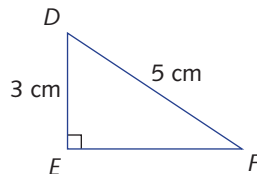
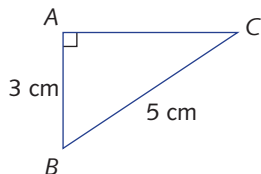
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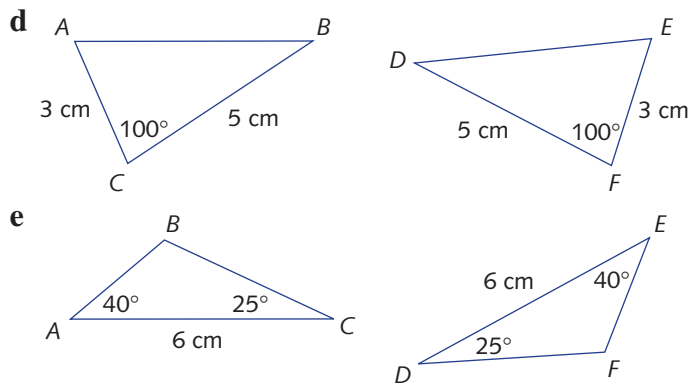


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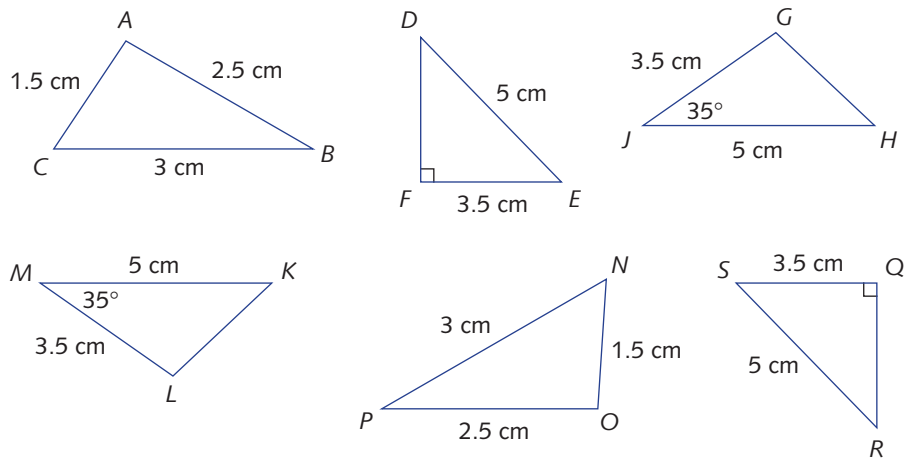


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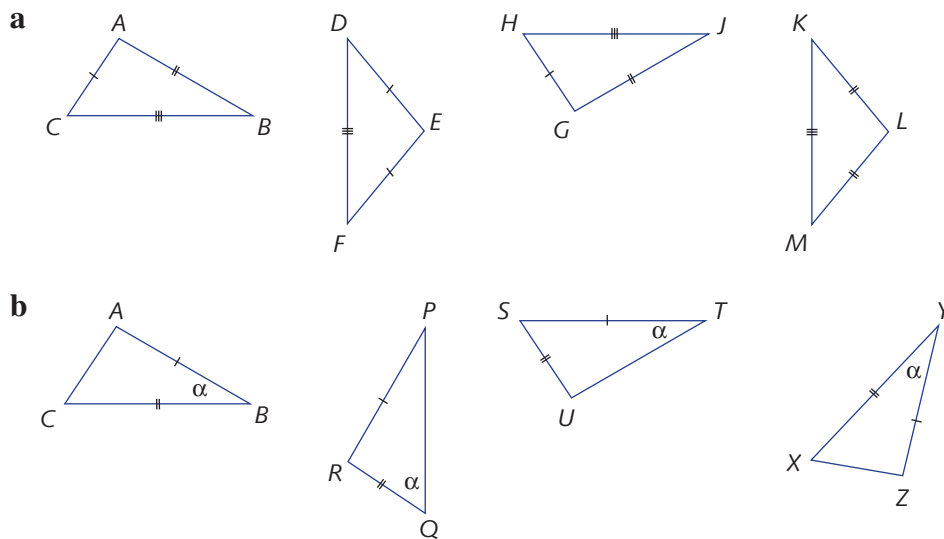


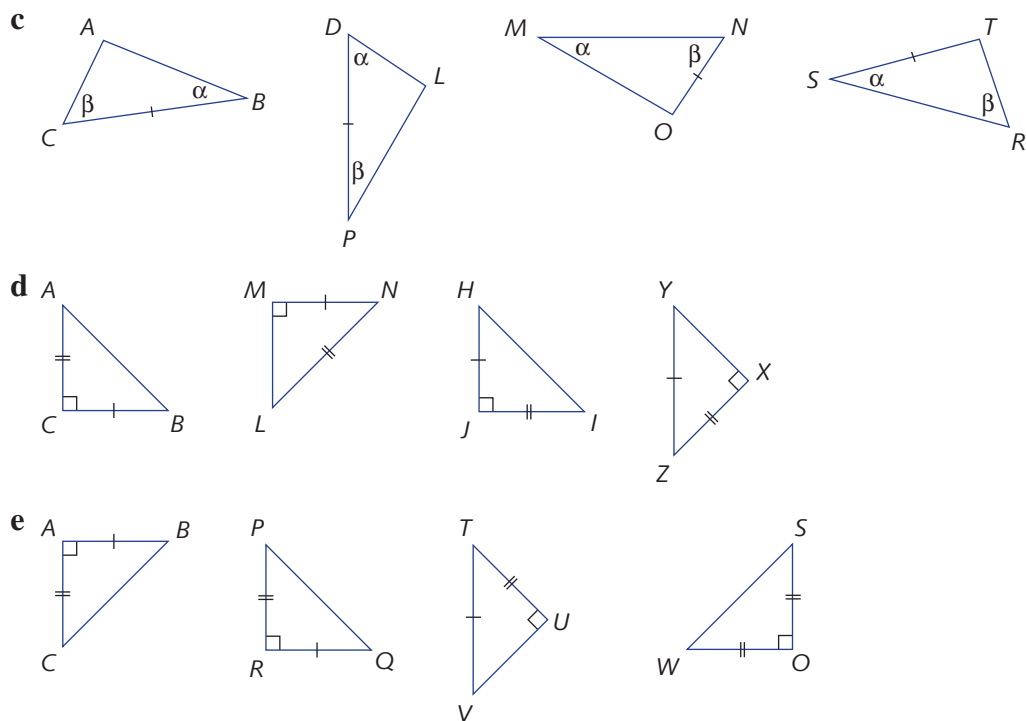


3 From the six triangles drawn, name three pairs of congruent triangles. Give reasons.



4 In each of the following, state which triangle is congruent to $\triangle ABC$, giving the appropriate congruence test as a reason.





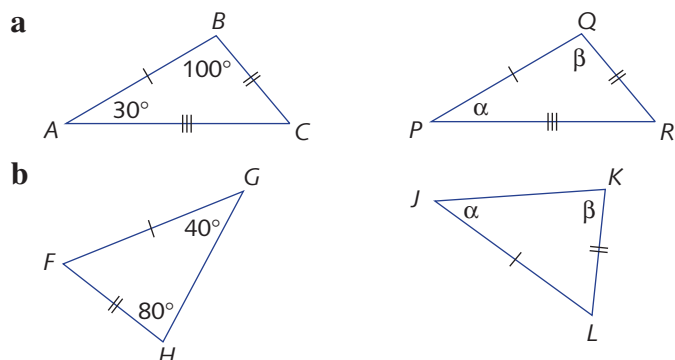
5 a If $\triangle ABC \equiv \triangle PQR$ (where the vertices are in matching order):

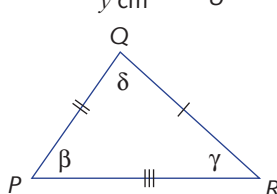
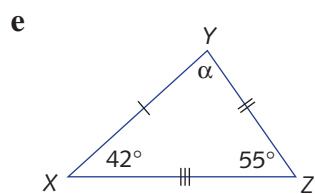
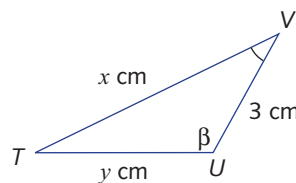
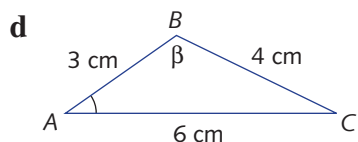
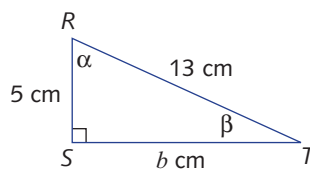
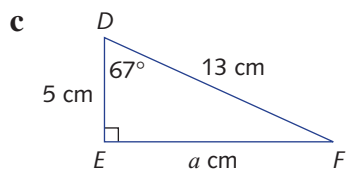
- i which angle in $\triangle PQR$ is equal in size to $\angle ABC$?
- ii which angle in $\triangle PQR$ is equal in size to $\angle CAB$?
- iii which side in $\triangle PQR$ is equal in length to AC ?
- iv which side in $\triangle ABC$ is equal in length to QR ?

b If $\triangle LMN \equiv \triangle TUV$:

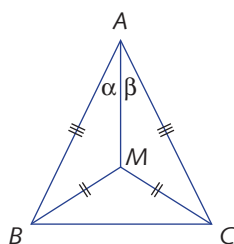
- i which angle in $\triangle TUV$ is equal in size to $\angle LNM$?
- ii which angle in $\triangle LMN$ is equal in size to $\angle UTV$?
- iii which side in $\triangle TUV$ is equal in length to MN ?
- iv which side in $\triangle LMN$ is equal in length to UT ?

6 Find the values of the pronumerals.

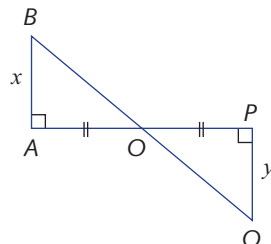




7 a

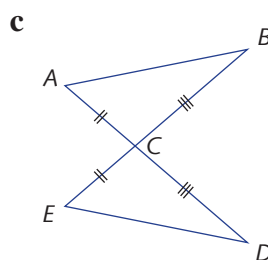


b

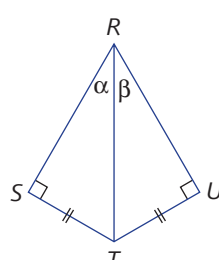


- i** Prove that $\triangle ABM \equiv \triangle ACM$.
ii Hence prove that $\alpha = \beta$.

- i** Prove that $\triangle OAB \equiv \triangle OPQ$.
ii Hence prove that $x = y$.

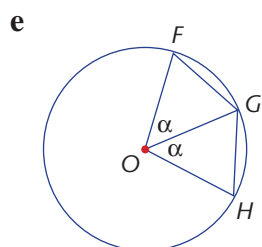


d

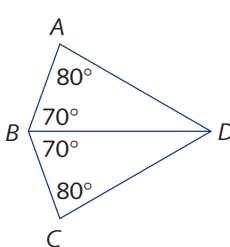


- i** Prove that $\triangle ABC \equiv \triangle EDC$.
ii Hence prove that $\angle B = \angle D$.

- i** Prove that $\triangle RTS \equiv \triangle RTU$.
ii Prove that $\alpha = \beta$.



f

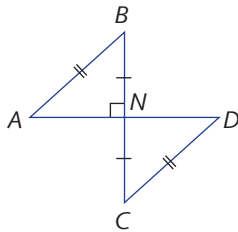


Use congruence to prove that $FG = GH$, given that O is the centre of the circle.

Use congruence to prove that $AD = DC$ and that $AB = BC$.

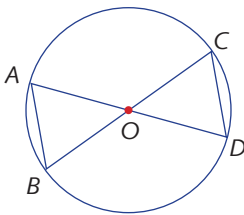
- 8 AB is a chord of a circle centre O and M is the midpoint of AB . Prove that OM is perpendicular to AB .

9 a



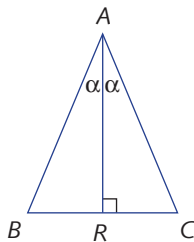
- i Prove that $\triangle NAB \equiv \triangle NDC$.
- ii Hence prove that $\angle A = \angle D$.
- iii Hence prove that $AB \parallel CD$.

c



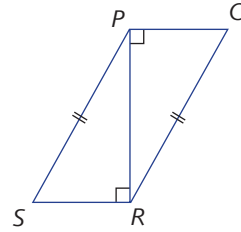
Use congruence to prove that $AB \parallel CD$, given that O is the centre of the circle.

10 a



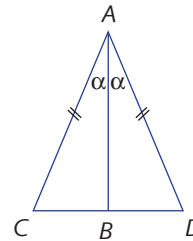
Use congruence to prove that $\triangle ABC$ is isosceles.

b



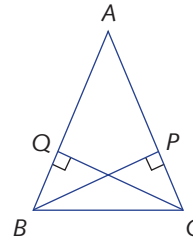
- i Give a reason why $PQ \parallel SR$.
- ii Prove that $\triangle PRS \equiv \triangle RPQ$.
- iii Hence prove that $PS \parallel QR$.

d



Prove that $\angle ABC = \angle ABD$ using congruence, and hence prove that $AB \perp CD$.

b

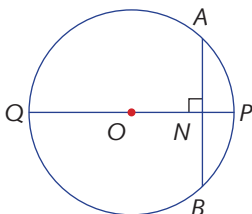


Use congruence to prove that $\triangle ABC$ is isosceles, given that $BP = CQ$.

- 11 In the triangle $\triangle ABC$, $AB = AC$. The bisectors of angles ABC and ACB meet at X . Prove that $XB = XC$.

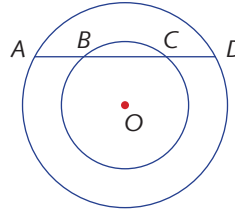
- 12 In each part, O is the centre of the circle or circles. You will need to construct radii and use congruence in these questions.

a



Prove that PQ bisects AB .

b



Prove that $AB = CD$.

13 This question reviews proofs of several theorems about isosceles and equilateral triangles.

a Prove that: *The base angles of an isosceles triangle are equal.*

The triangle ABC to the right is isosceles, with $AB = AC$.

The bisector of $\angle BAC$ meets the base BC at M .

i Prove that $\triangle ABM \equiv \triangle ACM$.

ii Hence prove that $\angle B = \angle C$.

b Conversely, prove that: *If two angles of a triangle are equal, then the sides opposite those angles are equal.*

In the triangle ABC to the right, $\angle B = \angle C$.

The bisector of $\angle BAC$ meets the base BC at M .

i Prove that $\triangle ABM \equiv \triangle ACM$.

ii Hence prove that $AB = AC$, that is, $\triangle ABC$ is isosceles.

c Prove that: *Each interior angle of an equilateral triangle is 60° .*

The triangle ABC to the right is equilateral. Let $\angle A = \theta$.

i Give a reason why $\angle B = \theta$.

ii Give a reason why $\angle C = \theta$. **iii** Explain why $\theta = 60^\circ$.

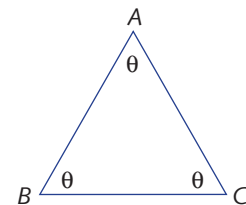
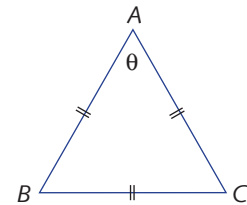
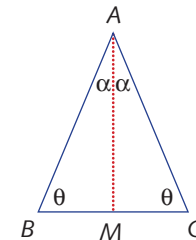
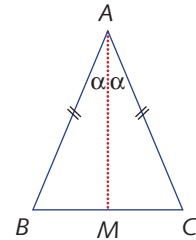
d Conversely, prove that: *If all angles of a triangle are equal, then the triangle is equilateral.*

In the triangle ABC to the right, all the interior angles are equal.

Let $\angle A = \angle B = \angle C = \theta$.

i Give a reason why $AB = AC$.

ii Give a reason why $AC = BC$.



14 This question reviews proofs by congruence of three well-known properties of parallelograms.

Recall that a **parallelogram** is a quadrilateral whose opposite sides are parallel.

a Prove that: *Opposite angles of a parallelogram are equal.*

A parallelogram $ABCD$ is drawn to the right. Let $\angle A = \alpha$.

i Find the sizes of $\angle B$ and $\angle D$ in terms of α .

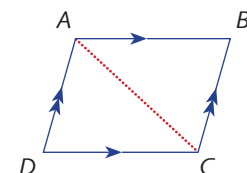
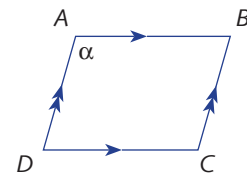
ii Hence find the size of $\angle C$ in terms of α .

b Prove that: *Opposite sides of a parallelogram are equal.*

Let $ABCD$ be a parallelogram with the diagonal AC joined.

i Prove that $\triangle ABC \equiv \triangle CDA$.

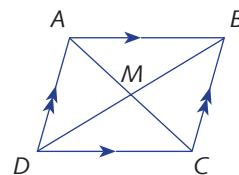
ii Hence prove that $AB = DC$ and $AD = BC$.



- c Prove that: *The diagonals of a parallelogram bisect each other.*

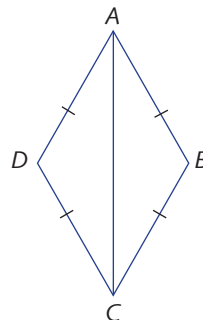
Let the diagonals of the parallelogram $ABCD$ meet at M .

- i Prove that $\triangle ABM \equiv \triangle CDM$.
- ii Hence prove that $AM = CM$ and $BM = DM$.



- 15 This question reviews proofs by congruence of some diagonal properties of rhombuses and rectangles. Recall that a **rhombus** is a quadrilateral with all sides equal.

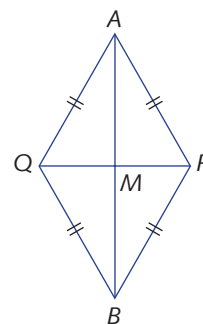
- a Prove that: *A rhombus is a parallelogram.*



- b Prove that: *The diagonals of a rhombus are perpendicular, and bisect the vertex angles through which they pass.*

Let the diagonals of the rhombus $APBQ$ meet at M .

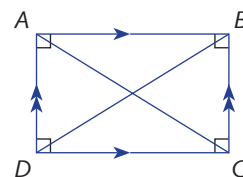
- i Prove that $\triangle APB \equiv \triangle AQB$.
- ii Hence prove that $\angle PAM = \angle QAM$.
- iii Prove also that $AM \perp PQ$.



- c Prove that: *The diagonals of a rectangle are equal.*

Let $ABCD$ be a rectangle with the diagonals joined.

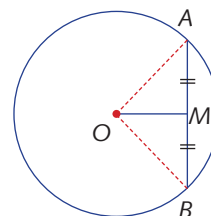
- i Prove that $\triangle ADC \equiv \triangle BCD$.
- ii Hence prove that $AC = BD$.



- 16 Prove that: *The line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord.*

Let M be the midpoint of a chord AB of a circle with centre O .

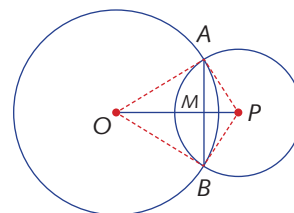
- a Prove that $\triangle OMA \equiv \triangle OMB$.
- b Hence prove that $OM \perp AB$.



- 17 Prove that: *When two circles intersect, the line joining their centres is the perpendicular bisector of their common chord.*

In the diagram to the right, AB is the common chord of two intersecting circles with centres O and P . The chord AB meets the line OP at M .

- a Prove that $\triangle AOP \equiv \triangle BOP$.
- b Hence prove that OP bisects $\angle AOB$.
- c Prove that $\triangle AOM \equiv \triangle BOM$.
- d Hence prove that $AM = BM$ and $AB \perp OP$.





In Year 8 we defined a parallelogram and discovered some of its important properties.

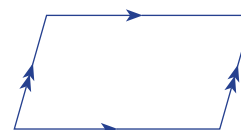
Parallelograms

Definition of a parallelogram

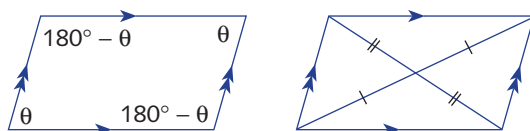
- A **parallelogram** is a quadrilateral whose opposite sides are parallel.

Properties of a parallelogram

- The opposite angles of a parallelogram are equal.
- The opposite sides of a parallelogram are equal.
- The diagonals of a parallelogram bisect each other.



The diagrams below illustrate these properties. Their proofs were reviewed in Exercise 7C, question 14.



The purpose of this section is to establish four well-known tests for a quadrilateral to be a parallelogram. The statement of each test has the form

‘If . . . , then the quadrilateral is a parallelogram.’

Four tests for a parallelogram

- If the opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.
- If the opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.
- If one pair of opposite sides of a quadrilateral are equal and parallel, then the quadrilateral is a parallelogram.
- If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

The first question in the following exercise reviews proofs of these tests. The second and third questions use the four tests to establish that various geometric and practical constructions do indeed result in parallelograms.



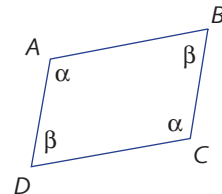
Exercise 7D

1 This question reviews the proofs of the four tests for a quadrilateral to be a parallelogram.

- a** Prove that: *If the opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.*

Let $ABCD$ be a quadrilateral with $\angle A = \angle C = \alpha$ and $\angle B = \angle D = \beta$.

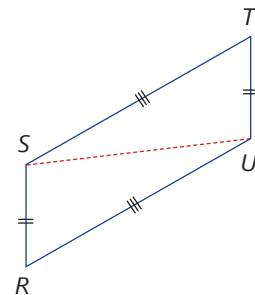
- Prove that $\alpha + \beta = 180^\circ$.
- Hence prove that $AB \parallel DC$ and $AD \parallel BC$.



- b** Prove that: *If the opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.*

Let $RSTU$ be a quadrilateral with $RS = UT$ and $RU = ST$.

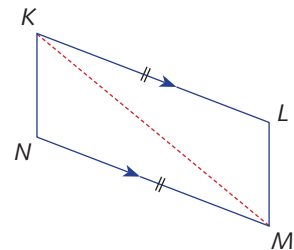
- Prove that $\triangle RSU \equiv \triangle TUS$.
- Hence prove that $RS \parallel UT$ and $RU \parallel ST$.
- Explain why this proves that $RSTU$ is a parallelogram.



- c** Prove that: *If one pair of opposite sides of a quadrilateral are equal and parallel, then the quadrilateral is a parallelogram.*

Let $KLMN$ be a quadrilateral with $KL = NM$ and $KL \parallel NM$.

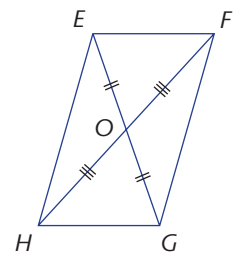
- Prove that $\triangle KLM \equiv \triangle MNK$.
- Hence prove that $KN \parallel LM$.
- Explain why this proves that $KLMN$ is a parallelogram.



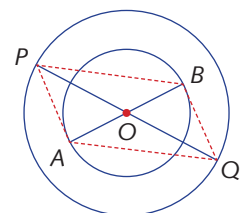
- d** Prove that: *If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.*

Let $EFGH$ be a quadrilateral whose diagonals meet at O and bisect each other.

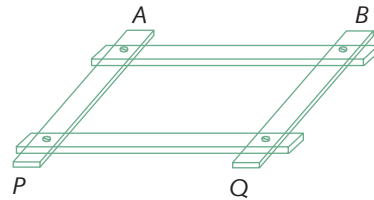
- Prove that $\triangle EHO \equiv \triangle GFO$.
- Hence prove that $EH = FG$ and $EH \parallel FG$.
- Explain why this proves that $EFGH$ is a parallelogram.



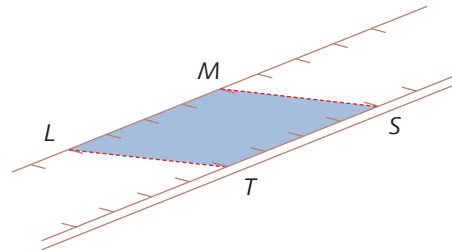
- 2 a** In the diagram to the right, both circles have centre O . The interval AOB is a diameter of the smaller circle, and POQ is a diameter of the larger circle. What sort of figure is $APBQ$, and why?



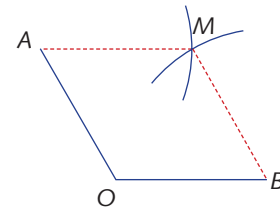
- b** The diagram shows four metal rods bolted together. The rods AB and PQ have equal length, and the rods AP and BQ have equal length. What sort of figure is $ABQP$, and why?



- c** The opposite edges of my ruler are parallel. I draw intervals LM and ST of length 4 cm on the opposite edges of the ruler, as shown in the diagram to the right. What sort of figure is $LMST$, and why?

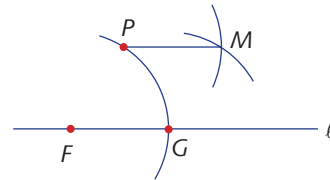


- 3 a** In the diagram to the right, $\angle AOB$ is any given angle. First I draw an arc with centre A and radius OB . Then I draw an arc with centre B and radius OA . The two arcs meet at M , and I claim that $AMBO$ is a parallelogram. Am I correct?



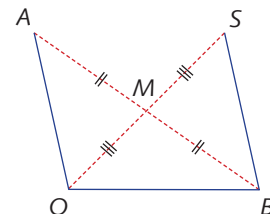
Explain why or why not.

- b** Let ℓ be any line and P any point not on the line. I choose any point F on the line and draw three arcs, each with the same radius FP . The first arc has centre F and meets the line again at G . The second and third have centres P and G and meet at M . Am I correct in claiming that $PMGF$ is a parallelogram?



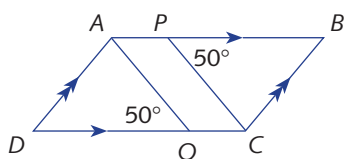
Why or why not?

- c** In the diagram to the right, $\angle AOB$ is any given angle. First I join AB and construct its midpoint M . Then I construct the point S on OM produced so that $OM = MS$. Is it true that $BSAO$ is a parallelogram?



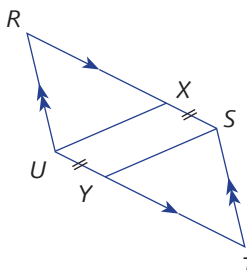
Why or why not?

4 a

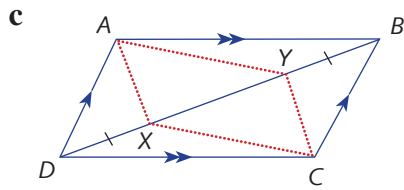


Given that $ABCD$ is a parallelogram, prove that $AP = QC$.

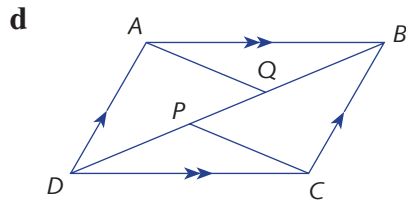
b



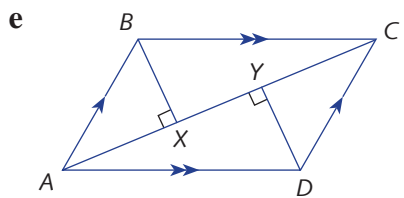
Given that $RSTU$ is a parallelogram, prove that $UX \parallel YS$.



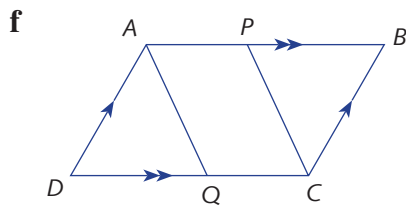
$ABCD$ is a parallelogram with X and Y points on the diagonal AC such that $BY = DX$. Prove that $AYCX$ is a parallelogram.



$ABCD$ is a parallelogram and $PB = DQ$. Prove that $AQ = CP$.



$ABCD$ is a parallelogram and BX and DY are perpendicular to AC . Prove that $BX = DY$.



$ABCD$ is a parallelogram with $BP = DQ$. Prove that $APCQ$ is a parallelogram.

7E Tests for rhombuses, rectangles and squares



Rhombuses, rectangles and squares are special types of parallelograms. Last year we defined these figures and proved some of their properties. This section establishes some important tests for these figures.

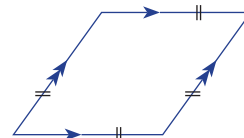
Rhombuses

Definition of a rhombus

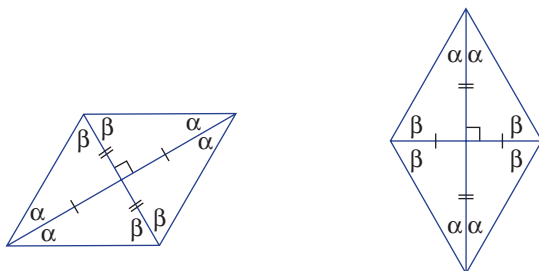
- A **rhombus** is a quadrilateral with four equal sides.

Diagonal properties of a rhombus

- The diagonals of a rhombus bisect each other at right angles.
- The diagonals of a rhombus bisect the vertex angles through which they pass.



These diagonal properties of a rhombus are far clearer when it is drawn in a diamond orientation, with its diagonals vertical and horizontal as illustrated below.



Tests for a rhombus

- If a quadrilateral is a parallelogram with two adjacent sides equal, then the parallelogram is a rhombus.
- If the diagonals of a quadrilateral bisect each other at right angles, then the quadrilateral is a rhombus.

The tests are proved, in question 1 of Exercise 7E.

Note: A rhombus is often defined to be a parallelogram with two adjacent sides equal.



Rectangles

Definition of a rectangle

- A **rectangle** is a parallelogram whose interior angles are right angles.

Diagonal properties of a rectangle

- The diagonals of a rectangle are equal and bisect each other.

Tests for a rectangle

- A parallelogram with one right angle is a rectangle.
- If all angles of a quadrilateral are equal, then the quadrilateral is a rectangle.
- If the diagonals of a quadrilateral are equal and bisect each other, then the quadrilateral is a rectangle.

The proof that a parallelogram with one right angle is a rectangle is proved in question **2a** of Exercise 7E. The second test is proved in question **2b** of Exercise 7E. The third test is proved in question **1c** of Exercise 7E.

Note: A rectangle is often defined to be a parallelogram with one right angle.



Squares

Definition of a square

- A **square** is a quadrilateral that is both a rectangle and a rhombus. Thus a square has all the properties of a rectangle and a rhombus.

Properties of a square

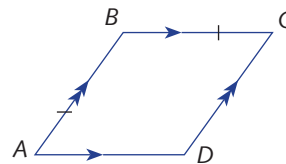
- All angles of a square are right angles.
- All sides of a square are equal.
- The diagonals of a square are equal and bisect each other at right angles.
- Each diagonal meets each side at 45° .

A square is often defined as a regular polygon with four vertices; that is, as a quadrilateral with four equal sides and four equal interior angles. Note that such a figure is a parallelogram, because its opposite sides are equal, so the two definitions agree.



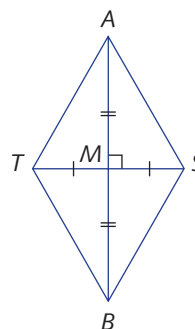
Exercise 7E

- 1 a Prove that: *If a parallelogram has two adjacent sides equal, then the parallelogram is a rhombus.*



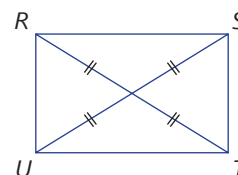
- b Prove that: *If the diagonals of a quadrilateral bisect each other at right angles, then the quadrilateral is a rhombus.*

The diagonals of the quadrilateral $ASBT$ meet at right angles at M and bisect each other.



- Why is $ASBT$ a parallelogram?
 - Prove that $\triangle AMS \equiv \triangle AMT$.
 - Hence prove that $AS = AT$, that is, $ASBT$ is a rhombus.
- c Prove that: *If the diagonals of a quadrilateral have equal length and bisect each other, then the quadrilateral is a rectangle.*

The diagonals of the quadrilateral $RSTU$ have equal length and bisect each other.

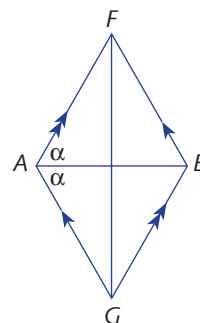


- Why is $RSTU$ a parallelogram?
 - Prove that $\triangle URS \equiv \triangle TSR$.
 - Hence prove that $\angle URS = \angle TSR$.
 - Hence prove that $\angle URS = 90^\circ$.
- 2 a Prove that: *A parallelogram with one right angle is a rectangle.*
- b Prove that: *If all angles of a quadrilateral are equal, then the quadrilateral is a rectangle.*
- 3 Prove that: *If one diagonal of a parallelogram bisects one vertex angle, then the parallelogram is a rhombus.*

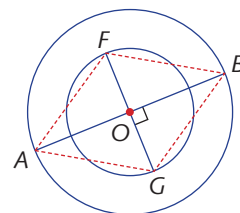
Let $AFBG$ be a parallelogram in which the diagonal AB bisects the vertex angle $\angle GAF$.

Let $\angle FAB = \angle GAB = \alpha$.

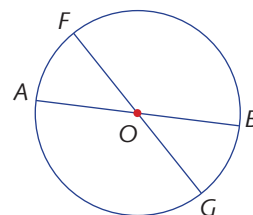
- Prove that $\angle ABF = \alpha$.
- Hence prove that $AFBG$ is a rhombus.



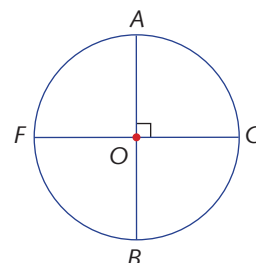
- 4 a In the diagram to the right, O is the centre of both circles. The diameters AB and FG are perpendicular. What sort of figure is $AFBG$, and why?



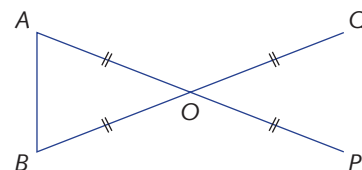
- b In the diagram to the right, O is the centre of the circle, and AB and FG are diameters of the circle. What sort of figure is $AFBG$, and why?



- c In the diagram to the right, O is the centre of the circle, and AB and FG are perpendicular diameters of the circle. What sort of figure is $AFBG$, and why?



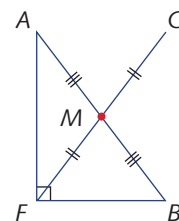
- 5 Let ABO be any isosceles triangle, with $AO = BO$. Mark a point P on AO produced so that $AO = OP$, and a point Q on BO produced, so that $BO = OQ$. What sort of figure is $ABPQ$, and why?



- 6 Prove that: *The circle with centre the midpoint of the hypotenuse of a right-angled triangle, and passing through the endpoints of the hypotenuse, also passes through the third vertex.*

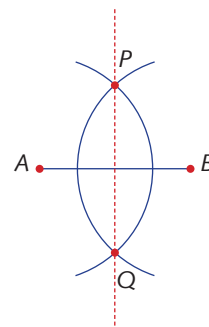
Let $\triangle ABF$ be right-angled at F , and let M be the midpoint of the hypotenuse AB . Produce FM to G so that $FM = MG$.

- a Explain why $AFBG$ is a rectangle.
b Explain why $AM = BM = FM = GM$.
c Hence explain why the circle with centre M and radius AM passes through F .

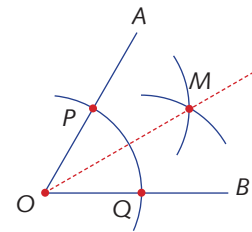


- 7 This question justifies some of the constructions that you learnt in earlier years.

- a The diagram shows the standard construction of the perpendicular bisector of a given interval AB . We draw two arcs with centres A and B respectively, equal radius, intersecting at P and Q .
i Explain why $APBQ$ is a rhombus.
ii Hence explain why PQ is the perpendicular bisector of AB .

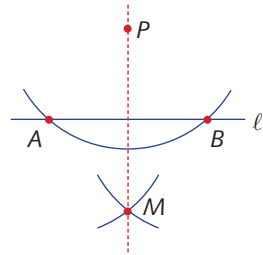


- b** The diagram shows the standard construction of the bisector of a given angle $\angle AOB$. We draw three arcs with centres O , P and Q respectively, and equal radius.



- i** Explain why $OPMQ$ is a rhombus.
- ii** Hence explain why OM is the bisector of $\angle AOB$.

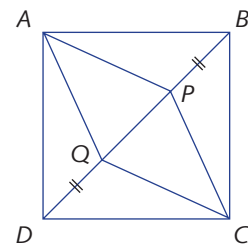
- c** The diagram shows the standard construction of the perpendicular from a given point P to a given line ℓ . First draw an arc with centre P cutting ℓ at A and B . Then with centres A and B and the same radius, draw arcs meeting at M .



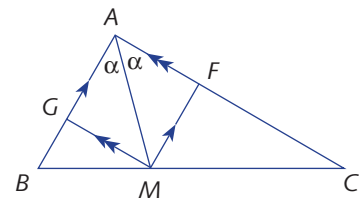
- i** Explain why $APBM$ is a rhombus.
- ii** Hence explain why PM is perpendicular to ℓ .

- 8 a** The points P and Q are chosen on the diagonal BD of the square $ABCD$ so that $BP = DQ$.

- i** Prove that the triangles ABP , CBP , ADQ and CDQ are all congruent.
- ii** Hence prove that $APCQ$ is a rhombus.



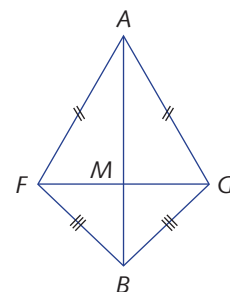
- b** In $\triangle ABC$ to the right, MA is the bisector of $\angle BAC$. The line MF is parallel to BA , and MG is parallel to CA . Prove that $AFMG$ is a rhombus.



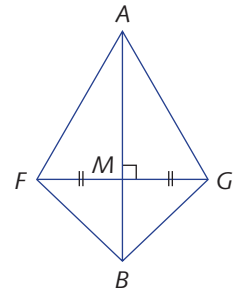
- 9** A **kite** is a quadrilateral with two pairs of equal adjacent sides. As with a rhombus, the properties of a kite are best seen when it is drawn in a diamond orientation, as in the diagram.

- a** Let $AFBG$ be a kite with $AF = AG$ and $BF = BG$. Let the diagonals AB and FG meet at M .

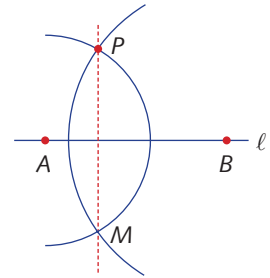
- i** Show that $\triangle ABF \equiv \triangle ABG$.
- ii** Hence show that AB bisects $\angle FAG$ and $\angle FBG$.
- iii** Show also that $\angle AFB = \angle AGB$.
- iv** Show that $\triangle AMF \equiv \triangle AMG$.
- v** Hence show that the diagonals are perpendicular.



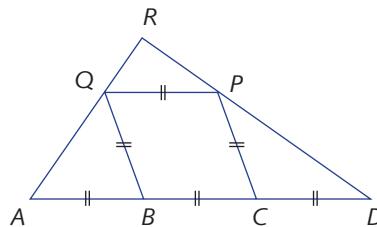
- b Conversely, let $AFBG$ be a quadrilateral in which the diagonal AB bisects the diagonal FG and is perpendicular to it. Prove that $AFBG$ is a kite.



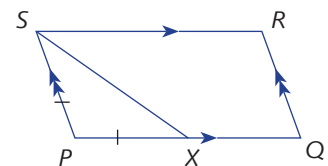
- c Let ℓ be a line and P a point not on the line. Choose two points A and B on ℓ . Draw an arc with centre A and radius AP . Then draw a second arc with centre B and radius BP . Let the arcs meet at M . Explain why PM is perpendicular to ℓ .



- 10 $ABCD$ is a straight line such that $AB = BC = CD$ and $BCPQ$ is a rhombus. The lines produced through AQ and DP meet at R . Prove that $\angle ARD$ is a right angle.



- 11 $PQRS$ is a parallelogram. X is a point on PQ such that $PX = SP$. Prove that SX bisects $\angle RSP$.



- 12 A **trapezium** is a quadrilateral with one pair of parallel lines.

- a The trapezium $ABCD$ has $\angle C = 70^\circ$ and $\angle D = 50^\circ$. Find the sizes of $\angle A$ and $\angle B$.

If the non-parallel sides of a trapezium are equal, then the figure is called an **isosceles trapezium**.

- b Prove that the base angles of an isosceles trapezium are equal.
c Prove that the diagonals of an isosceles trapezium are equal in length.
d Prove that if a trapezium has equal diagonals then it is isosceles.



Review exercise

1 Classify these angles according to their size.

- a 78° b 180° c 90° d 340° e 360° f 162°

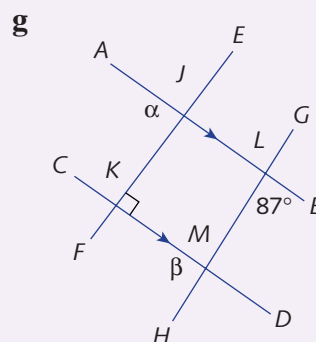
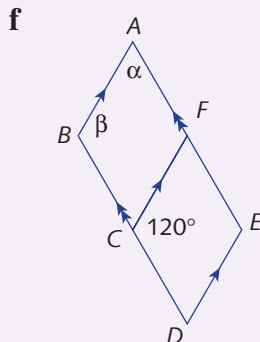
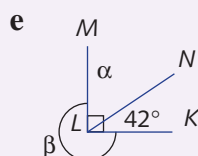
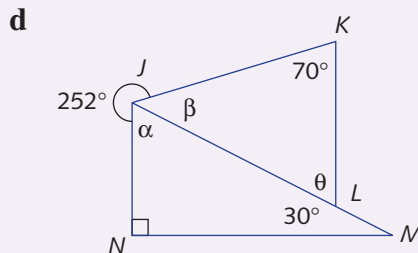
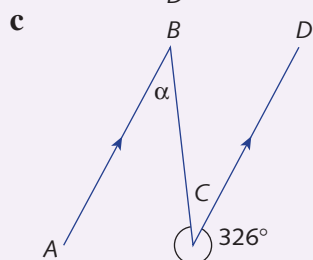
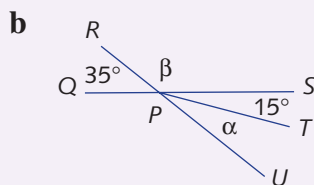
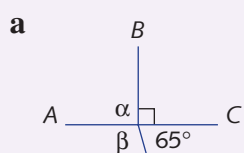
2 Write down the complements of:

- a 56° b 67° c 47° d 15° e 26° f 46°

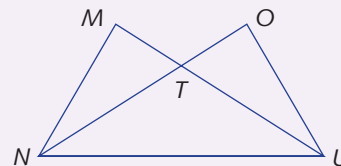
3 Write down the supplements of:

- a 154° b 89° c 34° d 90° e 113° f 116°

4 Find the angles marked with pronumerals. Give reasons for your answers.



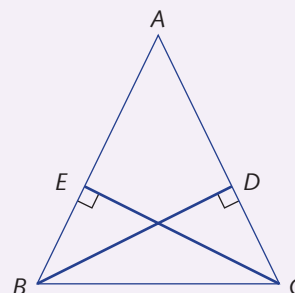
5 In the diagram to the right, $MT = OT$ and $\angle MNT = \angle OUT$. Prove that $NT = UT$.



6 In the diagram to the right, $\triangle ABC$ is isosceles with $AB = AC$, $BD \perp AC$ and $CE \perp AB$.

a Prove that $\triangle BDC$ is congruent to $\triangle BEC$.

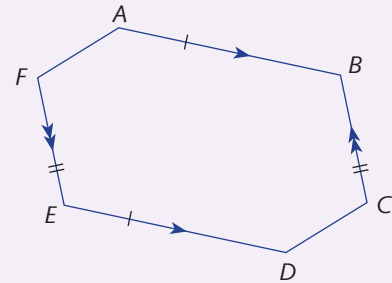
b Prove that $\triangle AED$ is isosceles.





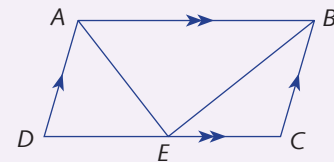
- 7 $ABCDEF$ is a hexagon in which the interval AB is parallel and equal in length to the interval ED , and the interval BC is parallel and equal in length to the interval FE .

- a Join B to E and prove that $\angle ABC = \angle FED$.
- b Prove that $ACDF$ is a parallelogram.



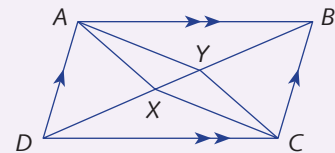
- 8 In a parallelogram $ABCD$, the point E lies on the side DC such that EA bisects $\angle BAD$ and EB bisects $\angle ABC$. Prove that:

- a $DE = EC$
- b $DC = 2CB$
- c $\angle AEB$ is a right angle.



- 9 In the parallelogram $ABCD$, the points X and Y are on the diagonal DB such that $DX = DA$ and $BY = BC$. Prove that:

- a $AX = CY$
- b $AY = CX$
- c $AYCX$ is a parallelogram.

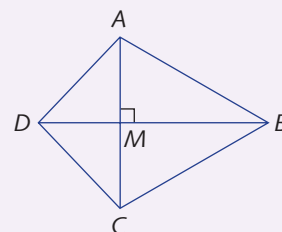




Challenge exercise

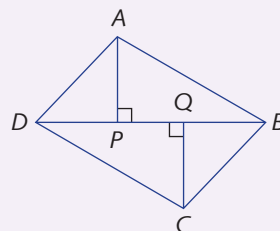
- 1 a Prove that: *If the diagonals of a quadrilateral are perpendicular, then the sums of the squares of opposite sides are equal.*

The diagonals of the convex quadrilateral $ABCD$, to the right, meet at right angles at M .
Prove that $AB^2 + CD^2 = AD^2 + BC^2$.



- b Conversely, prove that: *If the sums of the squares of opposite sides of a quadrilateral are equal, then the diagonals are perpendicular.*

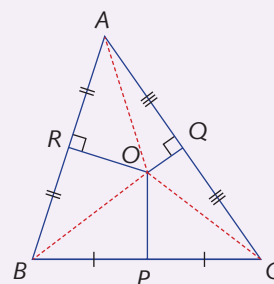
In the convex quadrilateral $ABCD$ to the right, $AB^2 + CD^2 = AD^2 + BC^2$. The perpendiculars to the diagonal BD from A and C meet BD at P and Q respectively. Prove that the points P and Q coincide.



- c What happens if the quadrilateral is non-convex?
- 2 The circle passing through all three vertices of a triangle is called a **circumcircle** of the triangle.

The following theorem not only shows that every triangle has a unique circumcircle, but also shows how to construct the circumcircle.

Prove that: *The three perpendicular bisectors of a triangle are concurrent. Their point of intersection (called the circumcentre) is the centre of a circle (called a circumcircle).*

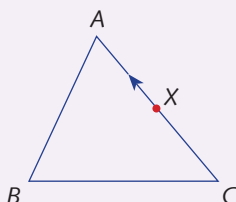


Prove this theorem using congruence. In the diagram above, the midpoints of BC , CA and AB are P , Q and R respectively.

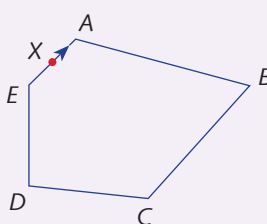
- 3 In Exercise 7B, you proved that the sum of the exterior angles of a convex polygon is 360° . This question introduces another way of looking at this theorem.

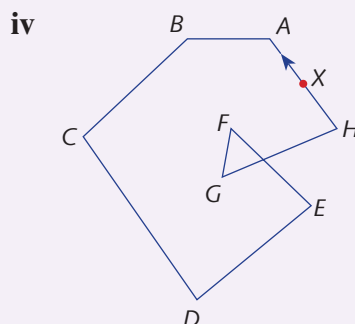
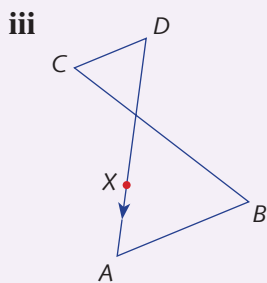
- a In each diagram below, imagine that you started at the point X , facing A , and walked around the figure back to X , passing through the vertices $A, B, C, \dots$ in alphabetical order. Through how many complete revolutions would you turn during your journey? Count clockwise rotation as negative, and anticlockwise revolutions as positive.

i



ii





- b** Use these ideas to develop a proof that the sum of the exterior angles of a convex polygon is 360° .
- 4** Suppose that you make a journey on the surface of the globe. You start facing east at Singapore, which lies on the Equator. Then you walk east one-quarter of the way around the globe, then turn 90° north and walk to the North Pole, then turn 90° towards Singapore and walk back to Singapore, and finally turn facing east just as you started. Through how many revolutions have you turned during this journey, and what is the sum of the exterior angles of your triangular path?
- 5** The diagonals of a convex quadrilateral dissect the quadrilateral into four triangles.
- a** Prove that: *If the quadrilateral is a trapezium, then a pair of opposite triangles have equal area.*
- b** Conversely, prove that: *If a pair of opposite triangles have equal area, then the quadrilateral is a trapezium.*
- 6 a** Let $ABCD$ be a square. Let Q be a point on DC and R a point on BC , and let $AQ \perp DR$. Prove that $AQ = DR$.
- b** Let $ABCD$ be a square. Let ℓ and m be perpendicular lines such that ℓ intersects the sides AB and CD of the square, and m intersects the sides AD and BC . Show that the intervals cut off on ℓ and m by the square are equal in length.
- 7** Let $ABCD$ be a square of side length 1. Extend the sides AB to E , BC to F , CD to G and DA to H so that $BE = CF = DG = AH = 1$.
- a** Prove that $EFGH$ is a square, and find its area.
- b** Show that the centre O of $ABCD$ is also the centre of $EFGH$.
- 8** Prove that: *Any line through the intersection of the diagonals of a parallelogram dissects the parallelogram into two figures of equal area.*
- 9** Prove that: *The perpendicular bisector of the base of an isosceles triangle passes through its vertex.*
- 10** H and K are the midpoint of the sides AB and AC of $\triangle ABC$. Points H and K are joined and the line produced to X so that $HK = KX$. Prove that:
- a** CX is equal and parallel to BH .
- b** $HK = \frac{1}{2}BC$ and HK is parallel to BC .



CHAPTER

8

Number and Algebra

Australian Curriculum
content descriptions

- ACMNA 209
- ACMNA 210
- ACMNA 212

Index laws

In Year 7 and Year 8 we introduced powers with whole number indices and developed the index laws for these powers.

We now extend our study to rational (fractional), including negative, indices.

We live in a world of very large and very small numbers. Population sizes, government spending, intergalactic distances and the size of computer memories are examples of very large numbers. Thickness of materials, circuit diagrams and subatomic particles are examples of very small numbers. Some examples of small and large numbers are:

- the time taken by light to travel one metre is approximately 0.000 000 003 seconds
- the radius of a hydrogen atom is approximately 0.000 000 000 025 metres
- the current Big Bang model of astronomy suggests that the Universe is about 13.7 billion years old.

This chapter introduces scientific notation, which is a convenient way of writing such numbers. Significant figures are also discussed.



Index notation



We recall the following from *ICE-EM Mathematics Year 8 Book 1*.

- A **power** is the product of a certain number of factors, all of which are the same.

For example,

$$2^4 = 2 \times 2 \times 2 \times 2 \text{ is the fourth power of 2.}$$

- The number 2 in 2^4 is called the **base**.
- The number 4 in 2^4 is called the **index** or **exponent**.
- For any number b , $b^1 = b$.
- In general, $b^n = \underbrace{b \times b \times b \times \dots \times b}_n$, where there are n factors in the product.

Here b is the **base** and n the **index**.

Example 1

Express as a power or as a product of powers:

a $5 \times 5 \times 5$

b $3 \times 3 \times 7 \times 7 \times 7 \times 7$

Solution

a $5 \times 5 \times 5 = 5^3$

b $3 \times 3 \times 7 \times 7 \times 7 \times 7 = 3^2 \times 7^4$

Example 2

Express each number as a power of a prime:

a 81

b 128

c 343

Solution

a $81 = 3 \times 3 \times 3 \times 3$
 $= 3^4$

b $128 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$
 $= 2^7$

c $343 = 7 \times 7 \times 7$
 $= 7^3$

Example 3

Express as a product of powers of prime numbers:

a 9000

b 66 150

Solution

By repeated division:

$$\mathbf{a} \quad 9000 = 2^3 \times 3^2 \times 5^3$$

$$\mathbf{b} \quad 66\,150 = 1323 \times 50$$

$$= 2 \times 3^3 \times 5^2 \times 7^2$$

The following laws for indices were discussed in Chapter 3 of *ICE-EM Mathematics Year 8 Book 1* for powers with whole number indices.

In the following, a and b are integers and m and n are non-zero whole numbers.



Index laws

Index law 1

To multiply powers of the same base, add the indices.

$$a^m a^n = a^{m+n}$$

Index law 2

To divide powers of the same base, subtract the indices.

$$\frac{a^m}{a^n} = a^{m-n} \quad \text{where } m > n \text{ and } a \neq 0$$

Index law 3

To raise a power to a power, multiply the indices.

$$(a^m)^n = a^{mn}$$

Index law 4

A power of a product is the product of the powers.

$$(ab)^m = a^m b^m$$

Index law 5

A power of a quotient is the quotient of the powers.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad \text{where } b \neq 0$$



Example 4

Simplify, expressing the answer in index form:

a $3^2 \times 3^4$

b $a^3 \times a^5$

c $3x^2 \times x^3$

d $2a^2b^3 \times 5ab^2$

Solution

a $3^2 \times 3^4 = 3^6$

b $a^3 \times a^5 = a^8$

c $3x^2 \times x^3 = 3x^5$

d $2a^2b^3 \times 5ab^2 = 10 \times a^{2+1} \times b^{3+2}$
 $= 10a^3b^5$

Example 5

Simplify, writing the answer as a number or a number times a power:

a $\frac{3^5}{3^2}$

b $\frac{8^7}{8^4}$

c $\frac{9^5}{9^4}$

d $10^6 \div 10^4$

e $a^7 \div a^4$

f $\frac{3y^4}{y}$

g $\frac{6x^5}{2x^3}$

Solution

a $\frac{3^5}{3^2} = 3^{5-2}$
 $= 3^3$
 $= 27$

b $\frac{8^7}{8^4} = 8^{7-4}$
 $= 8^3$
 $= 512$

c $\frac{9^5}{9^4} = 9^1$
 $= 9$

d $10^6 \div 10^4 = 10^{6-4}$
 $= 10^2$
 $= 100$

e $a^7 \div a^4 = a^{7-4}$
 $= a^3$

f $\frac{3y^4}{y} = 3 \times \frac{y^4}{y}$
 $= 3 \times y^{4-1}$
 $= 3y^3$

g $\frac{6x^5}{2x^3} = \frac{6}{2} \times \frac{x^5}{x^3}$
 $= 3 \times x^{5-3}$
 $= 3x^2$

Example 6

Simplify, expressing the answer in index form:

a $\frac{3x^3y^2}{4xy} \times \frac{6x^2y^3}{x^3y^2}$

b $\frac{8a^2b^3}{3a^3b} \div \frac{4ab^2}{9a^3b^5}$

Solution

a $\frac{3x^3y^2}{4xy} \times \frac{6x^2y^3}{x^3y^2} = \frac{18x^5y^5}{4x^4y^3}$
 $= \frac{9xy^2}{2}$

b $\frac{8a^2b^3}{3a^3b} \div \frac{4ab^2}{9a^3b^5} = \frac{8a^2b^3}{3a^3b} \times \frac{9a^3b^5}{4ab^2}$
 $= \frac{72 \times a^5 \times b^8}{12 \times a^4 \times b^3}$
 $= 6ab^5$

Example 7

Simplify by expanding the brackets:

a $\left(\frac{2}{3}\right)^2$

b $\left(\frac{m}{n}\right)^5$

c $\left(\frac{x^3}{y^2}\right)^2 \times \left(\frac{y}{x}\right)^4$

d $\left(\frac{2x^2}{3}\right)^2 \div \frac{4x^3}{9}$

Solution

a $\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2}$
 $= \frac{4}{9}$

b $\left(\frac{m}{n}\right)^5 = \frac{m^5}{n^5}$

c $\left(\frac{x^3}{y^2}\right)^2 \times \left(\frac{y}{x}\right)^4 = \frac{x^6}{y^4} \times \frac{y^4}{x^4}$
 $= x^2$

d $\left(\frac{2x^2}{3}\right)^2 \div \frac{4x^3}{9} = \frac{4x^4}{9} \times \frac{9}{4x^3}$
 $= x$

There are different possible interpretations of the word ‘simplify’. There may be more than one acceptable simplified form.

A number raised to the power zero

Clearly $\frac{4^3}{4^3} = 1$. If the index laws are to apply then $4^{3-3} = 4^0 = 1$. Hence we define $a^0 = 1$, for all non-zero numbers a .

Note: 0^0 is not defined. If x is a pronumeral, $x^0 = 1$ ($x \neq 0$).



Example 8

Simplify:

a $(5a^3)^0$

b $\frac{6x^2y}{xy^2} \times \frac{y^3x}{2y^2x^2}$

Solution

a $(5a^3)^0 = 1$

b $\frac{6x^2y}{xy^2} \times \frac{y^3x}{2y^2x^2} = \frac{6}{2} \times \frac{x^3}{x^3} \times \frac{y^4}{y^4}$
 $= 3x^0y^0$
 $= 3 \times 1 \times 1$
 $= 3$

Example 9

Simplify:

a $(mn^2)^0$

b $(a^4b^2)^3$

c $(2a^4)^3$

d $(x^2y)^0 \times (x^2y^3)^3$

Solution

a $(mn^2)^0 = 1$

b $(a^4b^2)^3 = a^{12}b^6$

c $(2a^4)^3 = 2^3 \times a^{12}$
 $= 8a^{12}$

d $(x^2y)^0 \times (x^2y^3)^3 = 1 \times x^6y^9$
 $= x^6y^9$



The zero power

For all non-zero numbers a , we define $a^0 = 1$.



Exercise 8A



1 State the base and index of:

a 6^4

b 7^3

c 8^2

d 10^4

e 5

f 6^0

2 Express as a power of a prime number:

a 8

b 27

c 64

d 243

e 125

f 81

3 Evaluate:

a 3^4

b 2^7

c 5^5

d 7^4

e $2^3 \times 3^5$

f $6^4 \times 3^2$

Example 2

Example 3

4 Express as a product of powers of prime numbers:

a 18 **b** 24 **c** 144 **d** 90 **e** 700 **f** 84

Example 4

5 Simplify, leaving the answer as a power or a product of powers:

a $2^2 \times 2^3$ **b** $2^7 \times 2^3$ **c** $3^4 \times 3^5$ **d** $3^4 \times 3^7$
e $3^2 \times 3 \times 3^4$ **f** $3^3 \times 3^4 \times 3^5$ **g** $a^3 \times a^8$ **h** $b^7 \times b^{12}$
i $3a^2 \times a^3$ **j** $3x^2 \times 4x^3$ **k** $2y \times 3y^4$ **l** $4b^2 \times 3b^4$

6 Simplify:

a $a^2b^3 \times b^2$ **b** $a^3b \times a^2b^3$ **c** $x^2y \times x^3y$
d $2xy^2 \times 3x^2y$ **e** $4a^3b^2 \times a^2b^4$ **f** $5a^4b \times 2ab^3$

Example 5

7 Simplify, leaving the answer as a power or a product of powers:

a $\frac{3^7}{3^2}$ **b** $\frac{2^6}{2^2}$ **c** $5^4 \div 5$ **d** $7^5 \div 7^3$
e $10^7 \div 10^2$ **f** $\frac{10^{12}}{10^4}$ **g** $\frac{a^4}{a}$ **h** $\frac{a^5}{a^3}$
i $\frac{2x^3}{x^2}$ **j** $\frac{6x^5}{2x^2}$ **k** $\frac{10y^{12}}{5y^3}$ **l** $\frac{27p^4}{9p}$

8 Simplify:

a $\frac{a^3b^2}{a^2}$ **b** $\frac{x^3y^2}{xy}$ **c** $\frac{a^5b^3}{a^4b}$ **d** $\frac{x^4y^7}{x^3y^2}$
e $\frac{12a^6b^2}{4a^2b}$ **f** $\frac{15xy^3}{3y^2}$ **g** $\frac{16a^4b^3}{12a^2b^2}$ **h** $\frac{27x^2y^3}{18xy^2}$

Example 6

9 Simplify:

a $\frac{a^3b^2}{ab} \times \frac{a^2b}{a}$ **b** $\frac{x^3y}{xy^2} \times \frac{x^4y^5}{x^2}$ **c** $\frac{2ab^2}{3a^2b^4} \times \frac{6a^4b^5}{ab}$ **d** $\frac{12x^4y^3}{3x^2y} \times \frac{x^2y^4}{x^3y^5}$
e $\frac{6ab^2}{5a^3b} \div \frac{12ab}{15a^5b}$ **f** $\frac{7x^3y^4}{2xy^2} \div \frac{21x^2y^3}{4x^3y^2}$ **g** $\frac{14a^4b^3}{3ab^2} \div \frac{7a^5b^4}{6a^3b^5}$ **h** $\frac{12x^2y}{x^3y^4} \div \frac{6xy^2}{x^6y^7}$

10 Copy and complete:

a $a^4 \times \dots = a^{10}$ **b** $b^7 \times \dots = b^{16}$ **c** $4a^3 \times \dots = 12a^7$
d $9d^5 \times \dots = 27d^6$ **e** $a^8 \div \dots = a^4$ **f** $x^{10} \div \dots = x^6$
g $b^7 \div \dots = b$ **h** $d^6 \div \dots = d^2$ **i** $15d^7 \div \dots = 3d^2$
j $9d^6 \div \dots = 3d$ **k** $a^2b \times \dots = a^5b^6$ **l** $m^4n^5 \times \dots = m^{10}n^7$

$$\begin{array}{lll} \text{m} & 8ab^4 \times \dots = 24a^2b^6 & \text{n} \quad a^7b^4 \div \dots = a^2b \\ \text{p} & \ell^6m^7 \div \dots = \ell^2m^5 & \text{q} \quad 9m^7n^4 \div \dots = 3m^2 \\ \text{r} & 18p^2q^6 \div \dots = 3pq \end{array}$$

- 11** Simplify each expression. Check your answer for part **a** by substituting $x = 2$ into both the original expression and the simplified expression. Repeat for $x = 3$ in part **b** and $x = -2$ in part **c**.

$$\begin{array}{llll} \text{a} & \frac{6x^2}{x} & \text{b} & \frac{6x^2}{3x} \\ \text{c} & \frac{6x^2}{6x} & \text{d} & \frac{8x^4}{4x} \\ \text{e} & \frac{3x^5}{x^3} & \text{f} & \frac{10x^4}{5x^3} \\ \text{g} & \frac{10x^4}{2x^4} & \text{h} & \frac{12x^4}{6x^2} \end{array}$$

Example 8

- 12** Simplify:

$$\text{a} \quad a^0 \quad \text{b} \quad 2x^0 \quad \text{c} \quad xy^0 \quad \text{d} \quad 7x^0y^0$$

Example 9

- 13** Simplify:

$$\begin{array}{llll} \text{a} & 3a^0 & \text{b} & 6a^0 \\ \text{c} & (4a)^0 & \text{d} & (3b)^0 \\ \text{e} & 4a^0 + 3b^0 & \text{f} & 6a^0 + 7m^0 \\ \text{g} & (2a + 1)^0 & \text{h} & (4a + 3b)^0 \\ \text{i} & (4b)^0 + 2b^0 & \text{j} & (3b)^0 - 5d^0 \\ \text{k} & (5m^0 + 7b)^0 & \text{l} & (6m - 2c^0)^0 \end{array}$$

- 14** Simplify, leaving the answer as a power:

$$\text{a} \quad (2^3)^4 \quad \text{b} \quad (3^2)^3 \quad \text{c} \quad (a^2)^5 \quad \text{d} \quad (y^5)^6$$

- 15** Simplify:

$$\begin{array}{lll} \text{a} & (a^3)^2 \times (a^3)^4 & \text{b} \quad (x^4)^2 \times (x^3)^3 \\ \text{c} & (b^4)^2 \div (b^3)^2 & \\ \text{d} & \frac{(y^3)^4}{(y^4)^2} & \text{e} \quad 2ab^2 \times 3a(b^3)^2 \\ \text{f} & \frac{3ab}{(b^2)^3} \times \frac{4b^7}{3a} & \\ \text{g} & \frac{4(x^3)^2y^4}{3x^4y^3} \times \frac{3x^3(y^2)^2}{8xy^5} & \text{h} \quad \frac{8a^2(b^3)^2}{3ab^2} \div \frac{16a^5b^3}{9(a^3)^2} \\ \text{i} & \frac{3(x^3y)^2}{(x^2y)^2} \div \frac{12x^4y^2}{(2x^3y)^2} & \end{array}$$

- 16** Copy and complete (using index law 3):

$$\begin{array}{lll} \text{a} & (a^6)^{\dots} = a^{24} & \text{b} \quad (b^3)^{\dots} = b^{21} \\ \text{c} & (m^5)^{\dots} = m^{10} & \\ \text{d} & (m^6)^{\dots} = m^{30} & \text{e} \quad (\dots)^6 = p^{36} \\ \text{f} & (\dots)^5 = p^{25} & \\ \text{g} & (\dots)^4 = a^8 & \text{h} \quad (\dots)^3 = m^{15} \\ \text{i} & (\dots)^{\dots} = m^{20} & \end{array}$$

- 17** **a** Is it true that $(a^2)^6 = (a^6)^2$? **b** Is it true that $(b^4)^7 = (b^7)^4$?
c Generalise your result.

18 Simplify by expanding the brackets:

a $(3a)^2$

b $(2x)^3$

c $(xy^3)^2$

d $(a^2b)^4$

e $\left(\frac{a}{5}\right)^2$

f $\left(\frac{2}{x}\right)^3$

g $\left(\frac{a}{b}\right)^5$

h $\left(\frac{x^2}{y}\right)^3$

19 Simplify:

a $(2a^2b)^2 \times 3ab^3$

b $(3xy^2)^3 \times (x^2y)^2$

c $(m^2n^3)^2 \times (mn)^3$

d $(5xy^2)^3 \times (x^2y^3)^2$

e $(2a^3b)^3 \times 3a^0$

f $(2xy^2)^0 \times (3x^2y)^3$

20 Simplify:

a $\left(\frac{x^2}{y}\right)^2 \times \left(\frac{y^2}{x}\right)^3$

b $\left(\frac{4a^2}{b}\right)^2 \times \left(\frac{b}{2a}\right)^3$

c $\left(\frac{x^3}{y^2}\right)^2 \div \left(\frac{x}{y^2}\right)^3$

d $\left(\frac{2x^4}{y}\right)^5 \div \left(\frac{4x^3}{y^3}\right)^2$

21 Simplify:

a $\frac{(3xy^2)^2 \times (2x^2y)^3}{(6x^2y)^2}$

b $\frac{3a^2b^4 \times (2ab^2)^3}{(4a^2b^3)^2}$

c $\frac{(2x^2y^3)^3 \times (5xy^2)^2}{(10x^2y)^2 \times (xy)^3}$

d $\frac{(6ab)^3 \times 2a^7b^4}{(2ab)^4 \times (3a^2b)^2}$

22 Copy and complete:

a $(\dots)^4 = a^8b^{12}$

b $(\dots)^6 = m^{30}n^{24}$

c $(a^7b^4)\dots = a^{14}b^8$

d $(p^3q)\dots = p^9q^3$

e $(x^4y^7)\dots = 1$

f $(\ell^3p^4)\dots = 1$

g $(\dots)^4 = 16a^8$

h $(\dots)^3 = 27q^9$

i $(\dots)^5 = 32m^5$

j $(\dots)^2 = 49m^6$

k $(\dots)^3 = 64\ell^9m^3$

l $(\dots)^2 = 25m^{10}n^6$



In the last section, we defined a^n to be the product of n factors of a , where a is any number and n is a positive integer. We also defined $a^0 = 1$ for all non-zero numbers a .

We now give meaning to negative integer indices. For example, we want to give a meaning to 2^{-4} , 3^{-100} and so on. To work towards a useful definition, look at the following pattern:

$$\begin{array}{lll} 2^5 = 32, & 2^4 = 16, & 2^3 = 8, \\ 2^2 = 4, & 2^1 = 2, & 2^0 = 1 \end{array}$$

Each index is one less than the preceding one, and each number to the right of an equal sign is half the number in the previous expression.

This can be continued purely as a pattern:

$$2^{-1} = \frac{1}{2}, \quad 2^{-2} = \frac{1}{4} = \frac{1}{2^2}, \quad 2^{-3} = \frac{1}{8} = \frac{1}{2^3} \quad \text{and so on.}$$

This suggests we define $2^{-n} = \frac{1}{2^n}$, for any positive integer n .



Negative indices

Define

$$a^{-n} = \frac{1}{a^n}$$

where a is a non-zero number and n is a positive integer.

Note that this tells us that $a^{-n} \times a^n = 1$. So $a^n = \frac{1}{a^{-n}}$.

For example,

$$2^{10} = \frac{1}{2^{-10}}.$$

So we can say that for all n , whether n is positive or negative, $a^{-n} = \frac{1}{a^n}$.

Example 10

Evaluate:

a 6^{-2}

b 4^{-3}

c 2^{-7}

d 10^{-3}

(continued on next page)

Solution

$$\begin{aligned} \mathbf{a} \quad 6^{-2} &= \frac{1}{6^2} \\ &= \frac{1}{36} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 4^{-3} &= \frac{1}{4^3} \\ &= \frac{1}{64} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 2^{-7} &= \frac{1}{2^7} \\ &= \frac{1}{128} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad 10^{-3} &= \frac{1}{10^3} \\ &= \frac{1}{1000} \end{aligned}$$

Fractions and negative indices

The **reciprocal** of a fraction such as $\frac{4}{3}$ is $\frac{3}{4}$. The index -1 means ‘the reciprocal of’, so

$$\left(\frac{3}{4}\right)^{-1} = \frac{4}{3}$$

When raising a fraction to other negative indices take the reciprocal first.

$$\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

Example 11

Evaluate:

$$\mathbf{a} \quad \left(\frac{1}{3}\right)^{-1}$$

$$\mathbf{b} \quad \left(\frac{2}{7}\right)^{-2}$$

$$\mathbf{c} \quad \left(4\frac{1}{4}\right)^{-2}$$

Solution

$$\begin{aligned} \mathbf{a} \quad \left(\frac{1}{3}\right)^{-1} &= \frac{3}{1} \\ &= 3 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \left(\frac{2}{7}\right)^{-2} &= \left(\frac{7}{2}\right)^2 \\ &= \frac{49}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \left(4\frac{1}{4}\right)^{-2} &= \left(\frac{17}{4}\right)^{-2} \\ &= \left(\frac{4}{17}\right)^2 \\ &= \frac{16}{289} \end{aligned}$$

Note that in general $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$.

The index laws we revised in Section 8A are also valid when using negative integer indices.

For all integers m and n and non-zero numbers a and b the following are true.		
Zero index		$a^0 = 1$
Negative index		$a^{-n} = \frac{1}{a^n}$
Index law 1	Product of powers	$a^m a^n = a^{m+n}$
Index law 2	Quotient of powers	$a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}$
Index law 3	Power of a power	$(a^m)^n = a^{mn}$
Index law 4	Power of a product	$(ab)^n = a^n b^n$
Index law 5	Power of a quotient	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

The results for negative indices m and n can be proved using the index laws for positive integer indices. An indication of the proof appears as question **10** in the Challenge exercise.

Example 12

Write as a single power and then evaluate:

a $3^4 \times 3^{-2}$

b $5^7 \times 5^{-8}$

c $13^{-8} \times 13^{15} \times 13^{-7}$

d $\left(\frac{2}{3}\right)^{-6} \times \left(\frac{2}{3}\right)^4$

Solution

a $3^4 \times 3^{-2} = 3^2$
 $= 9$

b $5^7 \times 5^{-8} = 5^{-1}$
 $= \frac{1}{5}$

c $13^{-8} \times 13^{15} \times 13^{-7} = 13^0$
 $= 1$

d $\left(\frac{2}{3}\right)^{-6} \times \left(\frac{2}{3}\right)^4 = \left(\frac{2}{3}\right)^{-2}$
 $= \left(\frac{3}{2}\right)^2$
 $= \frac{9}{4}$

Example 13

Write as a single power and then evaluate:

a $\frac{2^4}{2^5}$

b $\frac{3^4}{3^7}$

c $\frac{5}{5^3}$

d $\frac{3^4}{3^6}$

Solution

a $\frac{2^4}{2^5} = 2^{-1}$
 $= \frac{1}{2}$

b $\frac{3^4}{3^7} = 3^{-3}$
 $= \frac{1}{3^3}$
 $= \frac{1}{27}$

c $\frac{5}{5^3} = 5^{-2}$
 $= \frac{1}{5^2}$
 $= \frac{1}{25}$

d $\frac{3^4}{3^6} = 3^{-2}$
 $= \frac{1}{3^2}$
 $= \frac{1}{9}$

Example 14

Simplify, expressing the answers with positive indices.

a $a^2b^{-3} \times a^{-4}b^5$

b $\frac{x^2y^3}{x^3y^2}$

c $(2a^{-2}b^3)^{-2}$

d $\left(\frac{3m^2}{n}\right)^{-4}$

Solution

a $a^2b^{-3} \times a^{-4}b^5 = a^{2-4} \times b^{-3+5}$
 $= \frac{1}{a^2} \times b^2$
 $= \frac{b^2}{a^2}$

b $\frac{x^2y^3}{x^3y^2} = x^{-1}y^1$
 $= \frac{1}{x} \times y$
 $= \frac{y}{x}$

c $(2a^{-2}b^3)^{-2} = 2^{-2} \times a^4 \times b^{-6}$
 $= \frac{1}{2^2} \times a^4 \times \frac{1}{b^6}$
 $= \frac{a^4}{4b^6}$

d $\left(\frac{3m^2}{n}\right)^{-4} = \left(\frac{n}{3m^2}\right)^4$
 $= \frac{n^4}{81m^8}$



Exercise 8B

Example 10

1 Express with a positive index and then evaluate:

a 2^{-1}	b 5^{-1}	c 3^{-2}	d 6^{-2}	e 9^{-2}	f 10^{-2}
g 2^{-4}	h 2^{-5}	i 3^{-3}	j 5^{-3}	k 10^{-4}	l 10^{-6}
m 9^{-1}	n 3^{-4}	o 5^{-2}	p 8^{-2}	q 10^{-5}	r 2^{-7}

2 Write each fraction as a power of a prime with a negative index.

a $\frac{1}{8}$	b $\frac{1}{9}$	c $\frac{1}{27}$	d $\frac{1}{49}$	e $\frac{1}{121}$	f $\frac{1}{125}$
g $\frac{1}{16}$	h $\frac{1}{64}$	i $\frac{1}{512}$	j $\frac{1}{343}$	k $\frac{1}{4}$	l $\frac{1}{169}$
m $\frac{1}{625}$	n $\frac{1}{81}$	o $\frac{1}{256}$	p $\frac{1}{243}$	q $\frac{1}{31}$	r $\frac{1}{32}$

3 Express with positive indices, evaluating where possible:

a a^{-3}	b x^{-7}	c m^{-5}	d n^{-4}	e $3a^{-4}$	f $5x^{-7}$
g $4a^{-5}$	h $9m^{-7}$	i $\frac{1}{x^{-3}}$	j $\frac{1}{x^{-5}}$	k $\frac{3}{a^{-4}}$	l $\frac{5}{x^{-5}}$
m $3^{-2}a^{-2}$	n $4^{-2}x^{-2}$	o $7^{-3}a^{-5}$	p $5^{-2}x^{-7}$	q $\frac{13^{-2}}{n^{-6}}$	r $\frac{y^{-4}}{x^{-3}}$

Example 11, 12

4 Simplify:

a $\left(\frac{1}{4}\right)^{-1}$	b $\left(\frac{2}{5}\right)^{-2}$	c $\left(3\frac{1}{3}\right)^{-2}$	d $\left(\frac{2}{3}\right)^{-3}$
e $3^5 \times 3^{-2}$	f $5^{11} \times 5^{-8}$	g $7^3 \times 7^{-5}$	h $4^3 \times 4^{-5}$

Example 13

5 Write as a single power and then evaluate:

a $\frac{2^3}{2^6}$	b $\frac{4^2}{4^4}$	c $\frac{3^8}{3^9}$	d $\frac{6^5}{6^8}$	e $\frac{7^1}{7^3}$	f $\frac{5^7}{5^{10}}$
g $\frac{8^6}{8^7}$	h $\frac{20^4}{20^6}$	i $\frac{3^5}{3^9}$	j $\frac{2^7}{2^{13}}$	k $\frac{10^2}{10^6}$	l $\frac{12^{12}}{12^{14}}$

6 Express with negative index:

a $\frac{3}{x}$ **b** $\frac{4}{x}$ **c** $\frac{9}{x}$ **d** $\frac{5}{x^2}$ **e** $\frac{6}{x^3}$ **f** $\frac{8}{x^4}$
g $\frac{7}{x^{11}}$ **h** $\frac{3}{2x^4}$ **i** $\frac{2}{5x^6}$ **j** $\frac{4}{3x^7}$ **k** $\frac{3}{4x^4}$ **l** $\frac{2}{3x^5}$

7 Evaluate:

a $\left(\frac{1}{2}\right)^{-1}$ **b** $\left(\frac{2}{3}\right)^{-1}$ **c** $\left(\frac{4}{5}\right)^{-1}$ **d** $\left(\frac{11}{8}\right)^{-1}$ **e** $\left(\frac{1}{2}\right)^{-2}$ **f** $\left(\frac{1}{3}\right)^{-5}$
g $\left(\frac{2}{3}\right)^{-3}$ **h** $\left(\frac{4}{5}\right)^{-2}$ **i** $\left(2\frac{1}{4}\right)^{-2}$ **j** $\left(1\frac{1}{5}\right)^{-3}$ **k** $\left(2\frac{1}{2}\right)^{-2}$

Example 14

8 Simplify, expressing the answer with positive indices:

a $x^{-6}y^4 \times x^2y^{-2}$ **b** $a^{-3}b^{-5} \times a^5b^{-3}$ **c** $3x^{-2}y^5 \times 5x^{-7}y^{-2}$
d $2a^{-1}b^5 \times 7ab^{-3}$ **e** $7a^3m^{-4} \times 8a^{-5}m^{-3}$ **f** $3r^2s^3 \times 4r^{-3}s^{-5}$
g $\frac{8a^{-4}}{2a^6}$ **h** $\frac{16a^{-4}}{8a^5}$ **i** $\frac{18a^{-4}}{4a^5}$ **j** $\frac{27m^{-3}}{9m^{-2}}$ **k** $\frac{56t^{-7}}{8t^{-2}}$
l $\frac{36h^{-9}}{9h^{-4}}$ **m** $\frac{144x^7y^5}{12x^{-3}y^4}$ **n** $\frac{72a^4b^{-3}}{36ab^{-2}}$ **o** $\frac{7a^2b^{-3}c^{-4}}{21a^5b^{-7}c^{-9}}$ **p** $\frac{9m^3n^4p^{-5}}{21m^{-3}n^4p^2}$

9 Copy and complete:

a $6^4 \times \dots = 6^2$ **b** $9^5 \times \dots = 9^4$ **c** $b^9 \times \dots = b^7$
d $m^5 \times \dots = m^{-6}$ **e** $a^{11} \div \dots = a^{14}$ **f** $b^7 \div \dots = b^{15}$
g $d^{-7} \div \dots = d^{15}$ **h** $e^{-7} \div \dots = e^{-5}$ **i** $(m^{-2}) \dots = m^{10}$
j $(a^5) \dots = a^{-15}$ **k** $(\dots)^{-4} = \frac{m^8}{16}$ **l** $(\dots)^{-3} = \frac{1}{27a^9}$
m $(\dots)^{-2} = \frac{m^6}{25}$ **n** $(\dots)^{-3} = \frac{a^6}{b^9}$ **o** $(\dots)^{-6} = \frac{m^{12}n^{18}}{p^6}$
p $(\dots)^{-2} = p^4q^{-6}$ **q** $(\dots) \dots = a^6b^{-4}$ **r** $(\dots) \dots = \frac{m^6}{n^9}$

Write two possible alternatives for part **q** and part **r**.

10 Simplify, expressing the answers with positive indices. Evaluate powers where possible.

a $(3a^2b^{-2})^3 \times (2a^4)^{-2}$ **b** $(5x^4y^6)^{-3} \times (5^2xy^{-1})^3$ **c** $(5m^2n^{-3})^{-2} \times 2(m^{-2}n^3)^2$
d $(6a^5b^{-4})^{-3} \times 2(a^3b^{-3})^2$ **e** $\frac{(x^2)^2}{y} \times \frac{(y^2)^{-3}}{x^3}$ **f** $\frac{(2x^3)^{-2}}{y^4} \times \frac{(2x^7)^2}{3y^5}$
g $\frac{(2a^4b^{-2})^3}{c^2} \times \frac{(2^2a^{-3}b^2)^{-1}}{c}$ **h** $\frac{(m^2n^3)^2}{p^{-3}} \times (mnp^{-2})^{-3}$ **i** $\frac{(a^2)^3}{b^3} \div \left(\frac{a}{b^2}\right)^{-2}$
j $\frac{(2a^4)^2}{b^7} \div \frac{(a^2)^{-3}}{2b}$ **k** $\frac{(4c^4d^{-3})^2}{9} \div \frac{3c^{-2}}{d}$ **l** $\frac{(3m^2n^3)^{-2}}{p^4} \div \frac{p^{-3}}{m}$

**Fractional indices with numerator one**

We are now going to extend our study of indices by looking at fractional indices. We begin by considering what we mean by powers such as $3^{\frac{1}{2}}$, $2^{\frac{1}{3}}$ and $7^{\frac{1}{10}}$, in which the index is the reciprocal of a positive integer.

If a is a positive number, then $\sqrt{a}$ is the positive square root and $(\sqrt{a})^2 = a = a^1$.

We now introduce the alternative notation $a^{\frac{1}{2}}$ for $\sqrt{a}$. We do this because the third index law then continues to hold, that is:

$$\left(a^{\frac{1}{2}}\right)^2 = a^{2 \times \frac{1}{2}} = a^1.$$

Keep in mind that $a^{\frac{1}{2}}$ is nothing more than an alternative notation for $\sqrt{a}$.

For example, $49^{\frac{1}{2}} = 7$, $64^{\frac{1}{2}} = 8$, $100^{\frac{1}{2}} = 10$, and so on.

Every positive number a has a cube root $\sqrt[3]{a}$. It is the positive number whose cube is a .

For example:

$$\sqrt[3]{8} = 2 \text{ because } 2^3 = 8$$

$$\sqrt[3]{27} = 3 \text{ because } 3^3 = 27$$

We define $a^{\frac{1}{3}}$ to be $\sqrt[3]{a}$. The third index laws continues to hold.

$$\left(a^{\frac{1}{3}}\right)^3 = a^{3 \times \frac{1}{3}} = a^1$$

Similarly we define

$$a^{\frac{1}{4}} = \sqrt[4]{a}, a^{\frac{1}{5}} = \sqrt[5]{a}, \text{ and so on.}$$

**Fractional indices**

Let a be positive or zero and let n be a positive integer.

Define $a^{\frac{1}{n}}$ to be the n^{th} root of a . That is, $a^{\frac{1}{n}}$ is the positive number whose n^{th} power is a ,

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

For example, $a^{\frac{1}{2}} = \sqrt{a}$ and $a^{\frac{1}{3}} = \sqrt[3]{a}$

Square roots, cube roots, fourth roots and so on are usually irrational numbers, and a calculator can be used to obtain approximations. Using a calculator, we obtain:

$$10^{\frac{1}{2}} = \sqrt{10} \approx 3.1623, \quad 10^{\frac{1}{3}} = \sqrt[3]{10} \approx 2.1544, \quad 10^{\frac{1}{4}} = \sqrt[4]{10} \approx 1.7783,$$

$$10^{\frac{1}{5}} = \sqrt[5]{10} \approx 1.5849, \dots$$

Using our new notation, here are some other numerical approximations, all recorded correct to 4 decimal places. You can use your calculator to check these.

$$2^{\frac{1}{5}} \approx 1.1487, \quad 10^{\frac{1}{8}} \approx 1.3335, \quad 0.2^{\frac{1}{4}} \approx 0.6687, \quad 3.2^{\frac{1}{6}} \approx 1.2139$$

Example 15

Evaluate:

a $\sqrt[3]{27}$ **b** $\sqrt[4]{16}$ **c** $\sqrt[5]{243}$ **d** $\sqrt[3]{125}$

Solution

a $\sqrt[3]{27}$ is the number that when cubed is 27.

Thus $3^3 = 27$, because $\sqrt[3]{27} = 3$

b $\sqrt[4]{16} = 2$ because $2^4 = 16$,

c $\sqrt[5]{243} = \sqrt[5]{3^5}$
 $= 3$

d $\sqrt[3]{125} = \sqrt[3]{5^3}$
 $= 5$

Example 16

Evaluate:

a $9^{\frac{1}{2}}$ **b** $27^{\frac{1}{3}}$ **c** $16^{\frac{1}{4}}$ **d** $1000000^{\frac{1}{6}}$

Solution

a $9^{\frac{1}{2}} = 3$ (since $3^2 = 9$)

b $27^{\frac{1}{3}} = 3$ (since $3^3 = 27$)

c $16^{\frac{1}{4}} = 2$ (since $2^4 = 16$)

d $1000000^{\frac{1}{6}} = 10$ (since $10^6 = 1000000$)

Positive fractional indices

In the previous section, we defined $8^{\frac{1}{3}} = \sqrt[3]{8}$. For consistency with the index laws we now define $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2$.



Positive fractional indices

If a is a positive number or zero and, p and q are positive integers, define:

$$a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p = \left(\sqrt[q]{a}\right)^p$$

Example 17

Evaluate:

a $4^{\frac{3}{2}}$

b $8^{\frac{2}{3}}$

c $81^{\frac{3}{4}}$

Solution

a $4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3$
 $= 2^3$ (since $2^2 = 4$)
 $= 8$

b $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2$
 $= 2^2$ (since $2^3 = 8$)
 $= 4$

c $81^{\frac{3}{4}} = \left(81^{\frac{1}{4}}\right)^3$
 $= 3^3$ (since $3^4 = 81$)
 $= 27$

The index laws also hold for fractional indices. A proof of these more general results is given in question 10 of the Challenge exercise.

Example 18

Simplify:

a $(d^{21})^{\frac{1}{7}}$

b $p^{\frac{2}{3}} \times p^{\frac{4}{5}}$

c $q^{\frac{4}{5}} \div q^{\frac{2}{3}}$

(continued on next page)

Solution

$$\begin{array}{lll} \mathbf{a} & (d^{21})^{\frac{1}{7}} = d^{21 \times \frac{1}{7}} & \mathbf{b} \quad p^{\frac{2}{3}} \times p^{\frac{4}{5}} = p^{\frac{2}{3} + \frac{4}{5}} \\ & = d^3 & = p^{\frac{10}{15} + \frac{12}{15}} \\ & & = p^{\frac{22}{15}} \\ \mathbf{c} & q^{\frac{4}{5}} \div q^{\frac{2}{3}} = q^{\frac{4}{5} - \frac{2}{3}} & \\ & & = q^{\frac{12-10}{15}} \\ & & = q^{\frac{2}{15}} \end{array}$$

Negative fractional indices

Since $4^{-1} = \frac{1}{4}$, we define $4^{-\frac{3}{2}} = \frac{1}{4^{\frac{3}{2}}} = \left(\frac{1}{4}\right)^{\frac{3}{2}}$

We can now combine the definitions of negative indices and fractional indices.



Negative fractional indices

Let a be a positive number or zero, and let p and q be positive integers. Define

$$a^{-\frac{p}{q}} = \frac{1}{a^{\frac{p}{q}}} = \left(\frac{1}{a}\right)^{\frac{p}{q}}$$

Note: If $a > 0$ and $b > 0$, $\left(\frac{a}{b}\right)^{-\frac{p}{q}} = \left(\frac{b}{a}\right)^{\frac{p}{q}}$.

Example 19

Evaluate:

$$\mathbf{a} \quad 4^{-\frac{3}{2}} \qquad \mathbf{b} \quad \left(\frac{8}{27}\right)^{-\frac{1}{3}}$$

Solution

$$\begin{array}{ll} \mathbf{a} \quad 4^{-\frac{3}{2}} = \frac{1}{4^{\frac{3}{2}}} & \mathbf{b} \quad \left(\frac{8}{27}\right)^{-\frac{1}{3}} = \left(\frac{27}{8}\right)^{\frac{1}{3}} \\ & = \frac{27^{\frac{1}{3}}}{8^{\frac{1}{3}}} \\ & = \frac{3}{2} \end{array}$$

Example 20

Simplify, writing each answer with positive indices.

a $p^{-\frac{1}{4}} \times p^{\frac{2}{3}}$ **b** $x^{\frac{2}{3}} \div x^{-\frac{1}{2}}$ **c** $(125n^{-6})^{\frac{1}{3}}$ **d** $x^{\frac{1}{5}} \div x^{\frac{1}{3}}$

Solution

$$\begin{aligned}\text{a } p^{-\frac{1}{4}} \times p^{\frac{2}{3}} &= p^{-\frac{1}{4} + \frac{2}{3}} \\ &= p^{-\frac{3}{12} + \frac{8}{12}} \\ &= p^{\frac{5}{12}}\end{aligned}$$

$$\begin{aligned}\text{b } x^{\frac{2}{3}} \div x^{-\frac{1}{2}} &= x^{\frac{2}{3} - (-\frac{1}{2})} \\ &= x^{\frac{4}{6} + \frac{3}{6}} \\ &= x^{\frac{7}{6}}\end{aligned}$$

$$\begin{aligned}\text{c } (125n^{-6})^{\frac{1}{3}} &= 125^{\frac{1}{3}} \times n^{-6 \times \frac{1}{3}} \\ &= 5 \times n^{-2} \\ &= \frac{5}{n^2}\end{aligned}$$

$$\begin{aligned}\text{d } x^{\frac{1}{5}} \div x^{\frac{1}{3}} &= x^{\frac{3}{15} - \frac{5}{15}} \\ &= x^{-\frac{2}{15}} \\ &= \frac{1}{x^{\frac{2}{15}}}\end{aligned}$$

Exercise 8C

Example 15

1 Evaluate:

a $\sqrt[3]{8}$ **b** $\sqrt[5]{32}$ **c** $\sqrt[3]{216}$ **d** $\sqrt[4]{81}$ **e** $\sqrt[3]{64}$ **f** $\sqrt[5]{2^{10}}$

2 Write using fractional indices. Evaluate, correct to four decimal places.

a $\sqrt{14}$ **b** $\sqrt[4]{64}$ **c** $\sqrt[5]{7}$ **d** $\sqrt[7]{11}$ **e** $\sqrt[3]{2^7}$

Example 16

3 Evaluate:

a $4^{\frac{1}{2}}$	b $27^{\frac{1}{3}}$	c $243^{\frac{1}{5}}$	d $81^{\frac{1}{4}}$
e $64^{\frac{1}{2}}$	f $25^{\frac{1}{2}}$	g $125^{\frac{1}{3}}$	h $64^{\frac{1}{3}}$
i $32^{\frac{1}{5}}$	j $625^{\frac{1}{4}}$	k $216^{\frac{1}{3}}$	l $49^{\frac{1}{2}}$

Example 17

4 Evaluate:

a $4^{\frac{5}{2}}$

b $25^{\frac{3}{2}}$

c $125^{\frac{2}{3}}$

d $64^{\frac{5}{6}}$

e $32^{\frac{2}{5}}$

f $81^{\frac{3}{4}}$

g $216^{\frac{2}{3}}$

h $243^{\frac{3}{5}}$

i $(\sqrt[4]{16})^3$

j $(\sqrt[3]{27})^2$

k $\sqrt[5]{32^4}$

l $\sqrt[3]{2^6}$

Example 18

5 Simplify:

a $\left(\frac{1}{a^2}\right)^2$

b $\left(b^{\frac{1}{3}}\right)^6$

c $(c^{12})^{\frac{1}{4}}$

d $(c^{10})^{\frac{1}{5}}$

e $x^{\frac{1}{2}} \times x^{\frac{3}{2}}$

f $y^{\frac{1}{3}} \times y^{\frac{2}{3}}$

g $p^{\frac{3}{4}} \times p^{\frac{2}{5}}$

h $q^{\frac{3}{2}} \times q^{\frac{2}{3}}$

i $x^{\frac{3}{2}} \div x^{\frac{1}{2}}$

j $y^{\frac{2}{3}} \div y^{\frac{1}{3}}$

k $p^{\frac{3}{4}} \div p^{\frac{2}{5}}$

l $q^{\frac{3}{2}} \div q^{\frac{2}{3}}$

m $(4m^6)^{\frac{1}{2}}$

n $(27n^{12})^{\frac{1}{3}}$

o $\left(2x^{\frac{2}{3}}\right)^3$

p $\left(3y^{\frac{1}{2}}\right)^4$

Example 19

6 Evaluate:

a $4^{-\frac{1}{2}}$

b $25^{-\frac{1}{2}}$

c $\left(\frac{8}{125}\right)^{-\frac{1}{3}}$

d $\left(\frac{64}{27}\right)^{-\frac{1}{3}}$

e $32^{-\frac{2}{5}}$

f $\left(\frac{1}{81}\right)^{-\frac{1}{4}}$

g $81^{-\frac{1}{4}}$

h $\left(\frac{1}{25}\right)^{-\frac{1}{2}}$

i $\left(\frac{16}{81}\right)^{-\frac{1}{4}}$

j $\left(\frac{32}{243}\right)^{-\frac{1}{5}}$

Example 20

7 Simplify, expressing the answer with positive indices:

a $\left(\frac{1}{a^2}\right)^{-2}$

b $\left(b^{-\frac{2}{3}}\right)^6$

c $\left(2x^{\frac{2}{3}}\right)^{-3}$

d $\left(3y^{\frac{1}{2}}\right)^{-4}$

e $x^{\frac{1}{2}} \times x^{-\frac{3}{2}}$

f $y^{\frac{1}{3}} \times y^{-\frac{2}{3}}$

g $p^{\frac{3}{4}} \times p^{-\frac{2}{5}}$

h $q^{\frac{3}{2}} \times q^{-\frac{2}{3}}$

i $x^{\frac{3}{2}} \div x^{-\frac{1}{2}}$

j $y^{\frac{2}{3}} \div y^{-\frac{1}{3}}$

k $p^{\frac{3}{4}} \div p^{-\frac{2}{5}}$

l $q^{\frac{3}{2}} \div q^{-\frac{2}{3}}$

m $(4m^{-6})^{\frac{1}{2}}$

n $(27n^{-12})^{\frac{1}{3}}$

o $\left(2x^{-\frac{2}{5}}\right)^5$

8D Scientific notation



Scientific notation, or **standard form**, is a convenient way to represent very large or very small numbers. It allows such numbers to be easily read.



Here is an example with a very large number. The star Sirius A is approximately 81 362 000 000 000 km from the Sun. This is about 81 trillion kilometres. We can represent this number by placing a decimal point after the first non-zero digit and multiplying by an appropriate power of 10 to recover the original number. This distance can then be written as 8.1362×10^{13} km. If we move the decimal point thirteen places to the right, inserting the necessary zeroes, we arrive back at the number we started with.

We can also use this notation for very small numbers. For example, an Ångström (Å) is a unit of length equal to 0.000 000 000 1 m, which is the approximate diameter of a small atom. We place a decimal point after the first non-zero digit and multiply by the appropriate power of 10. Hence $0.000\,000\,000\,1\text{ m} = 1.0 \times 10^{-10}\text{ m}$ or $1 \times 10^{-10}\text{ m}$. When we move the decimal point 10 places to the left, inserting the zeroes, we arrive back at the original number. So, for example, the approximate diameter of a uranium atom is 0.000 000 000 38 m, or $3.8 \times 10^{-10}\text{ m}$, or 3.8 Å.



By definition, a positive number is in scientific notation if it is written as

$$a \times 10^b, \text{ where } 1 \leq a < 10 \text{ and } b \text{ is an integer.}$$

We also say that the above notation is the ‘**standard form**’ for a number.

Note: If the number to be written in scientific notation is greater than 1, then the index is positive or zero. If the number is positive and less than 1, then the index is negative.

The following examples demonstrate how to convert numbers from decimal form to scientific notation, and vice versa.

Example 21

Write in scientific notation:

a 610

b 21 000

c 46 000 000

d 81

e 0.0067

f 0.000 02

g 0.07

h 8.17

(continued on next page)



Solution

$$\begin{aligned}\mathbf{a} \quad 610 &= 6.1 \times 100 \\ &= 6.1 \times 10^2\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad 46\,000\,000 &= 4.6 \times 10\,000\,000 \\ &= 4.6 \times 10^7\end{aligned}$$

$$\begin{aligned}\mathbf{e} \quad 0.0067 &= 6.7 \div 1000 \\ &= 6.7 \times \frac{1}{1000} \\ &= 6.7 \times 10^{-3}\end{aligned}$$

$$\begin{aligned}\mathbf{g} \quad 0.07 &= 7 \div 100 \\ &= 7 \times 10^{-2}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad 21\,000 &= 2.1 \times 10\,000 \\ &= 2.1 \times 10^4\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad 81 &= 8.1 \times 10 \\ &= 8.1 \times 10^1\end{aligned}$$

$$\begin{aligned}\mathbf{f} \quad 0.00002 &= 2 \div 100\,000 \\ &= 2 \times \frac{1}{100\,000} \\ &= 2 \times 10^{-5}\end{aligned}$$

$$\mathbf{h} \quad 8.17 = 8.17 \times 10^0$$

Example 22

Write in decimal form:

$$\mathbf{a} \quad 2.1 \times 10^3$$

$$\mathbf{b} \quad 6.3 \times 10^5$$

$$\mathbf{c} \quad 5 \times 10^{-4}$$

$$\mathbf{d} \quad 8.12 \times 10^{-2}$$

Solution

$$\begin{aligned}\mathbf{a} \quad 2.1 \times 10^3 &= 2.1 \times 1000 \\ &= 2100\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad 5 \times 10^{-4} &= 5 \times \frac{1}{10\,000} \\ &= 5 \div 10\,000 \\ &= 0.0005\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad 6.3 \times 10^5 &= 6.3 \times 100\,000 \\ &= 630\,000\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad 8.12 \times 10^{-2} &= 8.12 \times \frac{1}{100} \\ &= 8.12 \div 100 \\ &= 0.0812\end{aligned}$$

Since numbers written in scientific notation involve powers, when these numbers are multiplied, divided or raised to a power, the index laws come into play.



Example 23

Simplify and write in scientific notation:

$$\mathbf{a} \quad (3 \times 10^4) \times (2 \times 10^6)$$

$$\mathbf{b} \quad (9 \times 10^7) \div (3 \times 10^4)$$

$$\mathbf{c} \quad (4.1 \times 10^4)^2$$

$$\mathbf{d} \quad (2 \times 10^5)^{-2}$$



Solution

$$\begin{aligned}\mathbf{a} \quad (3 \times 10^4) \times (2 \times 10^6) &= 3 \times 10^4 \times 2 \times 10^6 \\ &= 3 \times 2 \times 10^4 \times 10^6 \\ &= 6 \times 10^{10}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad (9 \times 10^7) \div (3 \times 10^4) &= \frac{9 \times 10^7}{3 \times 10^4} \\ &= \frac{9}{3} \times \frac{10^7}{10^4} \\ &= 3 \times 10^3\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad (4.1 \times 10^4)^2 &= 4.1^2 \times (10^4)^2 \\ &= 16.81 \times 10^8 \\ &= 1.681 \times 10^9\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad (2 \times 10^5)^{-2} &= 2^{-2} \times (10^5)^{-2} \\ &= \frac{1}{2^2} \times 10^{-10} \\ &= 0.25 \times 10^{-10} \\ &= 2.5 \times 10^{-11}\end{aligned}$$



Scientific notation

- **Scientific notation**, or **standard form**, is a convenient way to represent very large and very small numbers.
- To represent a number in scientific notation, insert a decimal point after the first non-zero digit and multiply by an appropriate power of 10.

For example:

$$81\,362\,000\,000\,000 = 8.1362 \times 10^{13} \text{ and } 0.000\,000\,000\,38 = 3.8 \times 10^{-10}$$



Some of the exercises in the rest of this chapter are best done using a calculator.



Exercise 8D

1 Write as a power of 10:

a 10

b 100

c 1000

d 10 000

e 1 000 000

f 1 000 000 000

g a googol – which is 1 followed by 100 zeroes.

Note: 10^6 is a million, 10^9 is a billion and 10^{12} is a trillion.

2 Write as a power of 10:

a $\frac{1}{10}$

b $\frac{1}{100}$

c $\frac{1}{1000}$

d 1 trillionth

e $\frac{1}{100\,000}$

f 1 millionth

Example 21

3 Write in scientific notation:

a 510

b 790

c 5300

d 26 000

e 64 000 000

f 796 000 000

g 576 000 000 000

h 4 000 000 000 000

i 0.008

j 0.06

k 0.000 72

l 0.000 041

m 0.000 000 006

n 0.000 000 206

Example 22

4 Write in decimal form:

a 3.24×10^4

b 7.2×10^3

c 8.6×10^2

d 2.7×10^6

e 5.1×10^0

f 7.2×10^1

g 5.6×10^{-2}

h 1.7×10^{-3}

i 8.72×10^{-4}

j 2.01×10^{-3}

k 9.7×10^{-1}

l 2.6×10^{-7}

5 The mass of the earth is approximately 6000 000 000 000 000 000 000 kg. Write this value in scientific notation.

6 The radius of the earth is about 6400 km. Write this approximate value in scientific notation.

7 Light travels approximately 299 000 km in a second. Express this in scientific notation.

8 The thickness of a piece of paper is about 0.0001 m. Express this in scientific notation.

9 The mass of a copper sample is 0.0089 kg. Express this in scientific notation.

10 The distance between interconnecting lines on a silicon chip for a computer is approximately 0.000 000 04 m. Express this in scientific notation.

Example 23

11 Simplify, expressing the answer in scientific notation:

a $(4 \times 10^5) \times (2 \times 10^6)$

b $(2.1 \times 10^6) \times (3 \times 10^7)$

c $(4 \times 10^2) \times (5 \times 10^{-7})$

d $(3 \times 10^6) \times (8 \times 10^{-3})$

e $(5 \times 10^4) \div (2 \times 10^3)$

f $(8 \times 10^9) \div (4 \times 10^3)$

g $(6 \times 10^{-4}) \div (8 \times 10^{-5})$

h $(1.2 \times 10^6) \div (4 \times 10^7)$

i $(2.1 \times 10^2)^4$

j $(3 \times 10^{-2})^3$

$$\mathbf{k} \quad \frac{(2 \times 10^5) \times (4 \times 10^4)}{1.6 \times 10^3}$$

$$\mathbf{m} \quad (4 \times 10^{-2})^2 \times (5 \times 10^7)$$

$$\mathbf{o} \quad \frac{(4 \times 10^5)^3}{(8 \times 10^4)^2}$$

$$\mathbf{l} \quad \frac{(8 \times 10^6) \times (4 \times 10^3)}{5 \times 10^7}$$

$$\mathbf{n} \quad (6 \times 10^{-3}) \times (4 \times 10^7)^2$$

$$\mathbf{p} \quad \frac{(2 \times 10^{-1})^5}{(4 \times 10^{-2})^3}$$

- 12** If light travels at 3×10^5 km/s and our galaxy is approximately 80 000 light years across, how many kilometres is it across? (A light year is the distance light travels in a year.)
- 13** The mass of a hydrogen atom is approximately 1.674×10^{-27} kg and the mass of an electron is approximately 9.1×10^{-31} kg. How many electrons, correct to the nearest whole number, will have the same mass as a single hydrogen atom?
- 14** If the average distance from the Earth to the Sun is 1.4951×10^8 km and light travels at 3×10^5 km/s, how long does it take light to travel from the Sun to the Earth?
- 15** The furthest galaxy detected by optical telescopes is approximately 4.6×10^9 light years from us. How far is this in kilometres? (Light travels at 3×10^5 km/s.)
- 16** In a lottery there are $\frac{45 \times 44 \times 43 \times 42 \times 41 \times 40}{720}$ different possible outcomes. If I mark each outcome on an entry form one at a time, and it takes me an average of 1 minute to mark each outcome, how long will it take me to cover all different possible outcomes?

8E Significant figures



Suppose that we are using a ruler marked in centimetres and millimetres to measure the length of a sheet of paper. We find that the length of the edge of the paper falls between 15.2 cm and 15.3 cm but is closer to 15.3 cm. The normal procedure is to use rounding and write the measurement as 15.3 cm.

Notice that there is uncertainty about the digit after the 3. If we had a more accurate way of measuring, we could remove that uncertainty and get another decimal place or two in the measurement of the length. In scientific notation, our measurement is 1.53×10^1 cm. We say that the length is 1.53×10^1 cm **correct to 3 significant figures**. This means that the length is between 1.525 cm and 1.535 cm.

The same applies to the way other measurements are reported. For example, you may read that the mass of an electron is about 0.000 000 000 000 000 000 000 000 910 938 26 g. This usually means that a measurement was made and the last digit 6 is a rounding digit and is therefore not completely accurate. The other digits are accurate. Writing this in scientific notation, we say that the mass of an electron is $9.109\,382\,6 \times 10^{-28}$ g **correct to 8 significant figures** because there are 8 digits in the factor 9.109 382 6 before the power of 10.

The mass of the earth is about 5.9736×10^{24} kg. How many significant digits are there in this measurement?

There are 5 digits in 5.9736, so the measurement is correct to 5 significant figures.

Notice that, to use the idea of significant figures, we must first express the number in scientific notation.

People sometimes apply the same kind of ideas to shorten a given decimal number containing many digits. This need not have anything to do with measurement. It is a way of abbreviating the information you are given. You might imagine that we begin to have doubts about the accuracy of the later digits in the number and so decide to ignore them. The procedure is called **writing the number to a certain number of significant figures**.

To write a number to a specified number of significant figures, first write the number in scientific notation and then round to the required number of significant figures.

For example, $0.000\,340\,61 = 3.4061 \times 10^{-4}$

$$\begin{aligned} &\approx 3 \times 10^{-4} && \text{(correct to 1 significant figure)} \\ &\approx 3.4 \times 10^{-4} && \text{(correct to 2 significant figures)} \\ &\approx 3.41 \times 10^{-4} && \text{(correct to 3 significant figures)} \\ &\approx 3.406 \times 10^{-4} && \text{(correct to 4 significant figures)} \end{aligned}$$

Example 24

Write in scientific notation and then round correct to 3 significant figures:

a 235.674

b 0.007 245 46

Solution

a $235.674 = 2.356\,746 \times 10^2$
 $\approx 2.36 \times 10^2$

b $0.007\,245\,46 = 7.245\,46 \times 10^{-3}$
 $\approx 7.25 \times 10^{-3}$

Example 25

Write in scientific notation and then round correct to 2 significant figures:

a 276 000 000

b 0.000 000 654

Solution

a $276\,000\,000 = 2.76 \times 10^8$
 $\approx 2.8 \times 10^8$

b $0.000\,000\,654 = 6.54 \times 10^{-7}$
 $\approx 6.5 \times 10^{-7}$



Significant figures

- A number may be expressed with different numbers of **significant figures**.
For example: 3.1 has 2 significant figures, 3.14 has 3 significant figures, 3.141 has 4 significant figures.
- To **round off** a number correct to a required number of significant figures, first write the number in scientific notation and identify the last significant digit required. Then:
 - if the next digit is 0, 1, 2, 3 or 4, round down
 - if the next digit is 5, 6, 7, 8 or 9, round up.
- To write a number correct to a specified number of significant figures, first write the number in scientific notation and then round to the required number of significant figures.



Exercise 8E

Example 24

1 Write in scientific notation, correct to 3 significant figures:

a 2.7043

b 634.96

c 8764.37

d 256 412

e 0.003612

f 0.024 186

Example 25

2 Write in scientific notation, correct to 2 significant figures:

a 368.2

b 278 000

c 0.004 321

d 0.000 021 906

3 Write in scientific notation correct to the number of significant figures indicated in the brackets:

- | | |
|--------------------------------------|--------------------------------------|
| a 274.62 (4) | b 274.62 (3) |
| c 274.62 (2) | d 274.62 (1) |
| e 0.041 236 (4) | f 0.041 236 (3) |
| g 0.041 236 (2) | h 0.041 236 (1) |
| i 1704.28 (4) | j 1704.28 (3) |
| k 1704.28 (2) | l 1704.28 (1) |
| m 1.9925×10^{27} (4) | n 1.9925×10^{27} (3) |
| o 1.9925×10^{27} (2) | p 1.9925×10^{27} (1) |

4 Use a calculator to evaluate the following, giving the answer in scientific notation correct to 3 significant figures:

- | | |
|---|---|
| a 3.24×0.067 | b $6.24 \div 0.026$ |
| c $4.736 \times 10^{13} \times 2.34 \times 10^{-6}$ | d $(5.43 \times 10^{-6}) \div (6.24 \times 10^{-4})$ |
| e $\sqrt{9.24 \times 10^7}$ | f $0.0276^2 \times \sqrt{0.723}$ |
| g $\frac{17.364 \times 24.32 \times 5.4^2}{3.6 \times 7.31^2}$ | h $\frac{6.54(5.26^2 + 3.24)}{5.4 + \sqrt{6.34}}$ |
| i $\frac{6.283 \times 10^8 \times 5.24 \times 10^6}{(4.37 \times 10^7)^2}$ | j $6.24 \times 3.27 \div (5.76 + 4.32)$ |
| k $52.4 \times 63.5 \div 24.7 \times 5 - 1076$ | |

5 Use a calculator to evaluate, giving the answer in scientific notation correct to 4 significant figures:

- | | |
|---|---|
| a 1.234×0.1988 | b $4.863 \times 10^{21} \times 7.919 \times 10^{-11}$ |
| c $1.234 \div 0.1988$ | d $(4.863 \times 10^{21}) \div (7.919 \times 10^{-11})$ |
| e 1.9346^3 | f $(7.919 \times 10^{21})^2$ |
| g $\sqrt{0.9741}$ | h $\sqrt{4.863 \times 10^{-12}}$ |
| i $\frac{177.41 \times 0.048}{16.23}$ | j $\frac{7.932 \times 10^{12} \times 9.4 \times 10^{-10}}{0.000\,000\,000\,416}$ |
| k $\frac{579.2 \times 0.6231}{79.05 \times 115.4}$ | l $\frac{74\,510\,000\,000}{6.4 \times 10^{-18} \times 4.4 \times 10^{23}}$ |

$$\mathbf{m} \quad \frac{79.99}{\sqrt{48.92 + 11.68^2}} \quad \mathbf{n} \quad \frac{15.62^2(79.1 + 111.7)}{12.46 + 4.48^3}$$

$$\mathbf{o} \quad 56.21 \times 12 + \frac{1}{2} \times 9.8 \times 12^2$$

- 6** Estimate each of the following, correct to one significant figure. In each case explain how you obtained your answer.

a the thickness of a sheet of paper

b the volume of your classroom

c the height of a six-storey building

d the area required for a car park for 500 cars

e the total printed area of a 600-page novel

f the volume of a warehouse that can store 300 000 pairs of shoes (still in their boxes)

g the length of a queue if every student in your school is standing in it

Discuss your answers with others in your class and your teacher.



Review exercise

1 Evaluate:

a 4^3 **b** 2^6 **c** 8^2 **d** 10^6

2 Express as a product of powers of prime numbers:

a 120^2 **b** 900^3 **c** 315^4 **d** 490^5

3 Simplify and evaluate where possible:

a $a^6 \times a^7$ **b** $b^4 \times b^9$ **c** $3a^4 \times 5a^5$ **d** $2x^3 \times 5x^6$

e $a^7 \div a^4$ **f** $m^{12} \times m^6$ **g** $\frac{12b^7}{6b^2}$ **h** $\frac{18p^{10}}{9p}$

i $(a^4)^3$ **j** $(b^6)^5$ **k** $(2a^7)^3$ **l** $(3m^2)^4$

m a^0 **n** $3b^0$ **o** $5m^0$ **p** $(3q)^0$

4 Simplify and evaluate where possible:

a $4a^2b^3 \times 5ab^4$ **b** $2m^4n^3 \times 5m^6n^7$ **c** $\frac{20a^4b^2}{5a^2b}$

d $\frac{24m^9n^4}{18m^6n^2}$ **e** $(3a^3b)^4$ **f** $(5a^2b)^2 \times 4a^4b^3$

g $\frac{5a^6b^7}{4a^3b^2} \times \frac{12a^{10}b^9}{a^6b^7}$ **h** $\frac{8m^4n^2}{7m^3n} \div \frac{3m^3n^5}{14m^9n^{16}}$ **i** $\frac{(2x^2y)^3}{5x^6y^2} \times \left(\frac{x^3}{2y^2}\right)^3$

j $\frac{(4ab)^3 \times 5a^2b}{(10ab^2)^2}$ **k** $\frac{(3x^2y)^3 \times 2(xy^2)^3}{(3xy)^4}$ **l** $\frac{(4a^2b^3)^2}{3a^6b^4} \div \frac{ab^5}{(3a^3b^2)^3}$

5 Evaluate:

a 6^{-2} **b** 8^{-3} **c** 2^{-7} **d** 4^{-3}

e $\left(\frac{4}{5}\right)^{-2}$ **f** $\left(\frac{2}{3}\right)^{-4}$ **g** $\left(2^{\frac{1}{2}}\right)^{-3}$ **h** $\left(16^{\frac{1}{4}}\right)^{-3}$

6 Simplify, expressing the answer with positive indices:

a $\frac{4m^2n^5p^{-6}}{16m^{-2}n^5p^3}$ **b** $(3x)^{-2}$ **c** $(4y)^{-3}$ **d** $(2^2y^3)^{-5}$

e $(5^{-2}x^3)^{-5}$ **f** $(7^{-2}ab^2)^{-2}$ **g** $(3^{-3}a^2b^{-1})^{-4}$ **h** $\left(\frac{a^3}{b^2}\right)^{-2}$

i $\left(\frac{a^4}{p^5}\right)^{-2}$ **j** $\left(\frac{3a^{-2}}{b^3}\right)^{-3}$ **k** $\left(\frac{5g^2}{h^{-3}}\right)^{-2}$ **l** $\left(\frac{m^{-3}}{(2n)^{-4}}\right)^{-2}$



7 Simplify, expressing the answer with positive indices:

a $4a^2 \times 5a^{-3}$

b $8m^2n^{-3} \times 5m^{-4}n^6$

c $14a^{-4} \div 7a^{-5}$

d $\frac{18m^2n^{-3}}{9m^4n^{-1}}$

e $(4m^2n^{-2})^{-3}$

f $\frac{(5m^2n)^{-2}}{10m^4n^3}$

g $(3a^4b^{-3})^{-3} \div 6a^2b^{-7}$ **h** $\frac{2m^3n^4}{(5m)^2} \times \frac{10m}{3n^{-4}}$

i $\frac{(5a^4b^{-3})^2}{a^{-2}b} \div \frac{5(a^{-1}b)^{-3}}{ab^{-4}}$

8 Evaluate:

a $49^{\frac{1}{2}}$

b $8^{\frac{1}{3}}$

c $125^{\frac{2}{3}}$

d $64^{\frac{2}{3}}$

e $9^{\frac{3}{2}}$

f $16^{-\frac{3}{2}}$

g $\left(\frac{1}{8}\right)^{-\frac{2}{3}}$

h $\left(\frac{64}{27}\right)^{-\frac{2}{3}}$

9 Simplify, expressing the answer with positive indices:

a $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$

b $3b^{\frac{2}{3}} \times 4b$

c $p^{\frac{2}{3}} \div p^{\frac{1}{2}}$

d $m^{-\frac{1}{2}} \div m^{-2}$

e $(4x^{\frac{1}{3}})^4$

f $(2x^{-\frac{1}{3}})^{-2}$

g $(9a^{\frac{1}{3}})^{\frac{1}{2}}$

h $(8p^{-2}q^3)^{\frac{1}{2}}$

10 Write in scientific notation:

a 47 000

b 164 000 000

c 0.0047

d 0.0035

e 840

f 0.840

11 Write in decimal form:

a 6.8×10^4

b 7.5×10^3

c 2.6×10^{-3}

d 9.4×10^{-2}

e 6.7×10^0

f 3.2×10^{-4}

12 Simplify, writing each answer in scientific notation:

a $(4 \times 10^6) \times (2 \times 10^2)$

b $(3.1 \times 10^4) \times (2 \times 10^{-2})$

c $(9 \times 10^4) \div (3 \times 10^{-2})$

d $(5.2 \times 10^{-7}) \div (2 \times 10^3)$

e $(3 \times 10^4)^3$

f $(5 \times 10^{-4})^2$

g $(4 \times 10^2)^2 \times (5 \times 10^{-6})$

h $\frac{(3 \times 10^4)^3}{9 \times 10^{-2}}$

13 Write in scientific notation correct to the number of significant figures indicated in the brackets.

a 18.62 (2)

b 18.62 (3)

c 18.62 (1)

d 0.004 276 (2)

e 5973.4 (2)

f 0.473 952 (4)

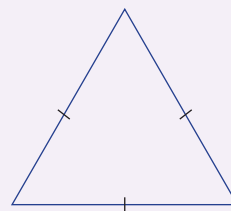


Challenge exercise

- 1 The following table gives information regarding the population, approximate rate of population growth, and land area of several countries in 1996. Use the information to answer the questions below.

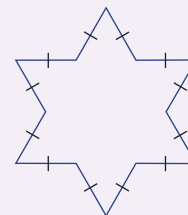
Country	Population (in millions)	Rate of population growth p.a.	Area of country (in km ²)
Australia	18.2	1.4%	7.6×10^6
China	1200	1.1%	9.6×10^6
Indonesia	201	1.6%	1.9×10^6
Singapore	2.9	1.1%	633
New Zealand	3.4	0.6%	2.7×10^5
United States	261	1.0%	9.2×10^6

- a How many times larger is Australia than Singapore, correct to the nearest whole number?
- b How many times larger is China than New Zealand, correct to one decimal place?
- c Express Indonesia's population as a percentage of the United States' population. (Calculate your answer correct to the nearest per cent)
- d The population density of a country is defined to be the average number of people for each square kilometre of land. Calculate the population density for each country and find:
- how many times larger the population density of China is compared to Australia
 - the country with the lowest population density
 - the country with the highest population density.
- e If each country maintains its present rate of population growth, what will the population of each country be (assume compounding growth):
- 5 years from 1996
 - 10 years from 1996
 - 100 years from 1996?
- f Find the population density of each country 100 years from 1996, assuming each country maintains its present growth rate.
- 2 a Consider the equilateral triangle shown opposite. If each side of the triangle is of length 3 cm, what is the perimeter of the triangle?





- b** Suppose that the following procedure is performed on the triangle above: ‘On the outside of each side of the triangle, draw a triangle with side lengths one third of the side lengths of the original triangle’.

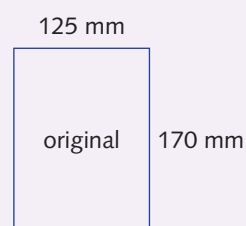


After this procedure is done, the star shape opposite is obtained.

- i** What is the length of each edge?
 - ii** What is the perimeter of the star?
 - iii** What is the ratio of the perimeter of this star to the original perimeter?
- c** Suppose that the procedure of part **b** is performed on this star (that is, each of the 12 sides has a triangle with side lengths one third of the previous ones drawn on it).
- i** How many sides will the new star have?
 - ii** What is the length of each edge of the new star?
 - iii** What is the perimeter of the new star?
 - iv** If the perimeter of the new star can be written in the form $a^2 \times 9$, find the value of a .
- d** Suppose that the procedure is repeated indefinitely. What will the perimeter of the shape be after the procedure has been performed:
- i** 3 times (expressing your answer in the form $a^3 \times 9$)
 - ii** 4 times (expressing your answer in a similar way)
 - iii** 10 times (expressing your answer using indices)
 - iv** n times (expressing your answer using indices)?
- e** Using your calculator, find, by trial and error, the number of times this procedure has to be done to produce a perimeter of at least:
- i** one metre **ii** ten metres **iii** one kilometre.
- f** If this procedure is performed indefinitely, what do you think happens to the perimeter? Does this surprise you?

Note: The shape produced by indefinitely repeating this procedure is called the Snowflake curve. You might like to try to draw what the snowflake looks like after each of the first few applications of the procedure.

- 3** A student is using a photocopy machine to reduce a picture of dimensions 170 mm $\times$ 125 mm. He sets the photocopier on 80% reduction (that is, each length in the photocopy will be 80% of its original length).



- a** What are the dimensions of the photocopy of the picture?



- b** If the student now reduces the photocopy to 80%, what are the dimensions of the photocopy of the photocopy?
- c** If the student continues to take 80% reduction photocopies, what are the dimensions of the picture after:

i 3 photocopies **ii** 4 photocopies **iii** n photocopies?

In each case, express your answer in terms of a power of 0.8.

- d** What is the area of the original picture?

- e** What is the area of the picture after:

i 1 photocopy **ii** 2 photocopies
iii 3 photocopies **iv** n photocopies?

In each case, express your answers in terms of a power of 0.8.

- f** If the student started with a picture of dimensions x cm $\times$ y cm, and successive photocopies are taken with reduction 80%:

i what are the dimensions of the picture after n photocopies
ii what is the area of the picture after n photocopies?

- 4** Write $2^{n+1} + 2^{n+1}$ as a single power of 2.

- 5** Simplify:

$$\begin{array}{ll} \text{a } \frac{8^n \times 3^{2n}}{2^n \times 6^n \times 9^n} & \text{b } \frac{2^{2n+1}}{(2^n)^{n+1}} \div \frac{(2^{2n+1})^{n-1}}{4^n} \\ \text{c } \frac{2^{n+3} - 4 \times 2^n}{2^{2n} - 4^{n-1}} & \text{d } \frac{a^2 x^m - b^2 x^{m+4}}{a - bx^2} \end{array}$$

- 6** If $10^x = p$, find the following in terms of p :

a 10^{2x+1} **b** 10^{1-3x}

- 7** Simplify $\frac{(yz)^{b+c} (zx)^{c+a} (xy)^{a+b}}{x^a y^b z^c}$

- 8** Expand and simplify:

a $(x - x^{-1})^2$ **b** $(x^2 + x^{\frac{1}{2}})^2$ **c** $(x^n - 3x^{-n})^2$

- 9** Simplify:

$$\begin{array}{ll} \text{a } \frac{2^{x+1} - 2^{x-2}}{2^{x-1} - 2^{x+2}} & \text{b } \frac{x^{\frac{1}{2}} y^{\frac{1}{2}} - x^{\frac{1}{2}} y^{\frac{1}{2}}}{x^{-1} - y^{-1}} \\ \text{c } \frac{x-1}{x - x^2 - 2} & \text{d } \frac{x-y}{\frac{1}{x^2} - \frac{1}{y^2}} \end{array}$$



- 10 a** Prove that the index laws hold for integer indices. (Use the laws for positive integer indices.)

For example, the product of powers result can be proved in the following way for negative integer indices.

Consider $a^{-p}a^{-q}$ where p and q are positive integers.

$$\begin{aligned} a^{-p}a^{-q} &= \frac{1}{a^p} \times \frac{1}{a^q} \\ &= \frac{1}{a^p a^q} \\ &= \frac{1}{a^{p+q}} \quad (\text{Index law 1 for positive integers.}) \\ &= a^{-(p+q)} \\ &= a^{-p+(-q)} \end{aligned}$$

- b** Prove that the index laws hold for fractional indices.

For example, $a^{\frac{1}{n}} \times a^{\frac{1}{m}} = a^{\frac{1}{n} + \frac{1}{m}}$ can be proved in the following way.

$$\begin{aligned} a^{\frac{1}{n}} \times a^{\frac{1}{m}} &= a^{\frac{m}{nm}} \times a^{\frac{n}{nm}} \\ &= \sqrt[nm]{a^m} \times \sqrt[nm]{a^n} \\ &= \sqrt[nm]{a^m \times a^n} \\ &= \sqrt[nm]{a^{m+n}} \\ &= a^{\frac{m+n}{nm}} \\ &= a^{\frac{1}{n} + \frac{1}{m}} \end{aligned}$$

The result can easily be extended to $a^{\frac{p}{n}} \times a^{\frac{q}{m}} = a^{\frac{p}{n} + \frac{q}{m}}$.

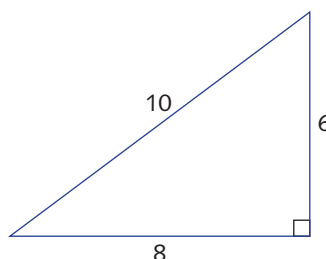
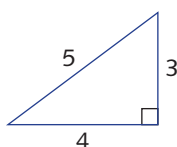
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Enlargements and similarity

This chapter introduces a transformation called an **enlargement**. This allows us to develop the idea of **similar** figures, which is a more general idea than congruence.

For example, the two right-angled triangles below are similar because they have exactly the same shape, but they are not congruent because they have different side lengths.



Similarity and proportion are closely related, as we shall see. This idea has been used in art and architecture for centuries. It is now used extensively in photography and in the zoom function of a computer application.

9A Enlargements



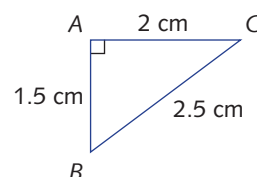
You have already met three types of transformations:

- translations
- rotations
- reflections.

This section introduces a fourth type of transformation called an **enlargement**, in which the lengths of the sides are increased or decreased by the same factor. In *ICE-EM Mathematics Year 8 Book 2*, we looked at these ideas under the topic heading of **scale drawings**. Now we are going to examine these ideas a little more closely.



The figure to the right is a right-angled triangle ABC , whose sides have length 1.5 cm, 2 cm and 2.5 cm.



O

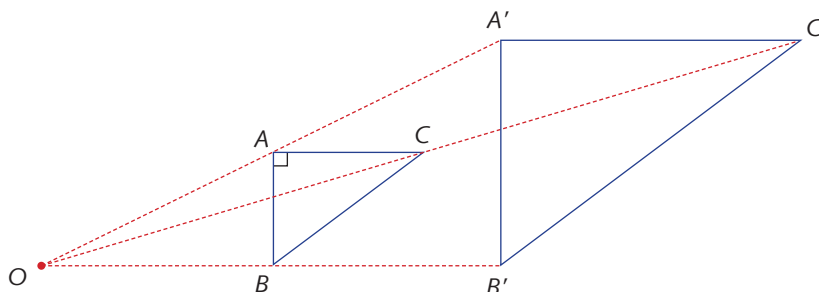
To define an enlargement of this figure, two things need to be specified.

First, we have to specify a **centre of enlargement** – in our example, it is the point O to the left of the triangle. The centre of enlargement stays fixed in the one place while the enlargement expands or shrinks everything else around it.

Secondly, we have to specify an **enlargement ratio** or **enlargement factor** – we will use an enlargement ratio of 2 here. This means that every point will double its distance from O . As a consequence, the side lengths of the triangle will double, as we saw in scale drawings.



The second diagram shows how this is done.



- The interval OA is joined, then produced to A' so that $OA' = 2 \times OA$.
- The interval OB is joined, then produced to B' so that $OB' = 2 \times OB$.
- The interval OC is joined, then produced to C' so that $OC' = 2 \times OC$.

The triangle $A'B'C'$ is then joined up, and is called the **image** of triangle ABC with enlargement factor 2 and centre of enlargement O .

The image triangle $A'B'C'$ is related to the original triangle in three important ways.

First, each side of the image triangle $A'B'C'$ is twice the length of the matching sides of the original triangle. We will usually write this using ratios:

$$\frac{A'B'}{AB} = 2$$

$$\frac{B'C'}{BC} = 2$$

$$\frac{A'C'}{AC} = 2$$

Secondly, each angle of the image is equal to the matching angle of the original:

$$\angle B'A'C' = \angle BAC$$

$$\angle A'C'B' = \angle ACB$$

$$\angle C'B'A' = \angle CBA$$

Thus the image $A'B'C'$ is a scale drawing of the original figure ABC , with ratio 2.

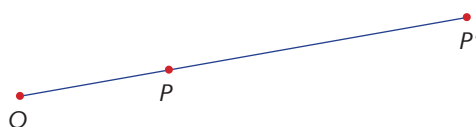
Thirdly, since matching angles are equal we can see that $A'B' \parallel AB$, $B'C' \parallel BC$ and $C'A' \parallel CA$.

The same kinds of results hold if we perform an enlargement by any positive factor.



Enlargement transformations:

- Each enlargement has a centre of enlargement O and an enlargement factor $k > 0$.
- The enlargement moves each point P to a new point P' on the ray with vertex O passing through P .
- The distance of P' from O is k times the distance of P from O .
That is, $OP' = k \times OP$.



When a figure is enlarged by a factor k :

- The image of each interval has k times the length of the original interval.
- The image of an angle has the same size as the original angle.
- The image of an interval is parallel to the original interval.

The first three dot points follow from the definition of an enlargement. If $k > 1$, then the image is larger than the original figure. If $k < 1$, then the image is smaller. If $k = 1$ then the original figure is unchanged by the transformation. In the challenge exercise, we will show how the last three points above can be proved using congruence. In the meantime, we will use all of the facts from the above box.

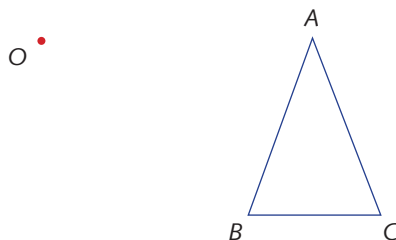
The main part of this chapter shows how to apply these properties of enlargement transformations. We shall develop the concept of similarity and introduce four similarity tests.

The following exercise explores enlargements and revisits scale drawings.

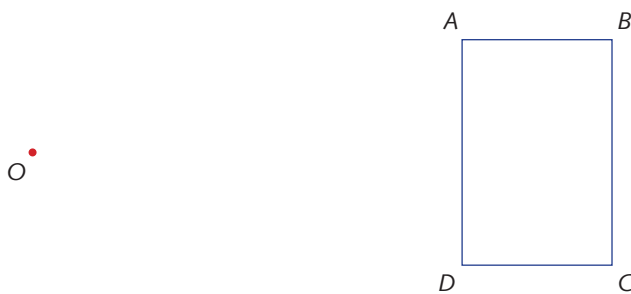


Exercise 9A

- 1 The following diagram shows an isosceles triangle ABC and a point O . Copy or trace the diagram into your exercise book, leaving plenty of room to the right and below your diagram.



- a Use ruler and compasses to apply an enlargement transformation with centre O and enlargement factor 3. Label your image $A'B'C'$.
- b Verify that each side length of $\triangle A'B'C'$ is three times the matching side length of $\triangle ABC$.
- c Use a protractor to verify that each angle of $\triangle A'B'C'$ is equal to the matching angle of $\triangle ABC$.
- 2 The diagram below shows a rectangle $ABCD$ and a point O . Copy or trace the diagram into your exercise book.



- a Now apply an enlargement with centre O and factor $\frac{1}{2}$ using these steps:
- Join OA and mark the midpoint A' of OA .
 - Join OB and mark the midpoint B' of OB .
 - Join OC and mark the midpoint C' of OC .
 - Join OD and mark the midpoint D' of OD .

- b** Use compasses to verify that each side length of $A'B'C'D'$ is half the matching side length of $ABCD$.
- c** Use a protractor to verify that each angle of $A'B'C'D'$ is equal to the matching angle of $ABCD$, that is, they are all right angles.
- 3** A ginger cat is sitting on a thin railing 2 metres horizontally from a desk lamp, producing a shadow of the cat on a wall 3 metres behind the cat.
- a** Draw a diagram showing how the cat's shadow can be characterised as an enlargement of the cat.
- b** What are the centre of enlargement and the enlargement factor of the transformation?
- 4** I have built a primitive slide projector by shining light from a point source through a transparent slide held up 10 cm away. The light then falls onto a wall that is parallel to the slide and distant 1 metre from it.
- a** Draw a diagram showing how the slide's image can be described as an enlargement of the slide.
- b** What are the centre of enlargement and the enlargement factor of the transformation?
- 5** A sprinkler is situated 1 metre from a gum tree whose trunk has a diameter of 80 cm, and is spraying water out in all directions. Two people standing 4 metres behind the tree remain dry. Draw a diagram, and use the enlargement factor to find the width of the 'safe dry zone' where they will not get wet.
- 6** Isaac is making a model of the solar system, in which the model of planet Mercury is 1 metre from the model of the Sun. Here are the mean distances of the eight planets from the Sun, given in Astronomical Units (1 AU is the mean distance from the Earth to the Sun, and is about 149.60 million kilometres).
- | | | |
|--------------------|-------------------|------------------|
| Mercury: 0.3871 AU | Venus: 0.7233 AU | Earth: 1 AU |
| Mars: 1.524 AU | Jupiter: 5.203 AU | Saturn: 9.539 AU |
| Uranus: 19.19 AU | Neptune: 30.06 AU | |
- Use your calculator to find the distances in metres of the other seven model planets from Isaac's model Sun, giving answers correct to two decimal places.
- 7** Ernest is making scale drawings of atoms, in which his drawing of a hydrogen atom is a circle of radius 1 cm. Here are the actual approximate radii of some of the atoms in his drawings, given in picometres ($1 \text{ pm} = 10^{-12} \text{ metres}$).
- Hydrogen: 25 pm Carbon: 70 pm Gold: 135 pm Radium: 215 pm
- Calculate the radii of Ernest's drawings of carbon, gold and radium atoms.
- 8** A map of a country is drawn to a scale of 1:12 500 000. Find the actual distance between two points whose separation on the map is:

- a** 1 cm **b** 4 cm **c** 0.6 cm **d** 1 mm



Two figures are called **similar** if we can enlarge one figure so that its enlargement is congruent to the other figure.

In simple terms this means that by enlarging or shrinking one figure, we get the other figure perhaps translated, rotated or reflected.

Thus similar figures have the same shape, but not necessarily the same size, just as a scale drawing has the same shape as the original, but has a different size.

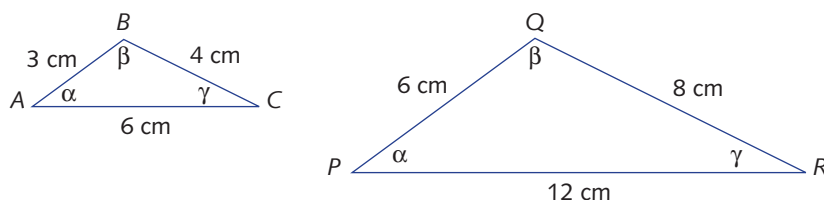
- Matching lengths of similar figures are therefore always in the same ratio, called the **similarity ratio**.
- Matching angles of similar figures are equal.



We will use the same convention for matching angles as we did for congruence.

If $\triangle ABC$ is similar to $\triangle PQR$ then, $\angle A = \angle P$, $\angle B = \angle Q$ and $\angle C = \angle R$

The two triangles in the diagram below are similar.

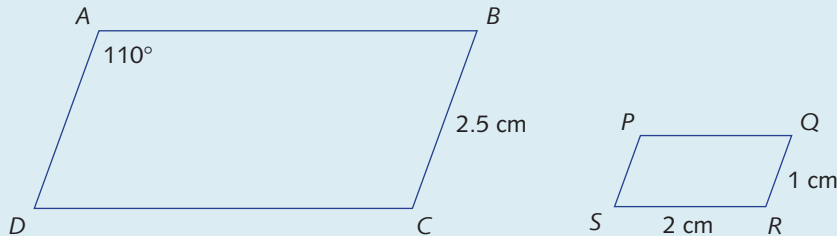


$\triangle ABC$ is similar to $\triangle PQR$.

Hence $\frac{PQ}{AB} = \frac{QR}{BC} = \frac{PR}{AC} = 2$ so the similarity ratio is 2.

Example 1

These two parallelograms are known to be similar.



- a** What is the size of $\angle P$?
- b** Find the base length of the large parallelogram.
- c** What is the similarity ratio of the transformation taking $PQRS$ to $ABCD$?

Solution

a $\angle P = 110^\circ$ (matching angles of similar figures)

b $\frac{DC}{SR} = \frac{BC}{QR}$ (ratio of matching sides)

$$\frac{DC}{2} = \frac{2.5}{1}$$

$$DC = 5 \text{ cm}$$

c The similarity ratio is 2.5.

Note 1: In part **b**, we could have written either

$$\frac{DC}{BC} = \frac{SR}{QR} \quad (\text{ratios within figures, 'larger' figure on the left, 'smaller' figure on the right})$$

or

$$\frac{DC}{SR} = \frac{BC}{QR} \quad (\text{one pair of matching sides on the left, the other pair on the right}).$$

It does not matter at all which method you choose to use. The subsequent algebra will be easier, however, if you always start by placing the unknown length on the top of the left-hand side of the equation.

Note 2: An alternative solution to part **b** would be to calculate $\frac{BC}{QR} = 2.5$ as the similarity ratio.

Thus $\frac{DC}{SR} = 2.5$ and $DC = 2.5 \times SR$.



Similar figures

- Two figures are called **similar** if an enlargement of one figure is congruent to the other figure.



Matching lengths and angles of similar figures

- Matching lengths in similar figures are in the same ratio, called the **similarity ratio**.
- Matching angles in similar figures are equal.

Example 2

The car shown in the photograph below is 3.8 metres long.



- a** Use your ruler and your calculator to find approximate ratios for the following, each correct to one decimal place. Your measurements are for the diagram but the ratio is for both the diagram and the real car.
- i** $\frac{\text{height of the car}}{\text{diameter of wheel}}$ **ii** $\frac{\text{length of the car}}{\text{height of the car}}$
- b** Find the height of the real car.

Solution

- a** In the picture, height of the car ≈ 2.8 cm
diameter of a wheel ≈ 1.2 cm
length of the car ≈ 8.2 cm.

$$\begin{array}{ll} \text{i} \quad \frac{\text{height of the car}}{\text{diameter of wheel}} \approx \frac{2.8}{1.2} & \text{ii} \quad \frac{\text{length of the car}}{\text{height of the car}} \approx \frac{8.2}{2.8} \\ & \approx 2.3 \qquad \qquad \qquad \approx 2.9 \end{array}$$

- b** Enlargement factor $\approx \frac{380}{8.2} \approx 46$

The height of the car $\approx 2.8 \times 46 \approx 128 \text{ cm} = 1.28 \text{ m}$



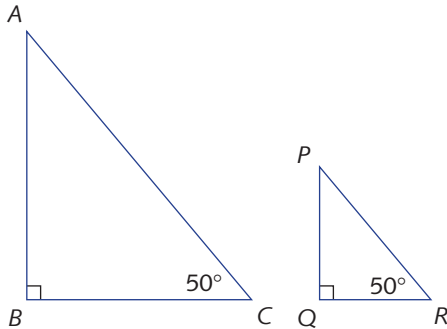
Exercise 9B



Example 1

- 1 The diagram in each part shows two figures that are known to be similar. Copy and complete the statements below each diagram.

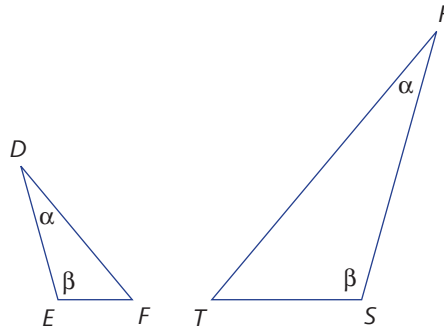
a



i $\frac{PR}{\dots} = \frac{PQ}{\dots}$

ii $\frac{QR}{\dots} = \frac{RP}{\dots}$

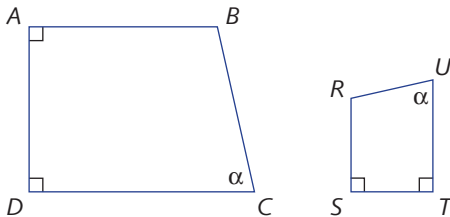
b



i $\frac{DE}{\dots} = \frac{DF}{\dots}$

ii $\frac{FE}{\dots} = \frac{FD}{\dots}$

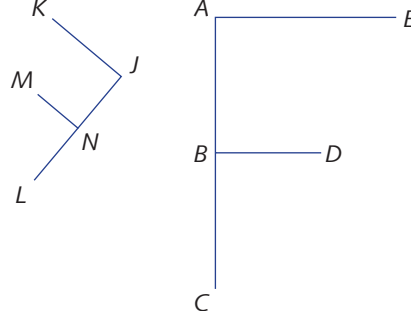
c



i $\frac{AB}{\dots} = \frac{AD}{\dots}$

ii $\frac{CB}{\dots} = \frac{CD}{\dots}$

d



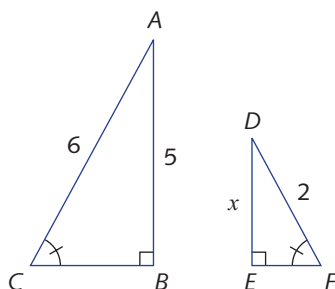
i $\frac{JK}{\dots} = \frac{MN}{\dots}$

ii $\frac{JL}{\dots} = \frac{MN}{\dots}$

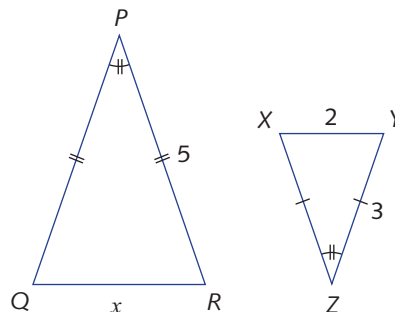
Example 1c

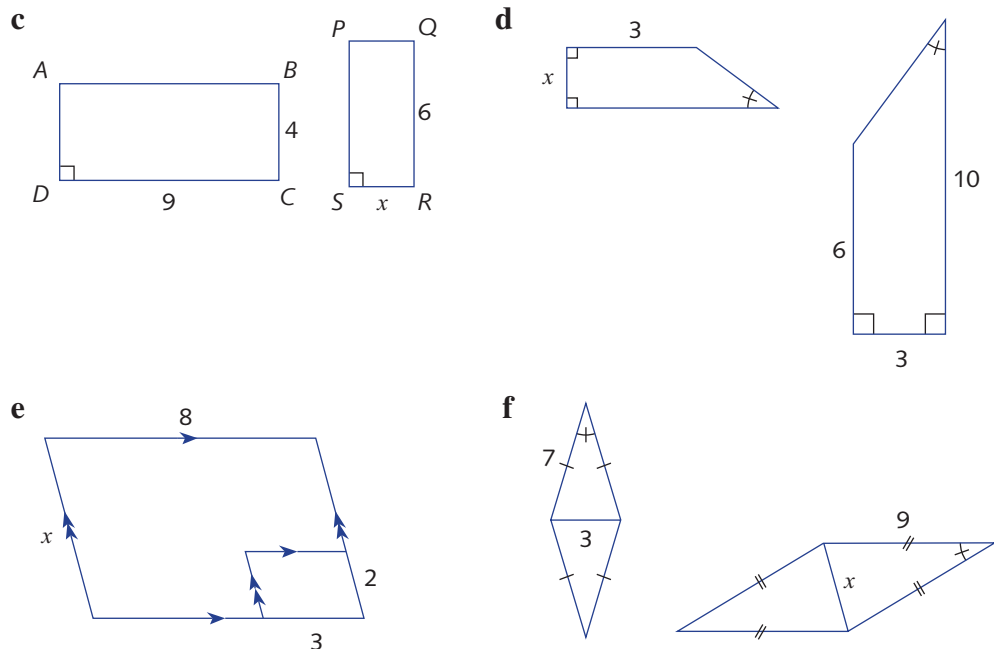
- 2 The diagram in each part shows two figures that are known to be similar. Write a ratio statement. Hence find the value of the pronumeral.

a



b





Example 2

- 3** Below is the Sydney Harbour Bridge, photographed from some distance away. By measuring lengths with your ruler, find approximations to:



- a** The ratio of the road distance between the towers and the height of the arch above the road.
- b** The ratio of the height of the left-hand tower and the height of the arch above the road.

- 4 The diagram below shows some of the keys of a piano. By measuring lengths with your ruler, find approximately:



- The ratio of the lengths of the black and the white keys.
- The ratio of the widths of the black and white keys at their front edges.
- The ratio of the length of a white key and the width of its front edge.
- If the length of a white key is 13.3 cm, find the width of its front edge.

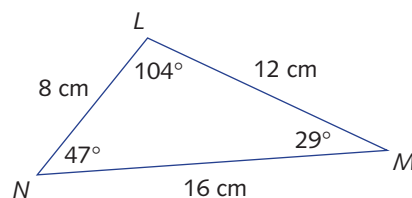
In question 5 to 8, assume that the triangles are named in matching order.

- 5 $\triangle ABC$ is similar to $\triangle PQR$. Copy and complete the following:

a $\angle ACB = \angle \dots$ b $\angle CAB = \angle \dots$ c $\frac{AB}{PQ} = \frac{AC}{\dots} = \frac{\dots}{QR}$

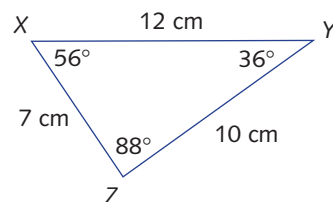
- 6 $\triangle LMN$ is similar to $\triangle TUV$, where $\triangle LMN$ is shown below, and $TU = 24$ cm, find:

- $\angle TUV$
- UV
- $\angle VTU$
- VT



- 7 $\triangle XYZ$ is similar to $\triangle GHK$, where $\triangle XYZ$ is shown below, and $HK = 18$ cm, find:

- $\angle GHK$
- $\angle GKH$
- GH
- GK



- 8 In two triangles ABC and LMN , $AB = 6$ cm, $LM = 12$ cm, $LN = 8$ cm and $MN = 16$ cm. What is the similarity ratio between the two triangles if:

- $\triangle ABC$ is similar to $\triangle LMN$
- $\triangle ABC$ is similar to $\triangle LNM$
- $\triangle ABC$ is similar to $\triangle MNL$
- $\triangle ABC$ is similar to $\triangle MLN$?

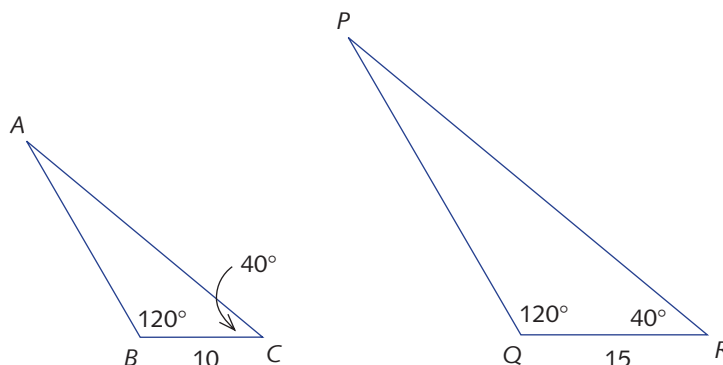
9C The AAA similarity test for triangles



As with congruence, most of the problems we treat involving similarity come down to establishing that two triangles are similar. In this section and the next two sections, we shall establish four **similarity tests** for triangles and use them in problems.

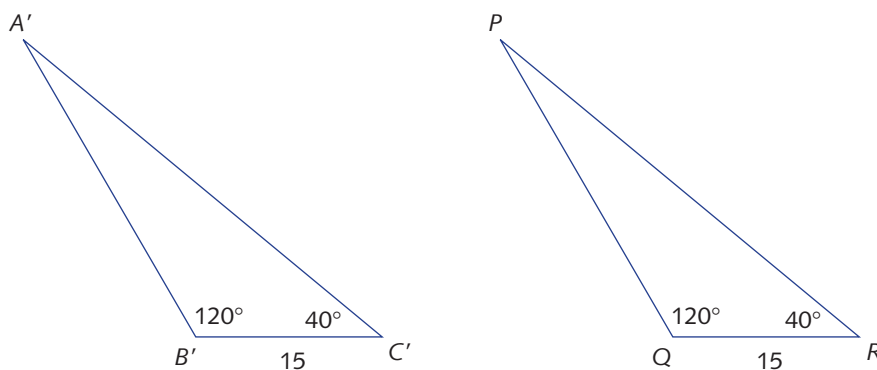


The two triangles below have the same interior angles, but have different sizes.



Notice that we only need to specify two angles of each triangle, since the third angle is determined using the fact that the angles of a triangle add to 180° .

Let us now enlarge the left-hand triangle by an enlargement factor of $\frac{3}{2}$ so that $B'C' = QR$, as in the diagram following. We have seen that the angles do not change, and that all three side lengths increase by the same factor.



By the AAS congruence test, the new triangle $\triangle A'B'C'$ is congruent to $\triangle PQR$.

This is because the two triangles have two pairs of angles equal, and the matching sides $B'C'$ and QR are also equal.

Hence the original triangle $\triangle ABC$ is similar to $\triangle PQR$, because its enlargement $\triangle A'B'C'$ is congruent to $\triangle PQR$.



The AAA similarity test

- If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.

Note: When using this test, it is sufficient to prove the equality of just two pairs of angles – the third pair must then also be equal since the angle sum of any triangle is 180° . Thus the test is sometimes called ‘the AA similarity test’.



Similarity statements

Arguments using similar triangles depend on a similarity statement that two triangles are similar. As with congruence, always give the similarity test in brackets afterwards, and be particularly careful to name the vertices in matching order.

Thus the conclusion of our argument above is written as

$$\triangle ABC \text{ is similar to } \triangle PQR \text{ (AAA).}$$

There are two common notations for similarity. These are

$$\triangle ABC \sim \triangle PQR \text{ and } \triangle ABC \equiv \triangle PQR.$$

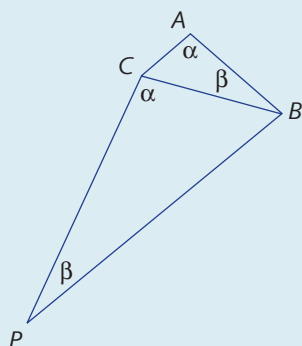
These are both read as ‘triangle ABC is similar to triangle PQR ’.

Example 3

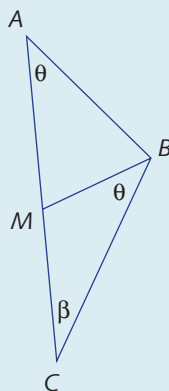
In each diagram below, write a similarity statement beginning with ‘ $\triangle ABC$ is similar to ...’

Be careful to name the vertices in matching order.

a



b



Solution

a $\triangle ABC$ is similar $\triangle CPB$ (AAA).

b $\triangle ABC$ is similar $\triangle BMC$ (AAA).

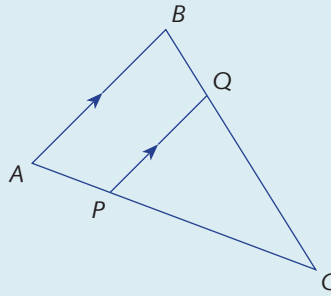
Notice that in part **b**, the angle $\angle C$ is **common** to both triangles.

Proving similarity

In many geometric problems, we need to prove that two triangles are similar. The proofs are set out just like congruence proofs.

Example 4

Prove that the two triangles in the diagram to the right are similar.



Solution

In the triangles ABC and PQC :

$$\angle BAC = \angle QPC \quad (\text{corresponding angles, } AB \parallel PQ)$$

$$\angle C = \angle C \quad (\text{common})$$

so $\triangle ABC$ is similar to $\triangle PQC$ (AAA).

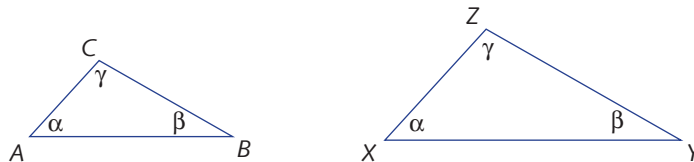
Using similar triangles to find lengths

Once we have established that two triangles are similar, we know that matching sides have the same ratio.

If $\triangle ABC$ is similar to $\triangle XYZ$ then:

$$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$$

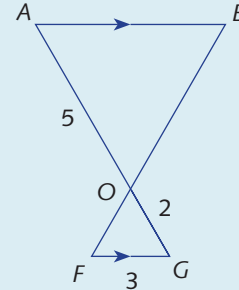
This ratio is the **similarity ratio**.



Example 5

In the diagram to the right:

- Prove that $\triangle ABO$ is similar to $\triangle GFO$.
- Hence find the length AB .
- Let X be a point on AB and Y be a point on FG such that XY passes through O . Prove that $\triangle AOX$ is similar to $\triangle GOY$.



Solution

- In the triangles ABO and GFO :

$$\angle AOB = \angle GOF \quad (\text{vertically opposite at } O)$$

$$\angle A = \angle G \quad (\text{alternate angles, } AB \parallel FG)$$

so $\triangle ABO$ is similar to $\triangle GFO$ (AAA)

- Hence $\frac{AB}{3} = \frac{5}{2}$ (matching sides of similar triangles)

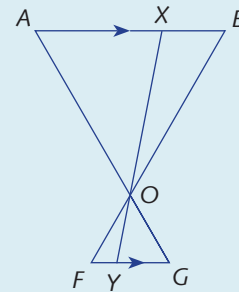
$$\begin{aligned} AB &= 3 \times \frac{5}{2} \\ &= 7\frac{1}{2} \end{aligned}$$

- In the triangles AXO and GYO :

$$\angle XAO = \angle YGO \quad (\text{alternate angles, } AB \parallel FG)$$

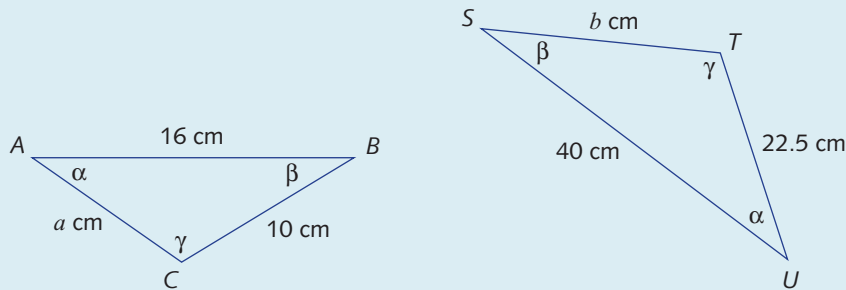
$$\angle AOX = \angle GOY \quad (\text{vertically opposite})$$

So $\triangle AXO$ is similar to $\triangle GYO$ (AAA)



Example 6

Find the values of a and b in the following diagrams.



Solution

$\triangle ABC$ is similar to $\triangle UST$ (AAA).

Hence $\frac{ST}{BC} = \frac{SU}{BA}$ (matching sides of similar triangles)

$$\frac{b}{10} = \frac{40}{16}$$

$$b = 25.$$

Also $\frac{AC}{UT} = \frac{BA}{SU}$ (matching sides of similar triangles)

$$\frac{a}{22.5} = \frac{16}{40}$$

so $a = 9.$

Alternatively, since $\frac{SU}{BA} = \frac{40}{16} = 2.5$, the similarity ratio from $\triangle ABC$ to $\triangle UST$ is 2.5; that is, every length in $\triangle UST$ will be 2.5 times the matching length in $\triangle ABC$.

Thus $ST = 2.5 \times BC$

$$b = 2.5 \times 10$$

$$= 25,$$

and $UT = 2.5 \times AC$

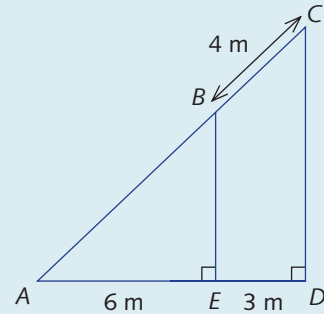
$$22.5 = 2.5 \times a$$

$$a = 9.$$

Example 7

In the diagram to the right:

- explain why $\triangle ABE$ is similar to $\triangle ACD$
- find AB
- find BE .



Solution

- Consider $\triangle ABE$ and $\triangle ACD$.

$$\angle BAE = \angle CAD \quad (\text{common angle})$$

$$\angle BEA = \angle CDA \quad (\text{both are right angles})$$

So $\triangle ABE$ is similar to $\triangle ACD$ (AAA).

- Since $\triangle ABE$ is similar to $\triangle ACD$,

$$\frac{AB}{AC} = \frac{AE}{AD} \quad (\text{matching sides of similar triangles})$$

Let $AB = x$ metres,

$$\begin{aligned} \text{then } \frac{x}{x+4} &= \frac{6}{9} \\ 9x &= 6(x+4) \\ 9x &= 6x+24 \\ 3x &= 24 \\ x &= 8. \end{aligned}$$

Thus $AB = 8$ metres.

- By Pythagoras' theorem,

$$AE^2 + BE^2 = AB^2$$

$$36 + BE^2 = 64$$

$$BE^2 = 28$$

$$BE = 2\sqrt{7} \text{ metres.}$$

Example 8

A tall vertical flagpole throws a 5 metre shadow. At the same time of day, a fencepost of height 2 metres throws a 30 cm shadow.

- Draw a diagram, identifying two similar triangles and proving that they are similar.
- Hence find the height of the flagpole.
- At another time of day, a 30 metre flagpole throws a 5 metre shadow. How long will the shadow of the 2 metre fencepost be?

Solution

- a** In the diagram, FC is the flagpole, PB is the fencepost, and T and S mark the end of the shadows.

In the triangles FCT and PBS :

$$\begin{aligned}\angle C &= \angle B = 90^\circ && \text{(pole and post are vertical)} \\ \angle T &= \angle S && \text{(both are the angle of elevation of the sun)}\end{aligned}$$

so $\triangle FCT$ is similar to $\triangle PBS$ (AAA).

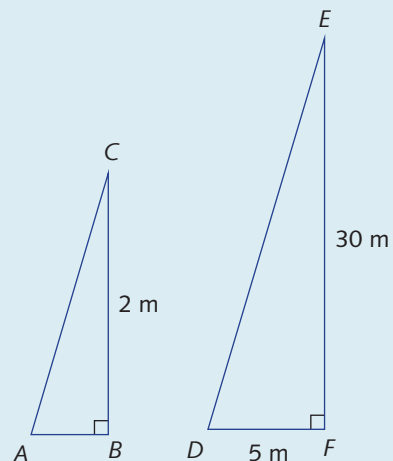
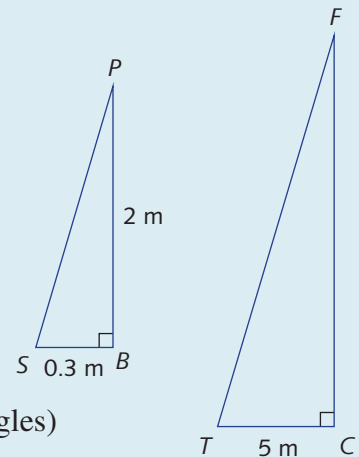
- b** Hence $\frac{FC}{2} = \frac{5}{0.3}$ (matching sides of similar triangles)
- $$FC = 33\frac{1}{3} \text{ metres}$$

Thus the height of the flagpole is $33\frac{1}{3}$ metres.

- c** $\triangle ACB$ is similar to $\triangle DEF$ (AAA)

$$\begin{aligned}\text{So } \frac{AB}{5} &= \frac{2}{30} \quad \text{(matching sides of similar triangles)} \\ AB &= \frac{1}{3}.\end{aligned}$$

Hence, the length of the shadow of the fencepost is $\frac{1}{3}$ metre.



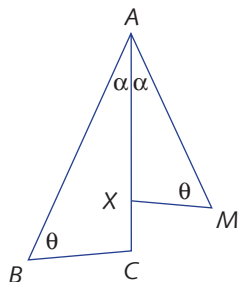


Exercise 9C

Example 3

- 1 Copy and complete each similarity statement, giving the similarity test and naming the vertices in matching order. Then copy and complete the ratio statement.

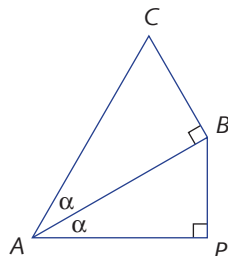
a



$\triangle ABC$ is similar to ... (...)

$$\frac{AB}{\dots} = \frac{BC}{\dots}$$

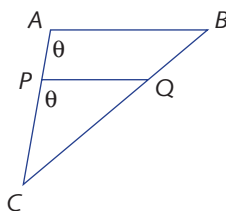
b



$\triangle ABC$ is similar to ... (...)

$$\frac{AB}{\dots} = \frac{BC}{\dots}$$

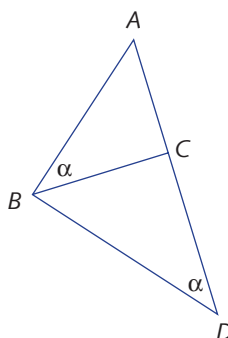
c



$\triangle ABC$ is similar to ... (...)

$$\frac{AB}{\dots} = \frac{BC}{\dots}$$

d



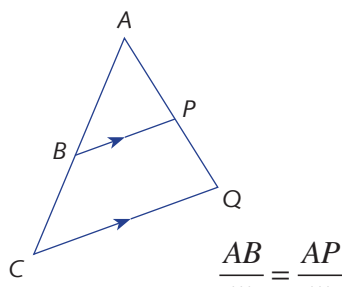
$\triangle ABC$ is similar to ... (...)

$$\frac{AB}{\dots} = \frac{BC}{\dots}$$

Example 4

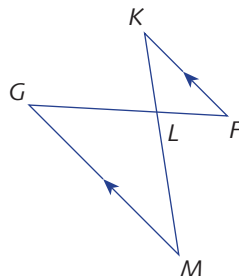
- 2 Prove that the two triangles in each diagram are similar, naming the vertices in matching order. Then copy and complete the ratio statement.

a



$$\frac{AB}{\dots} = \frac{AP}{\dots}$$

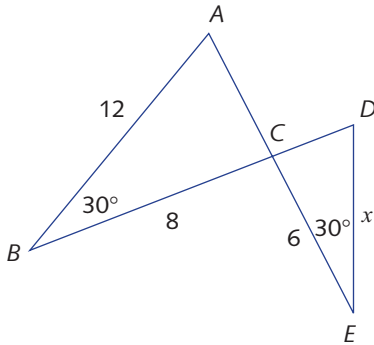
b



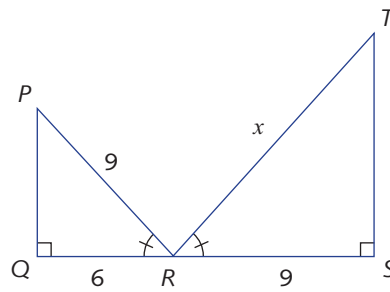
$$\frac{GL}{\dots} = \frac{GM}{\dots}$$

- 3 Prove that the two triangles in each diagram are similar. Then write a ratio statement and find the value of the pronumeral.

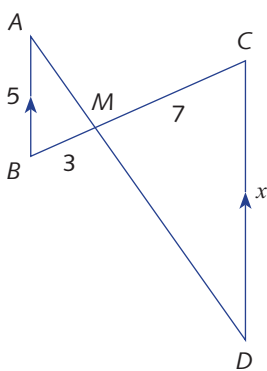
a



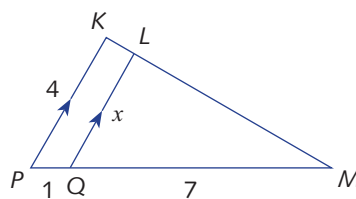
b



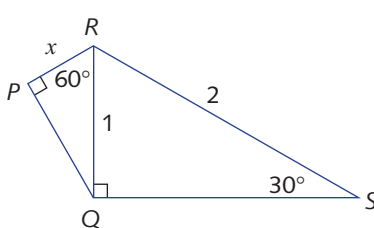
c



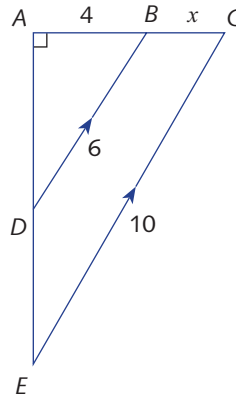
d



e



f



- 4 In each part, you will need to draw a diagram and prove that two triangles are similar.

- A stick 1.5 metres long leans against a wall and reaches 1 metre up the wall. How far up the wall will a ladder 10 metres long reach if it is parallel to the stick?
- A 3 metre high road-sign casts a shadow 1.4 metres long. How high is a telegraph pole that casts a shadow 3.5 metres long at the same time of day?
- A steep road rises 1.2 metres for every 6 metres travelled along the road surface. How far will a car have to travel to rise 600 metres?
- The front of an A-frame hut is an isosceles triangle of height 5 metres and base 8 metres. How wide will a model of the hut be if its height is 0.8 metres?

- 5 a Find the sizes of α , β and γ in the diagram to the right.

- b The three triangles in the diagram are similar.

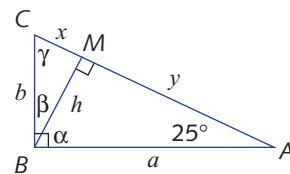
Copy and complete the statement:

$\triangle ABC$ is similar to ... which is similar to ... (AAA)

- c Hence copy and complete:

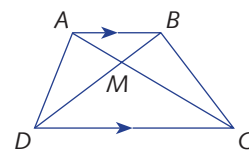
i $\frac{a}{b} = \frac{\dots}{\dots} = \frac{\dots}{\dots}$

ii $\frac{h}{a} = \frac{\dots}{\dots} = \frac{\dots}{\dots}$

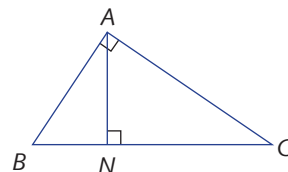


- 6 Prove that: *The diagonals of a trapezium dissect the trapezium into four triangles, two of which are similar.*

Let the diagonals of the trapezium $ABCD$ meet at M . Identify the two similar triangles and prove that they are similar.



- 7 Prove that: *The altitude to the hypotenuse of a right-angled triangle divides the hypotenuse into two intervals whose product equals the square of the altitude.*



Let AN be the altitude to the hypotenuse BC of the right-angled triangle $\triangle ABC$.

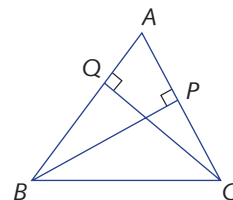
- a Prove that $\triangle ABN$ is similar to $\triangle CAN$.
b Hence prove that $AN^2 = BN \times CN$.
8 Let ABC be a triangle, and let BP and CQ be altitudes.

- a Prove that $\triangle ABP$ is similar to $\triangle ACQ$.

- b Hence prove that $\frac{BP}{CQ} = \frac{AB}{AC}$.

- c As an alternative approach, explain why $\frac{1}{2} \times AC \times BP$ and $\frac{1}{2} \times AB \times CQ$ are both equal to the area of $\triangle ABC$.

- d Hence prove that $\frac{BP}{CQ} = \frac{AB}{AC}$.

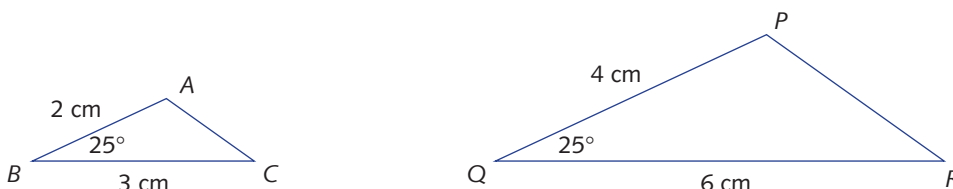


9D The SAS similarity test for triangles

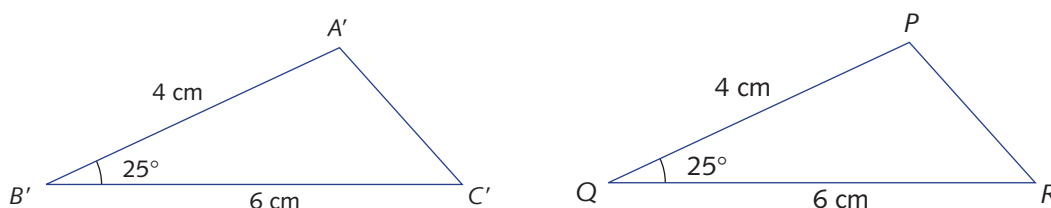


The two triangles below have a pair of equal angles, and the sides including this angle are in the same ratio 1:2.

$$\angle B = \angle Q, AB = \frac{1}{2}PQ \text{ and } BC = \frac{1}{2}QR.$$



Using the same procedure as before, we can enlarge the first triangle by a factor of 2 to produce a new triangle $\triangle A'B'C'$, and $\triangle A'B'C'$ is congruent to $\triangle PQR$ by the SAS congruence test.



Hence the original triangle $\triangle ABC$ is similar to $\triangle PQR$.

This gives us our second similarity test.



The SAS similarity test

- If the ratio of the lengths of two pairs of matching sides are equal and the included angles are equal then the two triangles are similar.



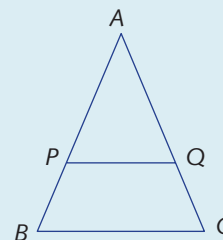
The statement that the two triangles above are similar is thus written as

$\triangle ABC$ is similar to $\triangle PQR$ (SAS).

Example 9

In triangle ABC ,
 $AB = AC = 10$ and $AP = AQ = 7$.

- Prove that $\triangle ABC$ is similar to $\triangle APQ$.
- Prove that PQ is parallel to BC .



(continued on next page)

Solution

a In the triangles ABC and APQ :

$$\frac{AB}{AP} = \frac{10}{7} \quad (\text{given})$$

$$\frac{AC}{AQ} = \frac{10}{7} \quad (\text{given})$$

$$\angle A = \angle A \quad (\text{common})$$

so $\triangle ABC$ is similar to $\triangle APQ$ (SAS).

b $\angle ABC = \angle APQ$ (matching angles of similar triangles)

So $BC \parallel PQ$. (corresponding angles are equal)

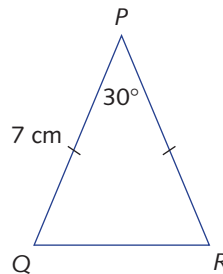
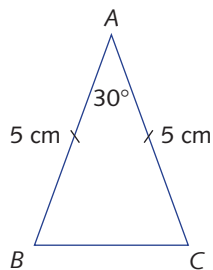


Exercise 9D

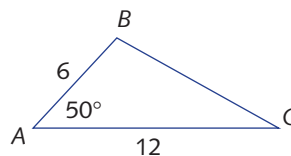
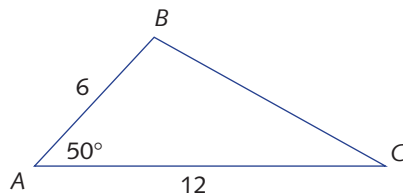


1 Prove that the two triangles in each diagram are similar, and write the similarity statement and the similarity factor (left to right).

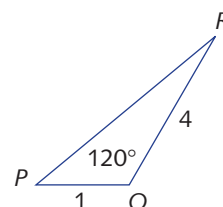
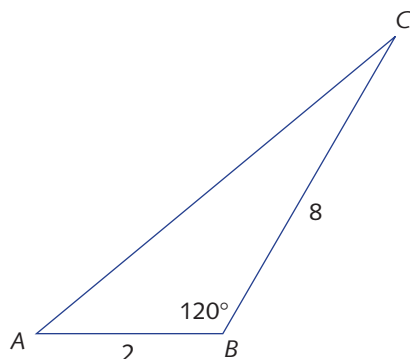
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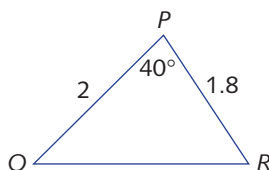
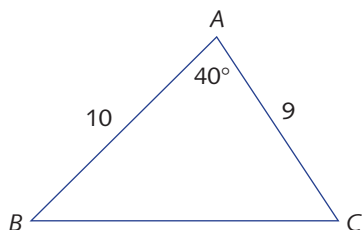
b



c



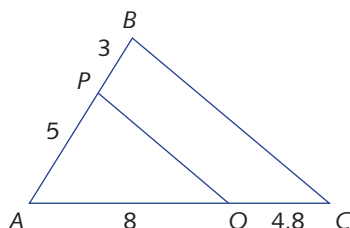
d



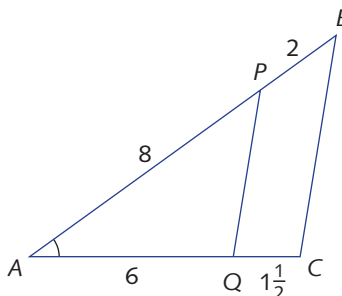
Example 9

- 2 In each part below, prove that $\triangle ABC$ is similar to $\triangle APQ$. Then prove that PQ is parallel to BC .

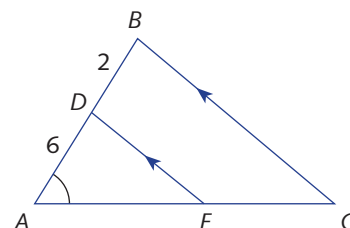
a



b

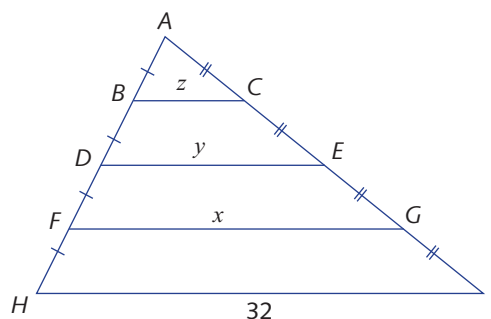


- 3 Prove that $\triangle ABC$ is similar to $\triangle ADE$ and copy and complete the ratio statement.



$$\frac{AB}{\dots} = \frac{AC}{\dots} = \dots$$

- 4 Prove that triangles ABC , ADE , AFG , and AHI are similar and find x , y and z .



- 5 The points P and Q lie on the sides AB and AC respectively of the triangle ABC such that $AP = 6$ cm and $AQ = 8$ cm. If $AB = 12$ cm and $AC = 16$ cm, prove that triangles APQ and ACB are similar. Is PQ parallel to BC ?
- 6 In triangles PQR and XYZ , $\angle P = \angle Z$ and $PQ \times YZ = PR \times XZ$. Name the other pairs of equal angles in the triangles.
- 7 Prove that if $ABCD$ is a quadrilateral such that $\angle A = \angle C$ and $\frac{DA}{AB} = \frac{BC}{CD}$, then $ABCD$ is a parallelogram.

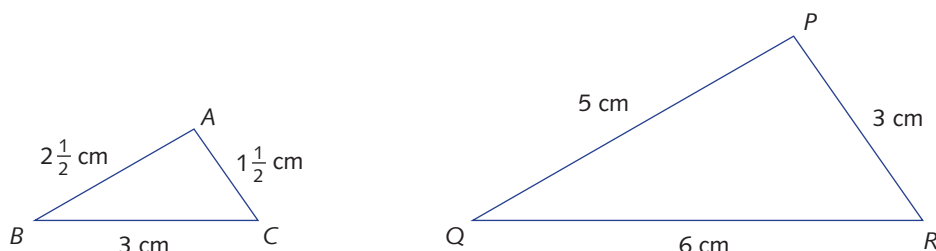


We now introduce two further similarity tests, giving four tests altogether.

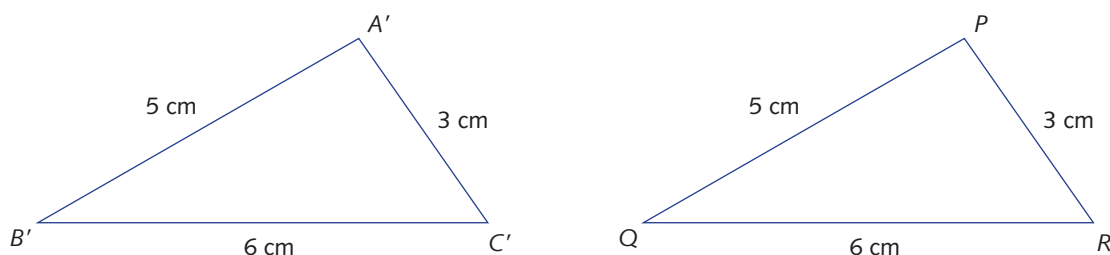
The SSS similarity test

The two triangles below have different sizes, but their sides can be paired up so that matching sides have the same ratio 1:2.

$$AB = \frac{1}{2}PQ, BC = \frac{1}{2}QR \text{ and } CA = \frac{1}{2}RP.$$



Let us now enlarge the first triangle by a factor 2, so that the side lengths all double.



By the SSS congruence test, the new triangle $\triangle A'B'C'$ is congruent to $\triangle PQR$.

Hence the original triangle $\triangle ABC$ is similar to $\triangle PQR$.



The SSS similarity test

- If we can match up the sides of one triangle with the sides of the other so that the ratio of matching lengths is constant, then the triangles are similar.

The statement that the two triangles above are similar is thus written as

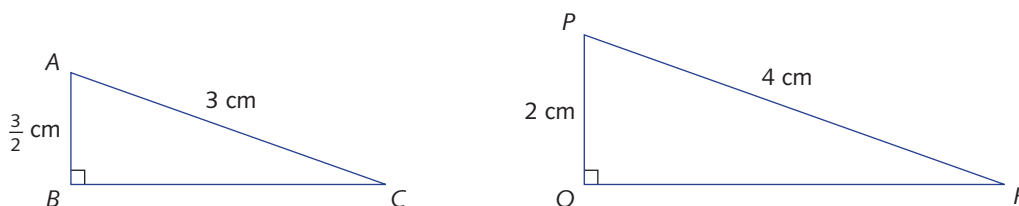
$\triangle ABC$ is similar to $\triangle PQR$ (SSS).

As an interesting example, the Pythagorean triples 3, 4, 5; 6, 8, 10; 9, 12, 15; ... give an infinite family of similar right-angled triangles.

The RHS similarity test



The two triangles below are both right-angled, and the hypotenuse and one side are in the same ratio 3 : 4.



Once again, we can enlarge the first triangle by a factor of $\frac{4}{3}$ to produce a triangle $\triangle A'B'C'$, and $\triangle A'B'C'$ is congruent to $\triangle PQR$ by the RHS congruence test.

Hence the original triangle $\triangle ABC$ is similar to $\triangle PQR$.

This gives us our fourth and last similarity test.



The RHS similarity test

- If the ratio of the hypotenuses of two right-angled triangles equals the ratio of another pair of sides, then the two triangles are similar.

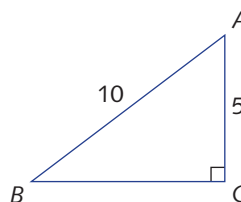
The statement that the two triangles above are similar is thus written as

$\triangle ABC$ is similar to $\triangle PQR$ (RHS).

Recall that given the hypotenuse and one side of a right-angled triangle, the third side can be found by Pythagoras' theorem.

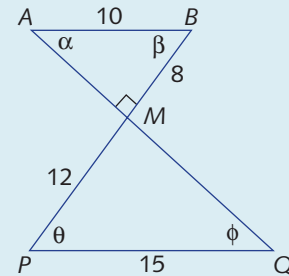
For example:

$$\begin{aligned} BC^2 &= AB^2 - AC^2 \\ &= 100 - 25 \\ &= 75 \\ BC &= \sqrt{75} \\ &= 5\sqrt{3} \end{aligned}$$



Example 10

- Prove that the two triangles in the diagram to the right are similar.
- Identify the equal angles in two triangles.
- Prove that $AB \parallel PQ$.



Solution

- In the triangles ABM and QPM

$$\angle AMB = \angle QMP = 90^\circ \text{ (vertically opposite)}$$

$$\frac{AB}{QP} = \frac{2}{3} = \frac{BM}{PM} \quad \text{(given)}$$

so $\triangle ABM$ is similar to $\triangle QPM$ (RHS similarity).

- Hence $\alpha = \phi$ and $\beta = \theta$ (matching angles of similar triangles).
- Hence $AB \parallel PQ$ (alternate angles are equal).

Using the four similarity tests in problems

Here are examples using the AAA, SSS, SAS and RHS similarity tests in problems.

Example 11

- Prove that the two triangles in the diagram are similar.
- Which of the marked angles are equal?

Solution

- In the triangles $\triangle ABC$ and $\triangle CBD$:

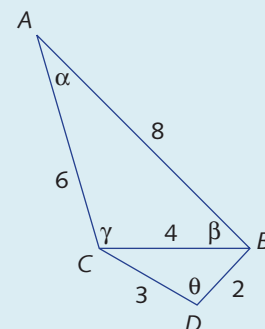
$$AB = 2 \times CB \quad \text{(given)}$$

$$BC = 2 \times BD \quad \text{(given)}$$

$$CA = 2 \times DC \quad \text{(given)}$$

so $\triangle ABC$ is similar to $\triangle CBD$ (SSS).

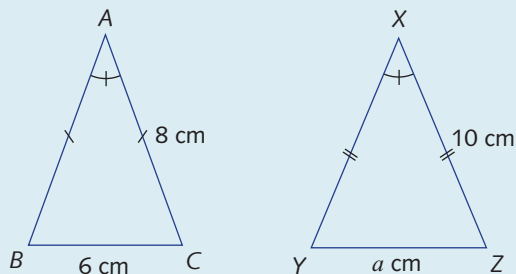
- Hence $\gamma = \theta$ (matching angles of similar triangles).



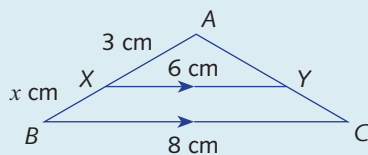
Example 12

Find the value of the pronumeral in each of the following.

a



b



Solution

a In the triangles ABC and XYZ :

$$\frac{AC}{XZ} = \frac{8}{10} \quad (\text{given})$$

$$\frac{AB}{XY} = \frac{8}{10} \quad (\text{given})$$

$$\angle A = \angle X \quad (\text{given})$$

so $\triangle BAC$ is similar to $\triangle YXZ$ (SAS).

$$\frac{YZ}{BC} = \frac{XZ}{AC} \quad (\text{matching sides of similar triangles})$$

$$\frac{a}{6} = \frac{10}{8}$$

$$a = \frac{5}{4} \times 6$$

$$a = \frac{15}{2}$$

b In the triangle ABC and AXY :

$$\angle A = \angle A \quad (\text{common})$$

$$\angle ABC = \angle AXY \quad (\text{corresponding angles, } XY \parallel BC)$$

so $\triangle ABC$ is similar to $\triangle AXY$ (AAA).

Hence $\frac{AB}{AX} = \frac{BC}{XY}$ (matching sides of similar triangles)

$$\frac{x+3}{3} = \frac{8}{6} = \frac{4}{3}$$

$$x+3=4$$

$$x=1$$

Example 13

- a** Prove that the two triangles in the diagram are similar.
b Hence prove that $LM \parallel BC$.

Solution

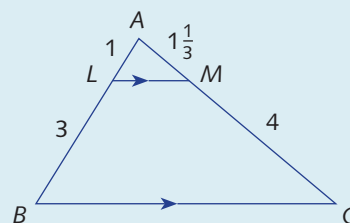
- a** In the triangles $\triangle ABC$ and $\triangle ALM$:

$$AB = 4 \times AL$$

$$AC = 4 \times AM$$

$$\angle BAC = \angle LAM \quad (\text{common})$$

so $\triangle BAC$ is similar to $\triangle LAM$ (SAS).

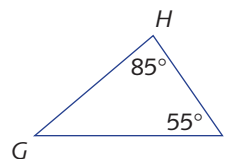
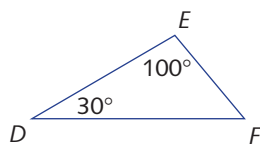
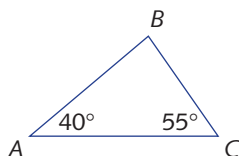


- b** Hence $\angle ABC = \angle ALM$ (matching angles of similar triangles) and so $LM \parallel BC$ (corresponding angles are equal).

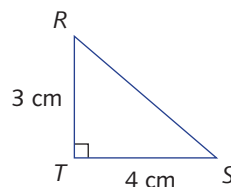
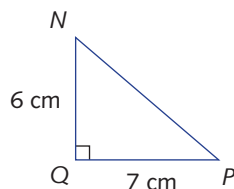
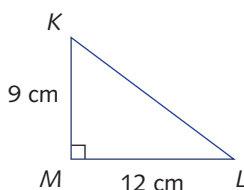
Exercise 9E

- 1** Name the pair of similar triangles in each part, naming the vertices in matching order and stating the similarity test.

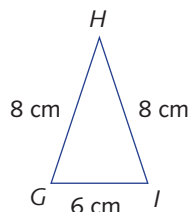
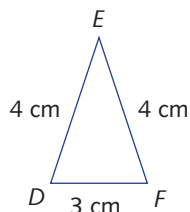
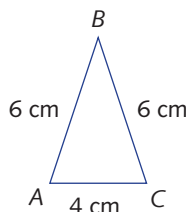
a



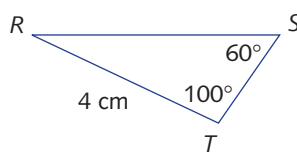
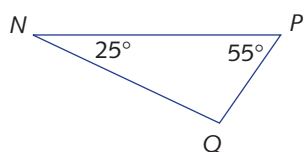
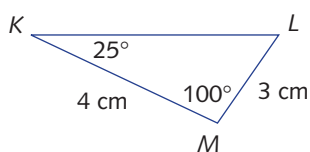
b



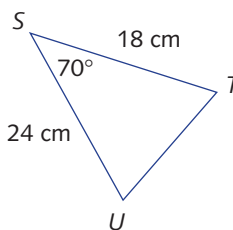
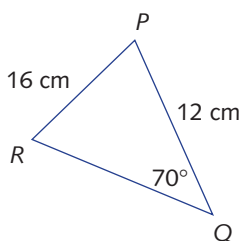
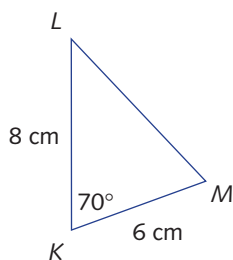
c



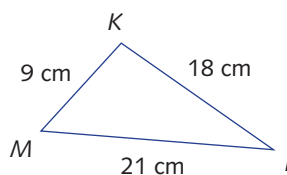
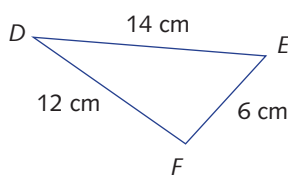
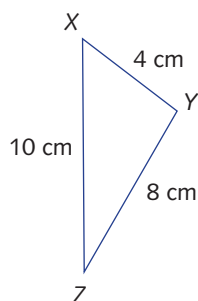
d



e

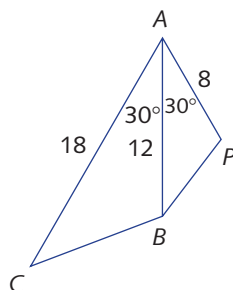


f



2 Copy and complete each similarity statement, stating the similarity test, and naming the vertices in matching order. They copy and complete the ratio statement.

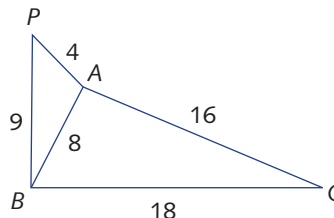
a



$\triangle ABC$ is similar to ... (...)

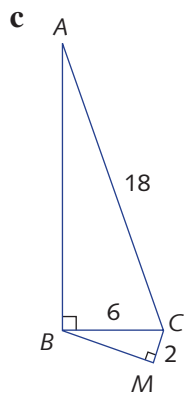
$$\frac{AB}{\dots} = \frac{BC}{\dots}$$

b



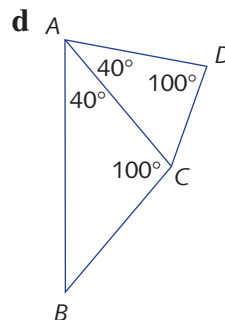
$\triangle ABC$ is similar to ... (...)

$$\frac{AB}{\dots} = \frac{BC}{\dots}$$



$\triangle ABC$ is similar to ... (...)

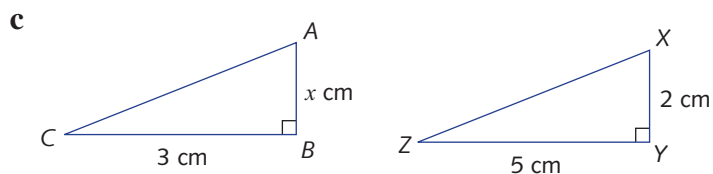
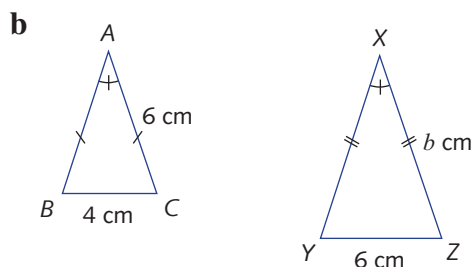
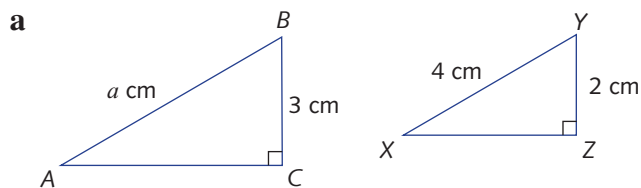
$$\frac{AB}{\dots} = \frac{BC}{\dots}$$



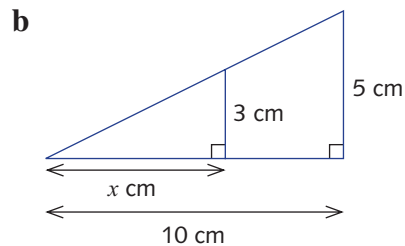
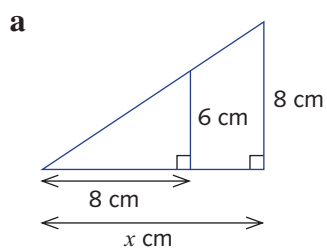
$\triangle ABC$ is similar to ... (...)

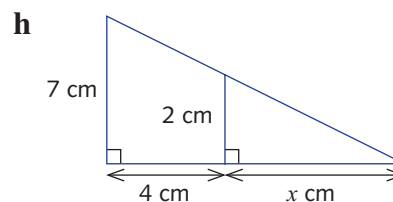
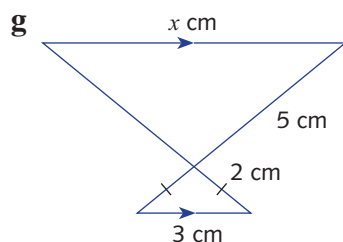
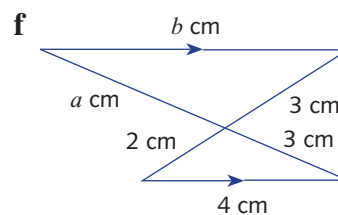
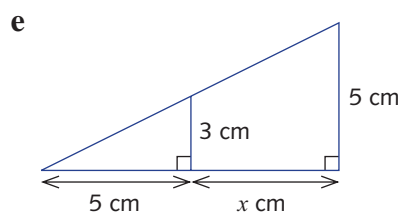
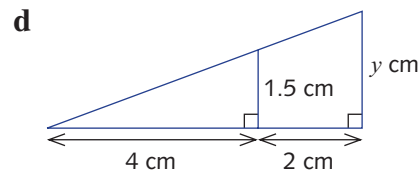
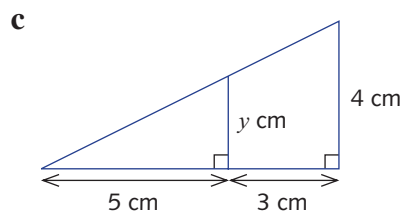
$$\frac{AB}{\dots} = \frac{BC}{\dots}$$

3 Find the values of the pronumerals in each of the following.



4 Find the values of the pronumerals in each part.





5 a Find the size of:

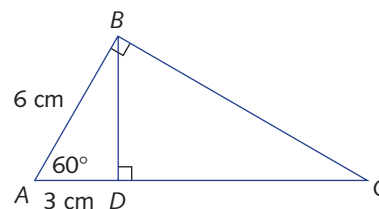
- i** $\angle ABD$ **ii** $\angle ACB$

b Justify the statement that $\triangle ABC$ is similar to $\triangle ADB$.

c Find AC .

d Show that $BD = 3\sqrt{3}$ cm.

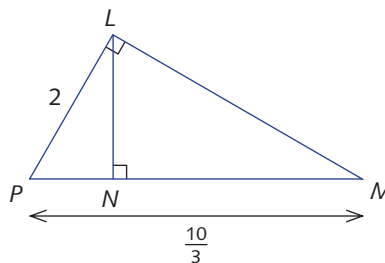
e Find BC .

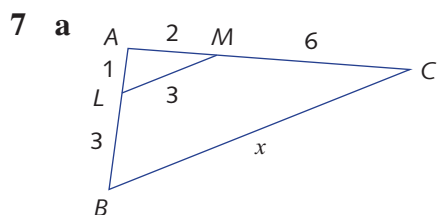


6 a Find LM .

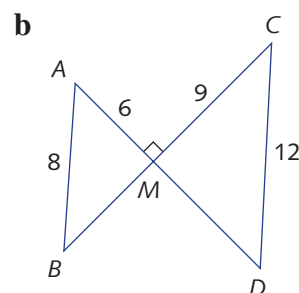
b Find LN .

c Find MN .

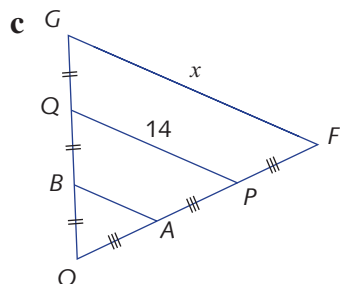




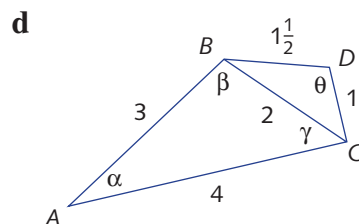
- i** Prove that $\triangle ALM$ is similar to $\triangle ABC$.
- ii** Hence prove that $LM \parallel BC$.
- iii** Find x .



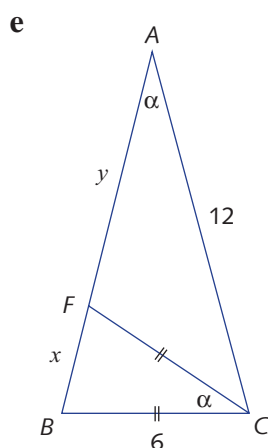
- i** Prove that $\triangle ABM$ is similar to $\triangle CDM$.
- ii** Hence prove that $\angle B = \angle D$
- iii** Is it true that $AB \parallel CD$?



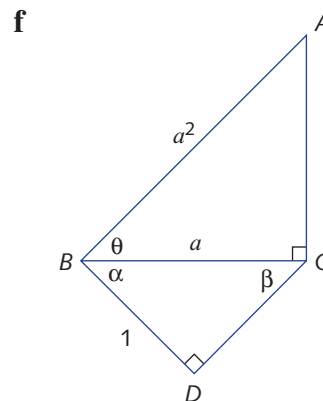
- i** Prove that $\triangle OPQ$ is similar to $\triangle OFG$.
- ii** Hence prove that $PQ \parallel FG$.
- iii** Find x .



- i** Prove that $\triangle ABC$ is similar to $\triangle...$
- ii** Which of α , β , and γ is equal to θ ?



- i** Prove that $\triangle ABC$ is similar to $\triangle CBF$.
- ii** Hence find x and y .

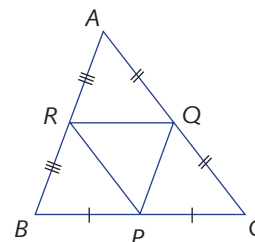


- i** Prove that $\triangle ABC$ is similar to $\triangle...$
- ii** Is α , or β equal to θ ?

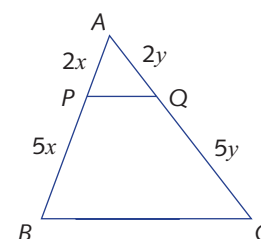
- 8 a** Prove that: *The line interval joining the midpoints of two sides of a triangle is parallel to the third side and half its length.*
- b** Prove that: *The line through the midpoint of one side of a triangle parallel to another side of the triangle meets the third side at its midpoint.*

- c** Prove that: *The midpoints of the sides of a quadrilateral form the vertices of a parallelogram.*

Let $ABCD$ be a quadrilateral, and let P, Q, R and S be the midpoints of the sides AB, BC, CD and DA respectively.



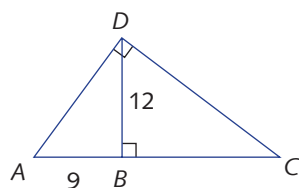
- 9** Prove that: *The intervals joining the midpoints of the sides of a triangle dissect the triangle into four congruent triangles each similar to the original triangle.*
- 10** The points P and Q lie on the sides AB and AC respectively of the triangle $\triangle ABC$, with $AP : PB = AQ : QB = 2 : 5$.
- Prove that $\triangle APQ$ is similar to $\triangle ABC$.
 - Hence prove that $BC = \frac{7}{2} \times PQ$.
 - Prove also that $PQ \parallel BC$.
- 11** If you think the statement is true, justify your answer using a similarity test. If you think it is false, draw two triangles that provide a counter-example.
- Any two equilateral triangles are similar.
 - Any two isosceles triangles are similar.
 - Any two isosceles triangles with equal apex angles are similar.
 - Any two isosceles triangles with equal length bases are similar.
 - Any two right-angled triangles are similar.
 - Any two right-angled triangles with equal length hypotenuses are similar.
 - Any two isosceles right-angled triangles are similar.
 - Any two right-angled triangles in which one other angle is 40° are similar.
 - Any two triangles in which one side is twice another side are similar.
 - Any two right-angled triangles in which one side is twice another side are similar.



12 If you think the statement is true, justify your answer. If you think it is false, draw two figures that provide a counterexample.

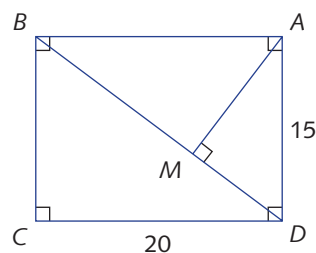
- a** Any two squares are similar.
- b** Any two rectangles are similar.
- c** Any two rectangles in which one side is three times another are similar.
- d** Any two rectangles whose diagonals meet at 30° are similar.
- e** Any two rectangles whose diagonals have length 6 cm are similar.
- f** Any two rhombuses are similar.
- g** Any two rhombuses with vertex angles 50° and 130° are similar.

13 a



Find AD , DC and BC .

b

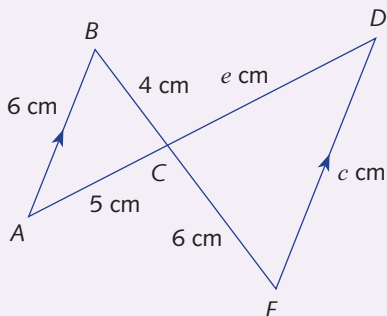


Find AM , BM and DM .

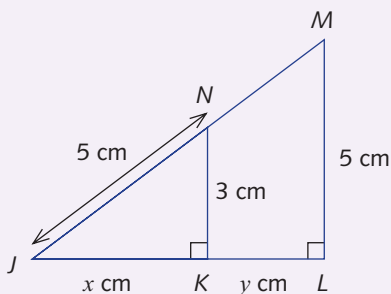
Review exercise



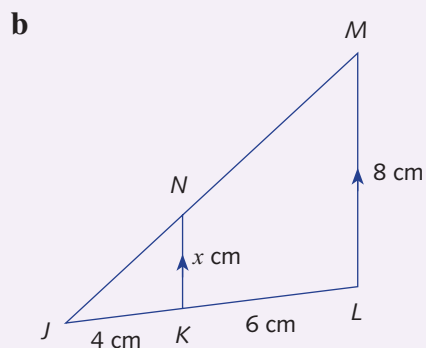
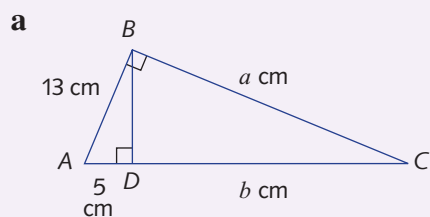
- 1 a Prove that triangle ABC is similar to triangle DEC and find the value of e and c .



- b Prove that triangle JNK is similar to triangle JML and find the values of x and y .

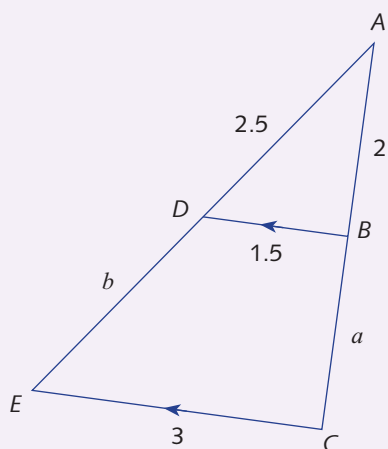


- 2 Write down a pair of similar triangles and find the value of each pronumeral.

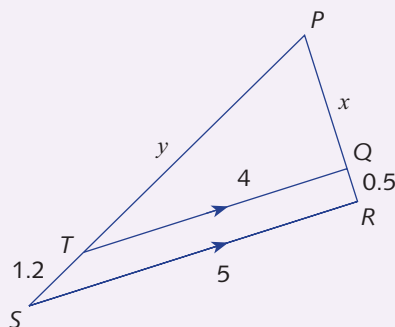


3 Write down a pair of similar triangles and find the value of each pronumeral.

a



b

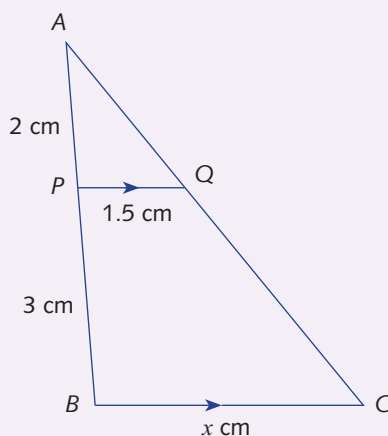


4 A vertical stick of length 30 cm casts a shadow of length 5 cm at 12 p.m. Find the length, in centimetres, of the shadow cast by a 1 metre ruler placed in the same position at the same time of the day.

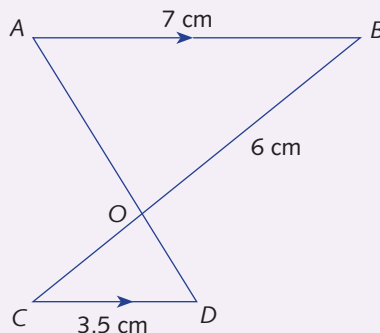
5 In the figure below $PQ \parallel BC$.

a Prove that $\triangle APQ$ is similar to $\triangle ABC$.

b Find the value of x .

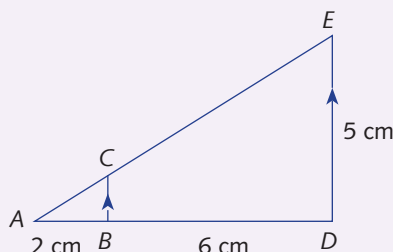


6 a In the figure below, $AB \parallel CD$ and $AB = 7$ cm, $BO = 6$ cm and $CD = 3.5$ cm. Find CB .



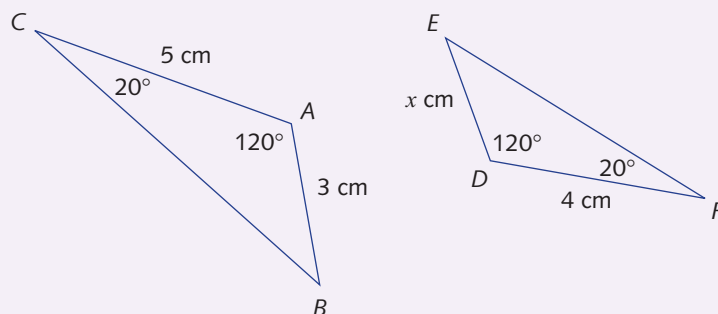


b In the diagram below, $BC \parallel DE$, $AB = 2$ cm, $BD = 6$ cm and $DE = 5$ cm. Find BC .



7 Prove that $\triangle ABC$ is similar to $\triangle DEF$.

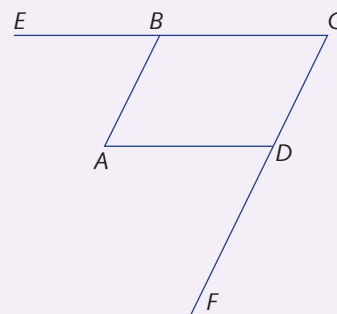
Find x .



8 $ABCD$ is a parallelogram with $\angle BAD$ acute. The point E lies on the ray CB such that $\triangle ABE$ is isosceles with $AB = AE$. The point F lies on the ray CD such that $\triangle ADF$ is isosceles with $AD = AF$.

a Prove that $\triangle ABE$ is similar to $\triangle ADF$.

b Prove that $DE = BF$.



9 ABC is a triangle, M is a point in the interval AB such that $AB = 3AM$, and N is a point in the line segment AC such that $AC = 3AN$.

a Prove that $BC \parallel MN$.

b If BN and CM intersect at P , prove that $BP = 3PN$.

10 A tower standing on level ground casts a shadow 40 m long. A vertical stick 3 m high is placed at the tip of the shadow. The stick is found to cast a shadow 6 m long. Find the height of the tower.

11 A line from the top of a church steeple to the ground just passes over the top of a pole 2.5 m high and meets the ground at a point A , 1.5 m from the base of the pole. If the distance from A to a point directly below the church steeple is 30 m, find the height of the steeple.



Challenge exercise

- 1 You proved in Exercise 9E, question 8, that the midpoints of the sides of a quadrilateral form the vertices of a parallelogram. Now prove that this parallelogram has area half the area of the original quadrilateral.

- 2 Let $ABCD$ be a trapezium with $AB \parallel CD$. Let the diagonals AC and BD meet at O .

Let the line through O parallel to AB meet AD at P and BC at Q .

By considering the pairs of triangles ADB and POD , BDC and QOB , and AOB and COD , prove that O is the midpoint of PQ .

- 3 Let $ABCD$ be a rectangle, and let K and L be the midpoints of AB and CD respectively. Let AC meet KL at M .

a Show that M is the midpoint of KL .

b Let DK meet AC at X . Find the ratio $AX : XC$.

c Find the ratio of the area of $\triangle KXA$ to the area of the rectangle $ABCD$.

- 4 Take any two points A and B in the plane and a piece of string longer than the distance AB . Imagine fixing one end of the string at A and the other at B , and pulling the string taut by using a pencil.

Now move the pencil around keeping the string taut and letting the string slide around the point of the pencil. In this way we trace out a curve called an **ellipse**.



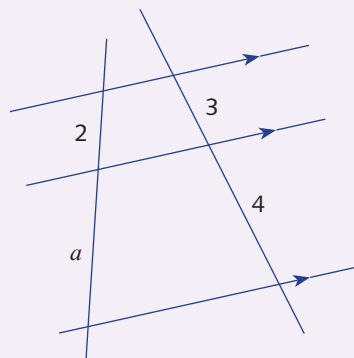
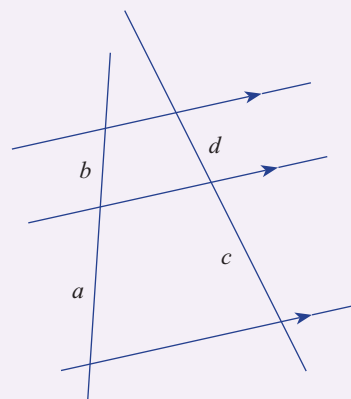
Prove that an enlargement transformation takes an ellipse to another ellipse.

- 5 a Take three parallel lines and any two transversals that are not parallel. Prove that the ratio of the lengths cut off by the parallel lines is the same for both transversals.

That is, prove that $\frac{a}{b} = \frac{c}{d}$ in the diagram opposite.

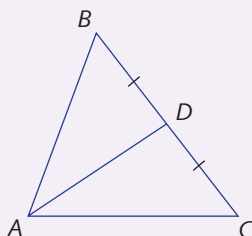
b Is the result still true if the transversals are parallel?

c In the diagram below, find a .



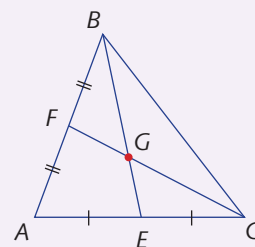


- 6 An **altitude** of a triangle is an interval from a vertex meeting the opposite side at right angles. Prove that if two altitudes of a triangle are equal, then the triangle is isosceles.
- 7 In this question we will prove that: *The medians of a triangle are concurrent and trisect each other.*



A **median** of a triangle is the line which joins a vertex to the middle of the opposite side.

- a Let the medians BE and CF meet at G . Show that $BG = 2GE$ and $CG = 2GF$.
- b Deduce that the third median AD also passes through G . The point G is called the **centroid** of the triangle ABC .
- c Prove that, if two of the medians of a triangle have equal length, then the triangle is isosceles.



8 Theorem

Let ABC and PQR be equiangular triangles, with $\angle A = \angle P = \alpha$, $\angle B = \angle Q = \beta$, $\angle C = \angle R = \gamma$. Let $BC = a$, $CA = b$ and $AB = c$, and let $QR = ka$, where k is a positive rational number. Then $RP = kb$ and $PQ = kc$,

Use congruence (not similarity) to prove the result for $k = 2$ and then $k = 3$.

This is a beginning of the proof that the four similarity tests can be proven from the congruence tests without enlargements when the similarity factor is a rational number.



CHAPTER

10

Review and problem-solving



10A Review

Chapter 1: Algebra

1 Evaluate $3x + 2y^2$ when:

a $x = 2$ and $y = 3$

b $x = 5$ and $y = -2$

c $x = 23$ and $y = 3$

d $x = \frac{1}{2}$ and $y = \frac{-3}{5}$

2 Simply each of these expressions by collecting like terms.

a $3a + 2b - a + 4b$

b $5x^2y - 3xy + 7xy - x^2y$

c $7m + 12n^2 + 2n^2 - 9m$

d $p^2 - 6p - p + 15$

3 Simplify:

a $7ab \times 2a$

b $-3x \times -2y$

c $\frac{20xy}{5x}$

d $25a \div 5 \times 3$

4 Write each expression as a single fraction.

a $\frac{a}{5} - \frac{2a}{3}$

b $\frac{3x}{8} - \frac{2x}{5}$

c $\frac{a}{5} \times \frac{2a}{3}$

d $\frac{a}{2b} \times \frac{2ab}{7}$

e $\frac{3x}{4} \div \frac{6x}{7}$

f $\frac{ab}{3} \div \frac{6b}{b}$

5 Expand:

a $3(a + 4)$

b $2(b + 6)$

c $4(b - 5)$

d $6(x - 1)$

e $2(3b + 2)$

f $5(4d - 1)$

g $-3(3d - 2)$

h $-2(5\ell - 4)$

i $-2x(3x + 1)$

j $4x(2x + 3)$

6 Expand and collect like terms for each of these expressions.

a $3(a + 2) + 4(a + 5)$

b $2(b + 1) + 3(b + 4)$

c $4(2x - 1) + 3(3x + 2)$

d $5(3d - 2) + 4(2d - 7)$

e $8(4e + 3) - 5(e - 1)$

f $6(f - 2) - 3(2f - 5)$

g $2x(x + 4) + 3(x - 2)$

h $3x(2x - 1) + 5x(4x - 3)$

i $x(3x + 2) - 4x(2x - 3)$

j $2x(5x + 4) - 6x(3x - 7)$

7 Simplify:

a $\frac{x + 1}{4} + \frac{x + 3}{3}$

b $\frac{x - 2}{2} + \frac{x - 1}{3}$

c $\frac{2x + 1}{3} - \frac{x + 1}{4}$

d $\frac{3x - 1}{4} - \frac{2x - 1}{6}$

8 Expand and simplify:

a $(x + 3)(x + 5)$

b $(x + 4)(x + 7)$

c $(x + 2)(x - 6)$

d $(x + 7)(x - 3)$

e $(x - 5)(x + 9)$

f $(x - 3)(x + 8)$

g $(2x + 1)(3x - 2)$

h $(4x + 3)(3x + 5)$

i $(5x - 2)(2x + 3)$

j $(3x - 4)(2x - 7)$

k $(x + 5)(x - 5)$

l $(x + 7)(x - 7)$

m $(2x + 3)(2x - 3)$

n $(3x - 5)(3x + 5)$

o $(x + 7)^2$

p $(x + 3)^2$

q $(2x - 5)^2$

r $(3x - 4)^2$

s $(x + 2)^2 - (x - 4)^2$

t $(2x + 3)^2 - (2x - 3)^2$

u $(x + 1)(2x + 3) + (2x - 1)(3x + 2)$

v $(x + 2)(2x - 5) - (3x + 1)(2x - 4)$

9 Copy and complete:

a $(x + 3)(x + \dots) = x^2 + 10x + \dots$

b $(x + 2)(x - \dots) = x^2 - x - \dots$

c $(x + 6)(\dots + \dots) = x^2 + 11x + 30$

d $(x + 4)(\dots + \dots) = x^2 + 10x + 24$

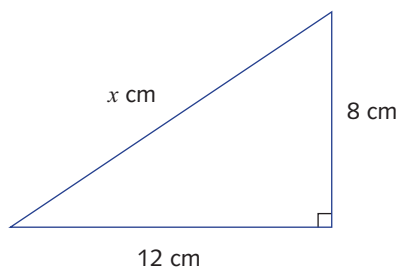
e $(2x - 1)(x + \dots) = 2x^2 + \dots - 6$

f $(3x + 2)(\dots + \dots) = 6x^2 + \dots + 14$

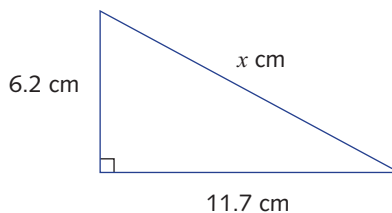
Chapter 2: Pythagoras' theorem and surds

1 For each of these right-angled triangles, find the value of the pronumeral, correct to one decimal place.

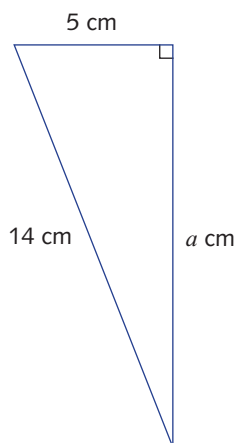
a



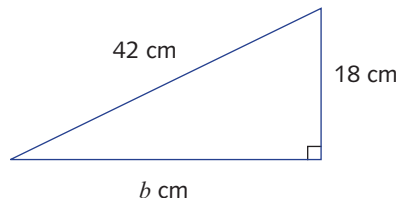
b



c



d



2 The lengths of the sides of a triangle are 8.2 cm, 11.6 cm and 14.3 cm. Is the triangle right-angled?

3 In each part below, the two shorter side lengths of a right-angled triangle are given. State the length of the hypotenuse.

- a** 3 cm, 4 cm **b** 5 cm, 12 cm **c** 4 cm, 7.5 cm
d 0.3 cm, 0.4 cm **e** 1 cm, 2.4 cm **f** 12 cm, 22.5 cm

4 A gardener is designing a rectangular lawn $ABCD$. If $AB = 4.2$ m and $BC = 3.15$ m, how far apart should A and C be to ensure $\angle ABC = 90^\circ$?



5 A plane takes off and after climbing on a straight line path for a distance of 1 km, it has flown a horizontal distance of 900 m. What is the plane's altitude, correct to the nearest metre?

6 Simplify each of these surds.

- a** $\sqrt{20}$ **b** $\sqrt{75}$ **c** $2\sqrt{18}$ **d** $4\sqrt{50}$ **e** $5\sqrt{108}$ **f** $9\sqrt{27}$

7 Write each number as the square root of a whole number.

- a** $2\sqrt{3}$ **b** $3\sqrt{2}$ **c** $10\sqrt{5}$ **d** $4\sqrt{7}$

8 Simplify:

- a** $4\sqrt{2} + 7\sqrt{2}$ **b** $8\sqrt{3} - 5\sqrt{3}$ **c** $4\sqrt{2} \times 5\sqrt{3}$ **d** $3\sqrt{5} \times 4\sqrt{7}$
e $\sqrt{18} + \sqrt{32}$ **f** $\sqrt{27} - \sqrt{12}$ **g** $4\sqrt{12} + 3\sqrt{75}$ **h** $8\sqrt{50} - 2\sqrt{98}$

9 Expand and simplify:

- a** $\sqrt{2}(\sqrt{3} + \sqrt{10})$ **b** $\sqrt{3}(4\sqrt{3} - 5)$ **c** $3\sqrt{5}(2\sqrt{2} - 4\sqrt{5})$
d $2\sqrt{2}(3\sqrt{3} + 4\sqrt{2})$ **e** $(2\sqrt{3} + 1)(3\sqrt{3} - 2)$ **f** $(4\sqrt{2} + 3)(5\sqrt{2} - 7)$
g $(3\sqrt{2} - 1)^2$ **h** $(\sqrt{5} + 1)^2$ **i** $(2\sqrt{5} + 7\sqrt{2})(2\sqrt{5} - 7\sqrt{2})$
j $(2\sqrt{3} + 5)(2\sqrt{3} - 5)$

10 Express each number with a rational denominator.

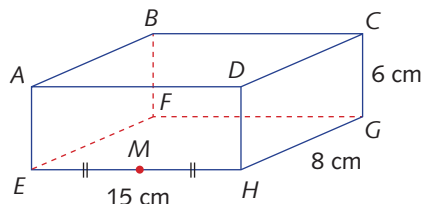
- a** $\frac{3}{\sqrt{3}}$ **b** $\frac{2\sqrt{5}}{\sqrt{5}}$ **c** $\frac{2}{4\sqrt{3}}$ **d** $\frac{5\sqrt{3}}{3\sqrt{2}}$

11 Express each number with a rational denominator.

- a** $\frac{3\sqrt{2}}{\sqrt{5} + 2}$ **b** $\frac{\sqrt{3}}{2\sqrt{3} - 1}$ **c** $\frac{3\sqrt{2} + 1}{\sqrt{5} + 2}$ **d** $\frac{\sqrt{2} + 1}{\sqrt{3} + \sqrt{2}}$

- 12 For the rectangular prism to right, calculate the length of each of these intervals. Give your answers as surds in simplest form.

a EG b EC c HC
d GM e CM f AM



- 13 A vase is in the shape of a cylinder with base radius 4 cm and height 10 cm. What is the length, correct to 1 decimal place, of the longest flower stem that can just fit in the vase?

Chapter 3: Consumer arithmetic

- 1 Express each percentage as a fraction in its simplest form.

a 18% b 64% c 2.6% d 8.5% e $37\frac{1}{2}\%$ f $6\frac{2}{3}\%$

- 2 Express each percentage as a decimal.

a 8% b 27% c 9.6% d 45.8% e $12\frac{1}{4}\%$ f $38\frac{1}{2}\%$

- 3 Express each rational number as a percentage.

a $\frac{2}{5}$ b $\frac{5}{8}$ c 0.61 d 0.23
e 0.02 f 0.62 g $\frac{4}{7}$ h $\frac{5}{9}$

- 4 Copy and complete the following table.

	Percentage	Fraction	Decimal
a	25%		
b		$\frac{3}{10}$	
c			0.26
d		$\frac{2}{3}$	
e	8%		
f			0.075

- 5 Calculate:

a 8% of 120 b 16% of 54 c 85% of \$400 d $9\frac{1}{2}\%$ of \$6000

- 6 There are 650 students at a high school, 54% of whom are boys. How many boys are at the school?

- 7 Cindy put an 8% deposit on a bracelet worth \$120. What deposit did she pay?

- 8 Netball is played by 6% of Australians. If the population of Australia is 22 500 000, how many Australians play netball?
- 9 In a class of 25 students, 8 travel to school by train. What percentage of the class travel to school by train?
- 10 In a survey of 1200 adults, it was discovered that 114 of them were unemployed. What percentage of the adults surveyed were unemployed?
- 11 Find the new value if:
- a 80 is increased by 40%
 - b 150 is increased by 6%
 - c 240 is decreased by 12%
 - d 160 is decreased by 4%.
- 12 During a sale, the price of a sofa bed is reduced by 20%. If the original price of the bed was \$650, what is its sale price?
- 13 A salesperson is given a salary increase of 4%. If her existing weekly salary is \$640, what will her new weekly salary be?
- 14 Joe's Electrical Store is having an 8% discount sale. The sale price of some items is given below. Calculate the price of the items before they were reduced.
- a Heater \$276
 - b Vacuum cleaner \$138
 - c Dishwasher \$690
 - d Microwave \$132.80
- 15 The enrolment of a school increased from 680 to 740. Calculate the percentage increase, correct to 2 decimal places.
- 16 During a sale the price of a suit is reduced from \$420 to \$370. Calculate the percentage discount, correct to 1 decimal place.
- 17 What single percentage change, correct to two decimal places, is equivalent to each of these multiple changes?
- a A 6% increase followed by a 12% increase
 - b A 10% increase followed by a 10% decrease
 - c A 16% decrease followed by a 8% decrease
 - d A 12% decrease followed by a 14% increase
- 18 Over the course of a year an employee is given successive salary increases of 4%, 6% and 5%.
- a If the employee's original monthly salary was \$2600, what is the employee's salary after the three increases?
 - b What single percentage change is equivalent to the three successive salary increases?
- 19 To obtain a bonus, a salesperson's sales must increase by 20% in a two-month period. If the salesperson's sales increase by 8% in the first month, by what percentage must they increase in the second month to ensure the bonus is obtained?

- 20** Mia invests \$6000 in the bank. How much will she have in her account after three years if the bank pays:
- a** 8% simple interest p.a. **b** 12% simple interest p.a.
c 4% compound interest p.a. **d** 9.6% compound interest p.a.?
- 21** The value of a new car depreciates at a compound rate of 6% each year. If the car has an initial value of \$19960, calculate its value after:
- a** one year **b** two years **c** four years
d five years **e** ten years **f** one hundred years.

Chapter 4: Factorisation

1 Factorise:

- a** $5a + 10$ **b** $6b + 18$ **c** $6c - 8$ **d** $9d - 24$
e $3e^2 + 9e$ **f** $6f^2 + 10f$ **g** $-4g^2 - 14g$ **h** $-3h^2 - 15h$
i $4a^2b + 6ab^2$ **j** $9mn^2 + 12mn$

2 Factorise:

- a** $x^2 + 7x + 12$ **b** $x^2 + 9x + 18$ **c** $x^2 - 5x - 6$
d $x^2 + 3x - 28$ **e** $x^2 - 11x + 30$ **f** $x^2 - 14x + 24$
g $x^2 + 3x - 70$ **h** $x^2 - 6x - 55$ **i** $3x^2 + 6x + 9$
j $4x^2 - 8x + 12$ **k** $x^2 - 100$ **l** $x^2 - 49$
m $9x^2 - 16y^2$ **n** $1 - 16a^2$

3 Write each expression as a single fraction.

- a** $\frac{1}{(x-1)^2} \div \frac{1}{x^2-1}$ **b** $\frac{x-4}{x^2+2x+1} \times \frac{x+1}{x^2-16}$
c $\frac{m-2}{4m} \times \frac{m}{m-2}$ **d** $\frac{p+1}{8(p-1)} \times \frac{4(p-1)}{(p+1)(p+2)}$
e $\frac{4}{a} \div \frac{2}{a^2}$ **f** $\frac{5a-7}{2a+4} \times \frac{12}{10a-14}$
g $\frac{(2x-1)(x+2)}{(2x-1)^2}$ **h** $\frac{(x+3)(x-2)}{x+6} \div \frac{x+3}{x+6}$
i $\frac{x^2+3x-4}{2x-2} \times \frac{6x-12}{x-1}$ **j** $\frac{x^3}{y^2} \times \frac{2y^3}{x}$

Chapter 5: Linear equations and inequalities

1 Write each of the following statements using symbols.

a two more than x

b four less than b

c half of c

d three times d

e one more than twice e

f two less than one third of f

g twice three more than s

h half of one less than h

2 Solve these equations.

a $4a = 36$

b $\frac{b}{4} = 2$

c $5x - 3 = 42$

d $3x + 19 = 46$

e $7 - 2y = -15$

f $8 - 3m = 2$

g $4(2x - 3) = 84$

h $7(1 - x) = 42$

i $5x - \frac{1}{2} = 17$

j $\frac{2x}{3} + \frac{1}{4} = 1$

k $3 - \frac{x}{5} = 4$

l $2 - \frac{3x}{4} = 5$

3 Solve the following equations.

a $2x + 7 = x + 9$

b $3x + 11 = 4x + 27$

c $2x - 9 = 3x + 15$

d $4x - 5 = 8x - 13$

e $5(x + 1) = 2(3x + 1)$

f $4(2x - 3) = 3(x + 4)$

g $2(x + 1) + 3(x + 2) = 17$

h $4(x - 3) - 3(2x + 1) = 18$

i $\frac{2a + 1}{3} = \frac{a - 1}{4}$

j $\frac{a - 3}{2} + 1 = \frac{a - 1}{4}$

k $\frac{3a}{2} + 1 = \frac{a + 1}{3}$

l $\frac{a - 1}{5} = \frac{2a + 3}{4}$

4 When $1\frac{1}{2}$ is added to twice a certain number, the result is $4\frac{3}{4}$. What is the number?

5 Gina is 4 years older than her sister. When their ages are added together the result is 22. How old is Gina?

6 Frankie buys four pencils and two rulers at the newsagent at a total cost of \$5.62. If each ruler costs 35 cents more than each pencil, what is the cost of a pencil?

7 A woman is three times as old as her daughter. In four years time she will be only two and a half times as old as her daughter. What is the woman's present age?

8 In an 80 km biathlon, a competitor completes the bicycle leg in 2 hours and the running leg in 40 minutes. If the competitor cycles 20 km/h faster than he runs, at what speed does he cycle?

9 Solve each equation for x .

a $7x + a = 3$

b $ax - b = c$

c $\frac{x}{a} + b = c$

d $a - x = b$

e $a - \frac{x}{b} = c$

f $x(a + b) = c$

g $a(x + b) = cx$

h $\frac{x}{a} + \frac{b}{c} = 0$

i $\frac{x}{a} - \frac{b}{c} = d$

j $\frac{x}{a} + b = \frac{cx}{d} + e$

10 If $ax + b = c$:

a solve the equation for x

b use the formula you found in part **a** to solve for x :

i $2x + 1 = 5$

ii $3x - 5 = 7$

iii $-3x - 1 = 5$

iv $\frac{x}{2} + 1 = 3$

v $\frac{x}{3} - 2 = -3$

vi $\frac{x}{5} - \frac{1}{2} = 2$

11 If $\frac{x}{a} + b = \frac{c}{d}$:

a solve the equation for x

b use the formula you found in part **a** to solve for x :

i $\frac{x}{2} + 3 = \frac{3}{7}$

ii $\frac{x}{5} - 1 = \frac{2}{5}$

iii $\frac{x}{3} + \frac{1}{3} = 1$

iv $2x + 1 = \frac{1}{2}$

v $\frac{x}{2} - \frac{1}{2} = \frac{1}{2}$

vi $3x - \frac{1}{3} = 3$

12 Solve each equation for x .

a $2x + a = 3b$

b $b(2x + a) = a + 2ab$

c $(x + a)(x - a) = (x - b)^2$

d $\frac{bx}{1 + bx} + \frac{ax}{a + ax} = 2$

13 Solve these inequalities.

a $2x + 9 < 23$

b $4x - 5 \geq 17$

c $1 - 5x \geq 21$

d $4 - 3m < 18$

e $3d + 5 \leq 5d + 7$

f $12m + 8 > 9m - 6$

g $3(m - 4) \geq 12$

h $2(3q + 1) \leq 18$

i $\frac{\ell + 2}{3} \leq \frac{\ell - 5}{7}$

j $\frac{2(m + 1)}{3} + 1 \leq \frac{m - 4}{2}$

Chapter 6: Formulas

- 1 Find the value of the subject of each of the following formulas when the pronumerals have the values indicated.

a $V = \ell bh$, when $\ell = 20$, $b = 4.6$ and $h = 2.8$

b $A = \frac{1}{2}bh$, when $b = 6.8$ and $h = 2.04$

c $s = ut + \frac{1}{2}at^2$, when $u = 4.6$, $a = 9.8$ and $t = 4$

d $E = \frac{p^2}{2m}$, when $p = 14.6$ and $m = 8$

e $A = P\left(1 + \frac{r}{100}\right)^n$, when $P = 1000$, $r = 8$ and $n = 4$

f $P = \frac{nRt}{V}$, when $n = 2\,100\,000$, $R = 6.1$, $t = 2000$ and $V = 810$

- 2 Rewrite each formula so that the pronumeral shown in the box is the subject.

a $A = \pi ab$ a

b $A = \pi r(r + \ell)$ ℓ

c $S = \frac{n(a-1)}{2}$ a

d $\ell = a + (n-1)d$ n

e $s = \frac{a+b+c}{2}$ c

f $v = at + bt$ t

g $ut = 2s - vt$ t

h $ap = b^2 - bp$ p

i $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ b

j $a = \frac{h}{R-r}$ r

k $T = 2\pi\sqrt{\frac{\ell}{g}}$ ℓ

l $m = \frac{3x}{may^2}$ y

- 3 **a** Given that $v = u + at$, find the value of a when $v = 26$, $u = 4$ and $t = 11$.
b Given that $F = BTI$, find the value of T when $F = 120$, $B = 4$ and $I = 6$.
c Given that $F = \frac{GMm}{V^2}$, find the value of G when $F = 200$, $M = 10$, $m = 4$ and $V = 2$.
d Given that $P = \frac{1}{r} + \frac{1}{q}$, find the value of r when $P = 8.6$ and $q = 0.4$.

- 4 a Conversion of a given temperature from the Fahrenheit scale to the Centigrade scale is done by means of the formula

$$C = \frac{5(F - 32)}{9}.$$

- i The melting point of gold is 2280°F . Convert this into Celsius (correct to the nearest degree).
- ii A healthy human being's temperature is approximately 36.9°C . What is the equivalent on the Fahrenheit scale, correct to one decimal place?
- b A formula used in life insurance is $Q = \frac{2m}{2 + m}$.
- i Calculate Q if $m = -0.7$ (correct to 4 significant figures).
- ii Calculate m if $Q = 3$.
- 5 For $x = -\frac{a}{2} - \frac{b}{c}$, find the exact value of x if:

a $a = 2, b = -2, c = -1$

b $a = 7, b = 3, c = -4$

c $a = -10, b = \sqrt{3}, c = \sqrt{3}$

d $a = \sqrt{2}, b = \sqrt{2}, c = 4$

e $a = 0, b = 0, c = 5$

f $a = \pi, b = -\pi, c = 7$

Chapter 7: Congruence and special quadrilaterals

- 1 In the diagram opposite, state whether the following angles are acute, obtuse, right or straight.

a $\angle BGC$

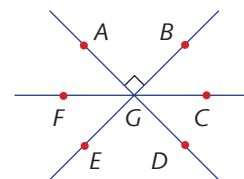
b $\angle AGE$

c $\angle FGD$

d $\angle EGB$

e $\angle CGD$

f $\angle EGD$

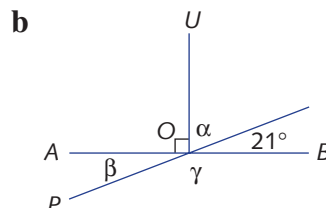
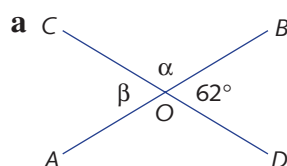


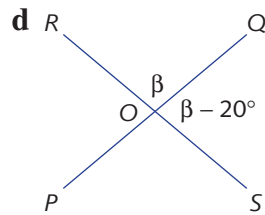
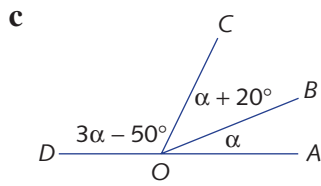
- 2 Given that $\angle ABC = 46^{\circ}$, $\angle PQR = 134^{\circ}$, $\angle LMN = 43^{\circ}$, $\angle DEF = 44^{\circ}$ and $\angle XYZ = 43^{\circ}$, which two angles are:

a complementary

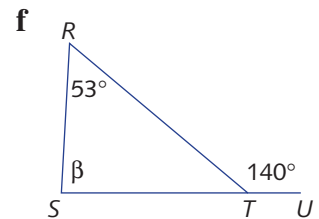
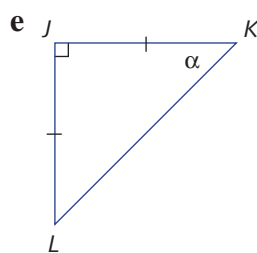
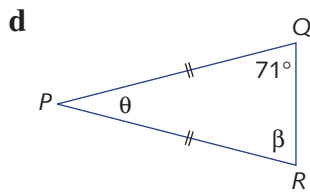
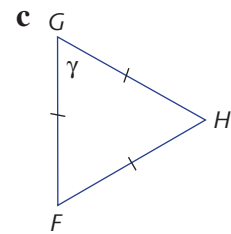
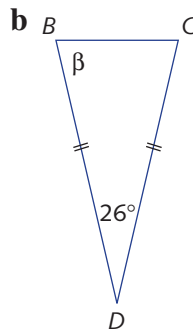
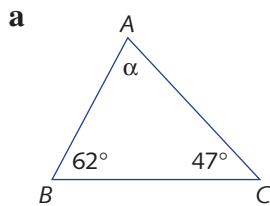
b supplementary?

- 3 Calculate the values of the pronumerals:

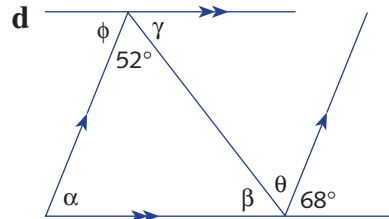
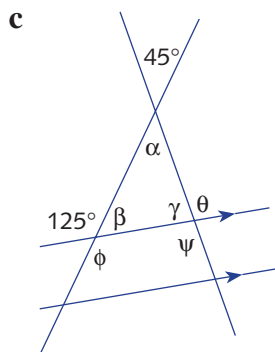
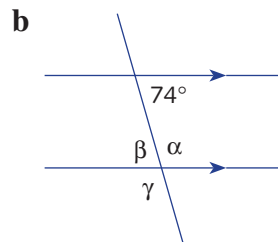
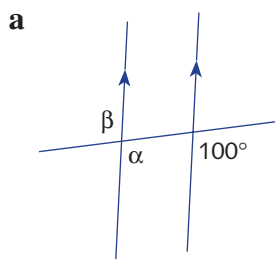




4 Calculate the values of the pronumerals:

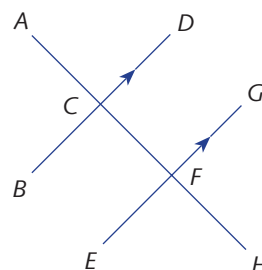


5 Find the values of the pronumerals:



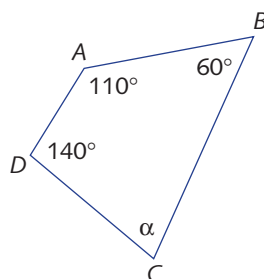
6 In the diagram opposite, which angle is:

- a corresponding to $\angle ACB$
- b vertically opposite $\angle BCF$
- c co-interior to $\angle CFE$
- d corresponding to $\angle CFG$
- e alternate to $\angle CFE$?

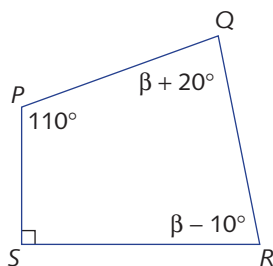


7 Find the values of the pronumerals:

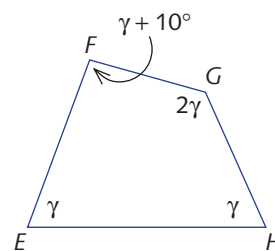
a



b

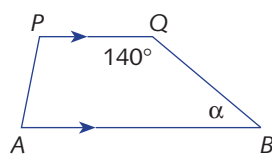


c

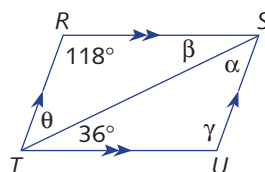


8 Find the values of the pronumerals:

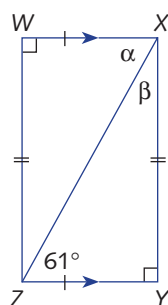
a



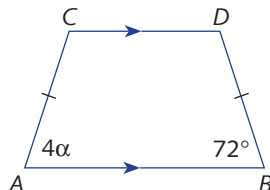
b



c

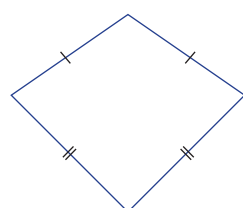


d

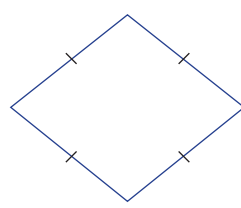


9 State the type of quadrilateral for each of the following.

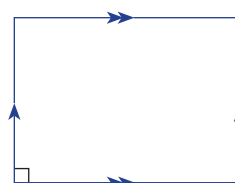
a



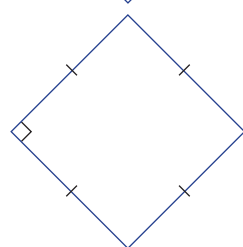
b



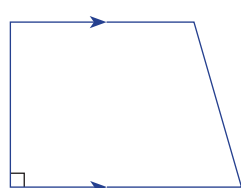
c



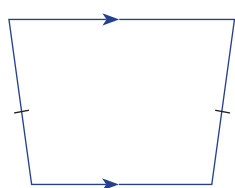
d



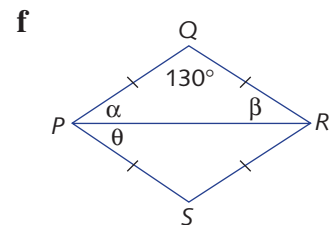
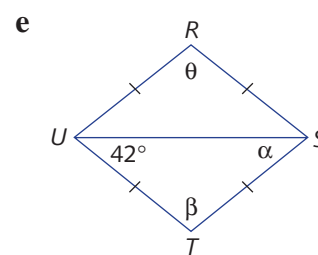
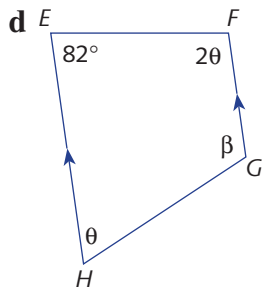
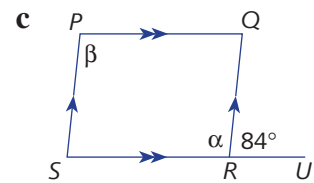
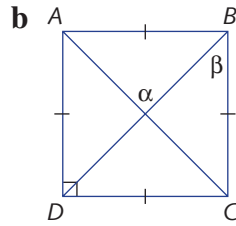
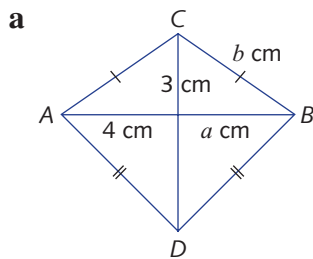
e



f



10 Find the values of the pronumerals:



11 From the list – trapezium, isosceles trapezium, parallelogram, rhombus, rectangle, square, kite – name the quadrilaterals that have:

a opposite angles equal

b diagonals of equal length

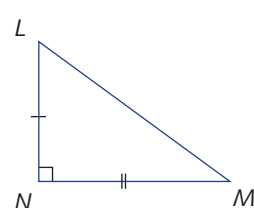
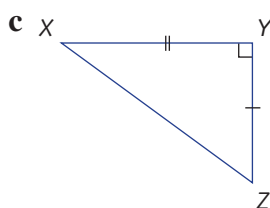
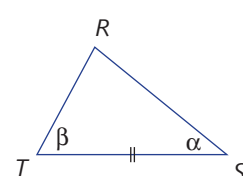
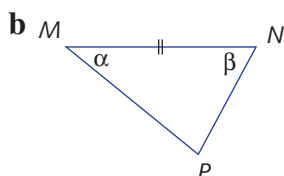
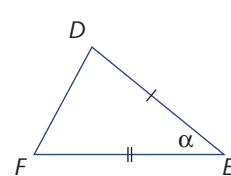
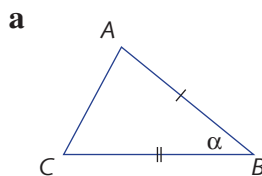
c diagonals that intersect at 90°

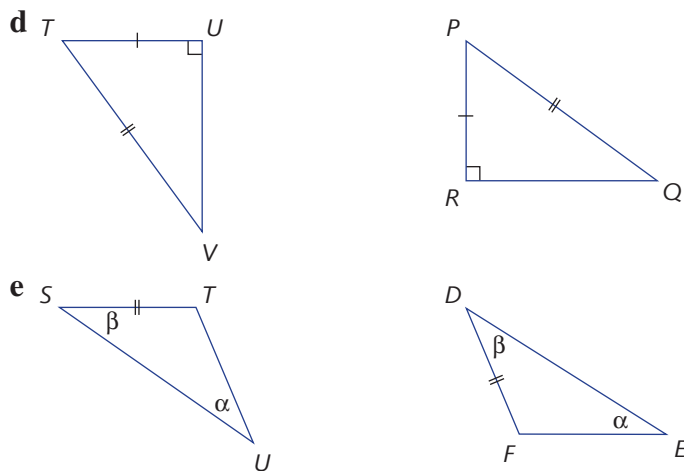
d both diagonals as angle bisectors

e opposite sides parallel

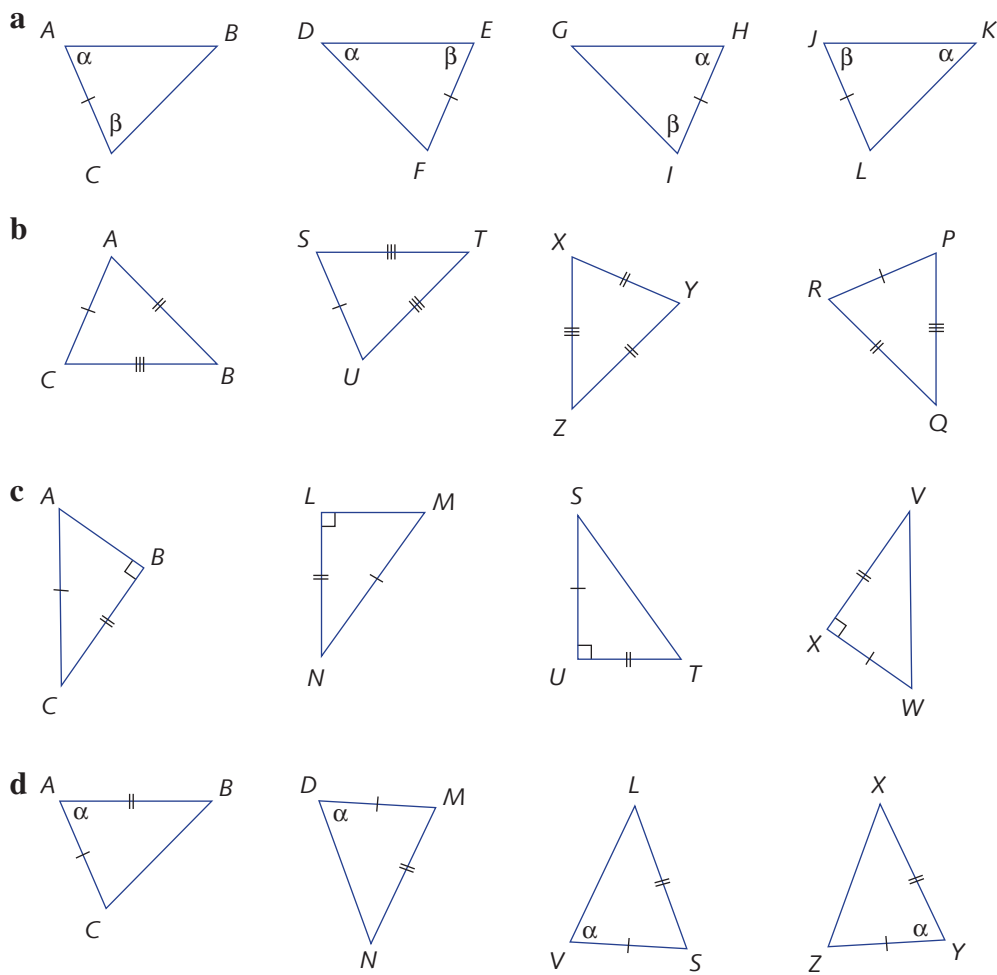
f adjacent angles supplementary.

12 Each pair of triangles is congruent. State the congruence test and name the two triangles with the vertices in correct order.

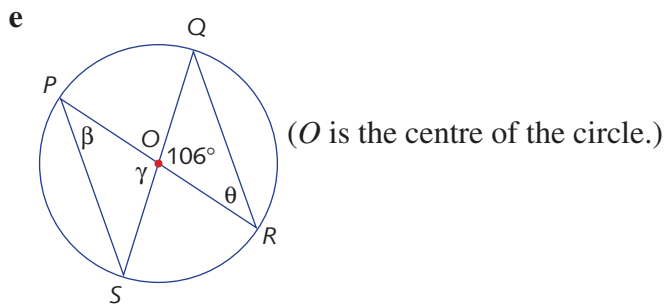
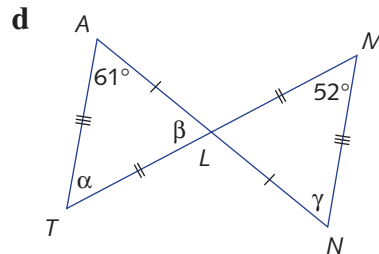
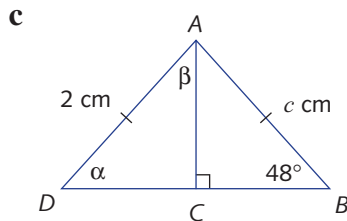
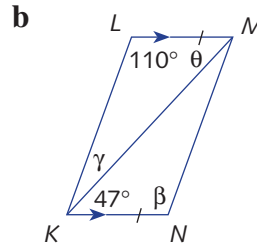
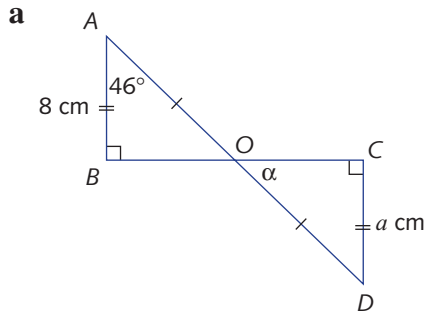




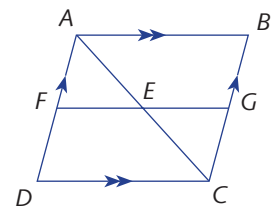
13 In each part, name the triangle congruent to $\triangle ABC$. State the congruence test used.



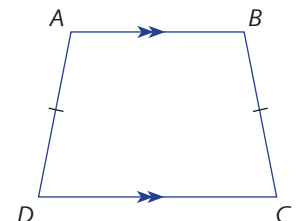
- 14** Each pair of triangles is congruent. Name the two triangles with the vertices in the correct order and then find the value of each pronumeral.



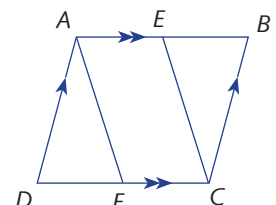
- 15** In the parallelogram $ABCD$, F is the midpoint of AD and G is the midpoint of BC . If AC and FG intersect at E , prove that E is the midpoint of AC .



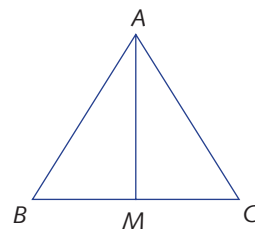
- 16** In the isosceles trapezium $ABCD$, E is the midpoint of AB and F is the midpoint of CD . Prove that $EF \perp DC$.
(Hint: Join AF and BF .)



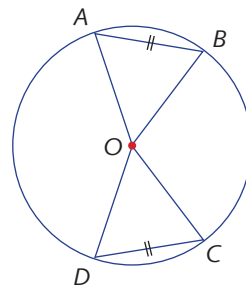
- 17** In the parallelogram $ABCD$, E is the midpoint of AB and F is the midpoint of CD . Prove that $AECF$ is a parallelogram.



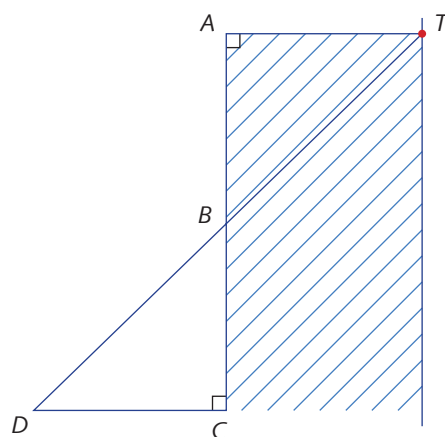
- 18 In the diagram to the right, $\triangle ABC$ is isosceles with $AB = AC$ and M is the midpoint of the interval BC . Prove that $\angle ABC = \angle ACB$. (Only congruence is to be used.)



- 19 AB and CD are chords of a circle with centre O . If $AB = CD$, prove that $\angle AOB = \angle COD$.



- 20 A scout wants to estimate the width of a river. On the opposite river bank there is a tree T . The scout marks a point A on his side of the river directly opposite T . He then goes 6 paces along the river bank perpendicular to AT and marks a point B , then continues a further 6 paces to mark a point C . The scout then walks away from the river, at a right angle to BC , to a point D where B and T are in a line. From C to D is 7 paces.



- a Approximately how many paces wide is the river?
- b If the scout's pace is 0.8 m long, approximately how wide is the river in metres?

Chapter 8: Index laws

- 1 Express in index form:

a $3 \times 3 \times 3 \times 3$

b $5 \times 5 \times 5$

c $a \times a \times a \times a \times a \times a$

d $m \times m \times m \times m \times m \times m$

- 2 Evaluate:

a 6^3

b 5^4

c 2^7

d 4^5

- 3 Evaluate:

a 2.6^3

b 1.7^2

c 8.9^4

d 0.047^2

4 Simplify, writing each answer in index form:

a $5^6 \times 5^7$	b $3^5 \times 3^7$	c $2x^4 \times 5x^2$	d $3m^4 \times 5m^6$
e $\frac{5^7}{5^2}$	f $\frac{7^5}{7^4}$	g $\frac{24a^6}{16a^2}$	h $\frac{15n^{10}}{10n^7}$
i $(5^3)^2$	j $(5^2)^5$	k $(2n^6)^3$	l $(4b^4)^2$
m $2a^0$	n $(4b)^0$	o $(3ab^2)^3$	p $(5x^2y)^0$

5 Simplify:

a $\frac{m^5 n^4}{m^2 n} \times \frac{m^6 n^7}{m^4 n^8}$	b $\frac{p^4 q^5}{pq^2}$	c $(3m^2 n)^3 \times 2m^4 n$
d $\frac{(2m^4 n)^3}{4m^5 n}$	e $\frac{16q^4 r^2}{3q^2} \div \frac{8qr^3}{pq^5 r^4}$	f $\frac{7x^2 y^4}{2xy} \div \frac{21xy^5}{4x^3 y^6}$

6 Express with a positive index and evaluate as a fraction:

a 4^{-2}	b 5^{-2}	c 6^{-3}	d 4^{-4}
e 10^{-4}	f 8^{-1}	g $\left(\frac{2}{3}\right)^{-2}$	h $\left(\frac{3}{4}\right)^{-3}$

7 Express with positive indices:

a a^{-2}	b b^{-6}	c $a^{-2}b^{-4}$	d $m^{-1}n^{-3}$	e $\frac{1}{x^{-4}}$
f $\frac{2}{y^{-3}}$	g $3m^{-2}$	h $5m^{-6}$	i $\frac{m^{-4}}{n^{-6}}$	j $\frac{3p^{-1}}{q^{-2}}$

8 Simplify, expressing your answer with positive indices:

a $x^{-4}y^{-2} \times x^2y^{-6}$	b $p^{-5}q^{-3} \times pq^{-2}$	c $4n^{-2}b \times 3a^4b^{-6}$
d $2\ell^2m^{-1} \times 5\ell^{-4}m^{-2}$	e $\frac{56t^{-4}}{8t^2}$	f $\frac{25\ell^{-2}}{15\ell^{-6}}$
g $\frac{28x^6y^2}{21x^{10}y}$	h $\frac{16u^{-2}v}{12u^{-6}v^{-7}}$	i $(2mn^{-1})^{-2}$
j $(3m^{-1}n^2)^{-4}$	k $(2a^{-4}b^{-1})^{-3} \times 3a^6b^{-2}$	l $(a^{-1}b^2)^{-4} \times (a^2b^{-3})^3$
m $\frac{(3a^{-2}b^{-1})^{-3}}{a^2b^{-4}} \times \frac{a^{-3}b^4}{12a^6b^{-2}}$	n $\frac{(a^{-1}b^4)^{-3}}{(a^2b^{-1})^4} \times \frac{a^{-2}b}{(ab^{-1})^3}$	o $\frac{(a^2)^4}{b^4} \div \frac{ab^{-2}}{(b^{-1})^3}$
p $\frac{ab^{-1}}{4a^2} \div \frac{(2a^{-3}b)^{-2}}{12a^{-1}b^3}$	q $\frac{(4xy)^0}{(4xy)^{-1}} \times \frac{3x^2y^{-6}}{(x^{-2}y^3)^{-4}}$	r $\frac{2m^{-3}n^{-2}}{(2mn^4)^{-2}} \times \frac{3m^{-3}}{(mn^2)^0}$

9 Evaluate without using a calculator:

a $9^{\frac{1}{2}}$ **b** $8^{\frac{1}{3}}$ **c** $16^{\frac{1}{4}}$ **d** $243^{\frac{1}{5}}$
e $4^{\frac{3}{2}}$ **f** $27^{\frac{2}{3}}$ **g** $49^{\frac{3}{2}}$ **h** $8^{\frac{4}{3}}$

10 Simplify, expressing the answer with positive indices:

a $\sqrt[3]{8a^2}$ **b** $\sqrt[5]{32a^5b}$ **c** $\sqrt[4]{16a^2}$ **d** $\sqrt[3]{5x^5} \times \sqrt[3]{25x^3}$
e $3x^{\frac{3}{2}} \times 2x^{\frac{5}{2}}$ **f** $(32a^{10}b^5)^{\frac{2}{5}}$ **g** $\frac{(12x^3y^5)^{\frac{2}{3}}}{(4y)^{\frac{1}{3}}}$ **h** $6a^{\frac{3}{2}}b^{\frac{1}{3}} \times 2a^{\frac{1}{2}}b^{\frac{5}{3}}$

11 Write in scientific notation or standard form:

a 21 000 **b** 410 **c** 61 000 000 000
d 2400 **e** 0.0062 **f** 0.0471
g 0.000 000 7 **h** 0.000 38 **i** 46 **j** 2.9

12 Write as a decimal:

a 7.2×10^4 **b** 3.8×10^2 **c** 9.7×10^{-1} **d** 2.06×10^{-2}
e 1.52×10^2 **f** 4.07×10^3 **g** 1.6×10^{-4} **h** 8.7×10^{-2}

13 Simplify, giving your answer in standard form:

a $(5 \times 10^2) \times (4 \times 10^6)$ **b** $(2.1 \times 10^2) \times (3 \times 10^6)$
c $(6 \times 10^{-4}) \div (2 \times 10^{-3})$ **d** $(9.6 \times 10^4) \div (4.8 \times 10^{-2})$
e $(4.2 \times 10^3)^2$ **f** $(1.1 \times 10^{-2})^3$
g $\frac{(5 \times 10^4)^3}{(15 \times 10^2)^2}$ **h** $\frac{(3 \times 10^{-2})^2}{1.8 \times 10^{-4}}$

14 Write in standard form, correct to the number of significant figures indicated in the brackets.

a 241 000 (2) **b** 627 (1) **c** 0.000 649 26 (3)
d 0.008 716 (2) **e** 27 654 321 (4) **f** 0.267 83 (3)

15 Give a one significant figure estimate for:

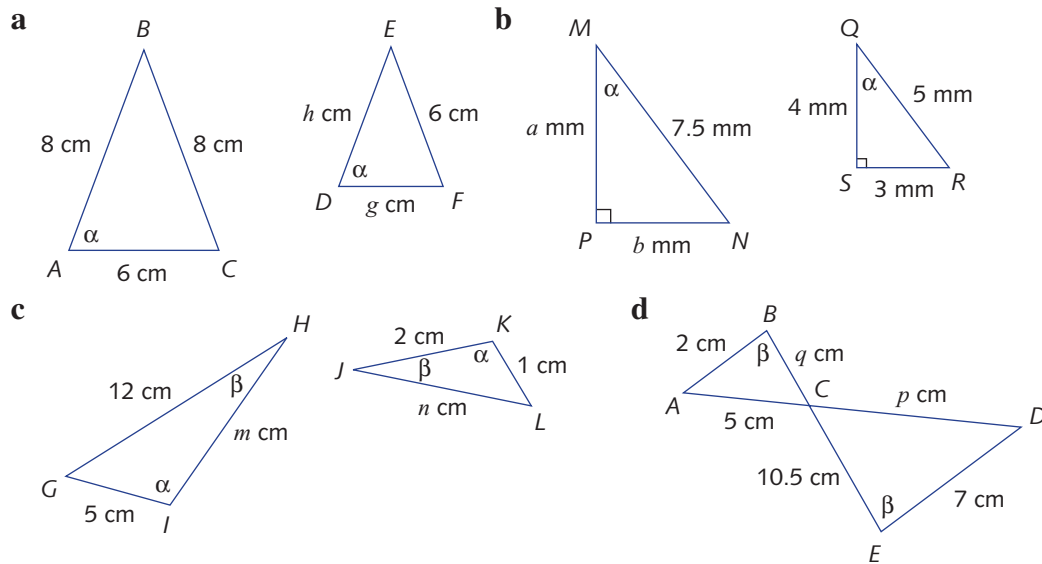
a $210\,000 \times 0.006\,13$ **b** $470\,000\,000 \div 216\,000$
c $\frac{0.000\,473}{0.009\,91}$ **d** $(0.003\,14)^2$

16 Evaluate, giving your answer to the number of significant figures indicated in the brackets.

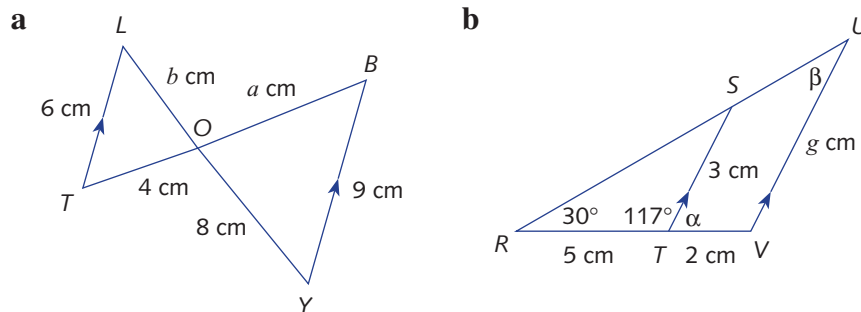
a 4.671×1.304 (2) **b** 8.96×10.2 (3) **c** 8.206^3 (4)
d 0.041^2 (3) **e** $\frac{18.6 \times 1.75}{0.26}$ (1) **f** $\frac{87.01 \times 0.003}{2.765}$ (2)
g $\frac{\sqrt{47.8}}{2.73}$ (4) **h** $\frac{18.7^2 + 9.21^2}{18.7^2 - 9.21^2}$ (4) **i** $\frac{0.03^2}{\sqrt{0.03}}$ (2)

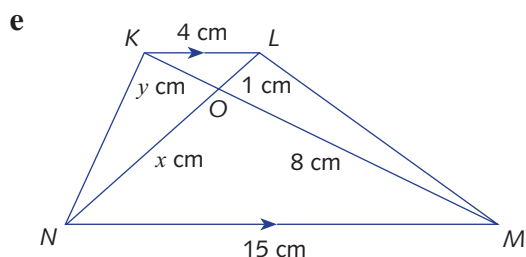
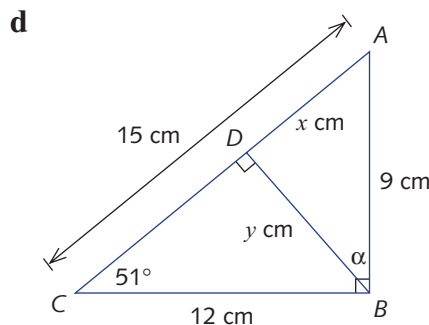
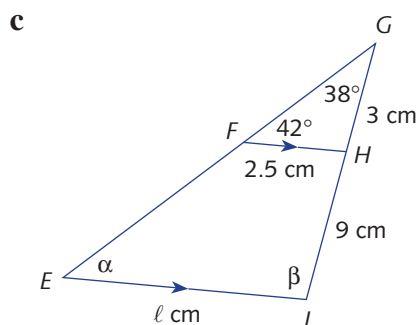
Chapter 9: Enlargements and similarity

1 For each part, write a similarity statement with the vertices in the correct order. Then find the value of each unknown side length.



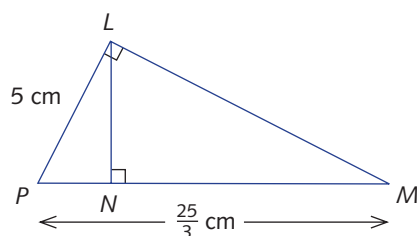
2 Name each pair of similar triangles, giving the similarity test used, and then find the value of each pronumeral.





- 3 At a particular time of the day, a tree casts a shadow of length 5.2 m and a 30 cm ruler casts a shadow of length 16 cm. Find the height of the tree.
- 4 A girl is walking along a footpath and notices that the end of her shadow coincides with the end of the shadow of a lamp post. If the girl is 1.4 m tall and she is standing 5 metres from the lamp post, find the height of the lamp post if the length of her shadow is 80 cm.
- 5 In the diagram to the right:

- a find LM
b find LN
c find MN .



- 6 A builder is constructing a ramp AB , as drawn in the diagram, with $BC = 2$ metres and $AC = 6$ metres. Two vertical supports ED and GF need to be placed at 2-metre intervals.

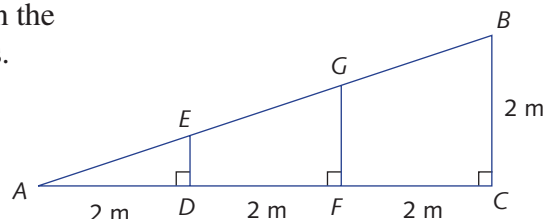
- a Find:
i DE ii FG .

- b Find AB .

- c The builder needs to construct two timber frames (as drawn in the diagram) to support the ramp.

- i What length of timber, to the nearest metre, will he need?

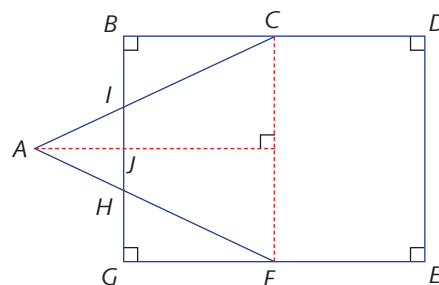
- ii Timber costs \$4.60 per metre and is only sold in lengths which are a multiple of one metre. How much will the timber cost the builder?



- 7 A ladder leans against a vertical wall. The base of the ladder is 2.4 metres from the wall and the top of the ladder is 3.2 metres above the ground. A man of height 1.7 metres climbs 1 metre up the ladder and stands in a vertical position.

- How long is the ladder?
- How far off the ground is the top of the man's head?
- How far from the wall is the man?

- 8 An engineer is building a trailer. The tray $BDEG$ of the trailer is 2 metres long and 1.5 metres wide – that is, $BD = 2$ m and $DE = 1.5$ m. The intervals AC and AF represent the draw-bar of the trailer, where $AC = AF$, C is the midpoint of BD and F is the midpoint of GE .



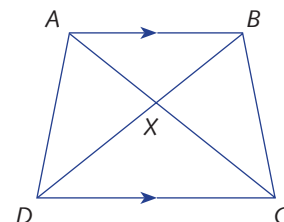
- As part of the design, $AJ = 1$ metre. Find, correct to the nearest millimetre:

- AC
- BI

- The engineer changes the distance of A from BG so that $BI : IJ = 7 : 8$. Find, correct to the nearest millimetre:

- BI
- AJ
- AC

- 9 $ABCD$ is an isosceles trapezium with $AB \parallel CD$ and $AD = BC$. The diagonals of $ABCD$ intersect at X .



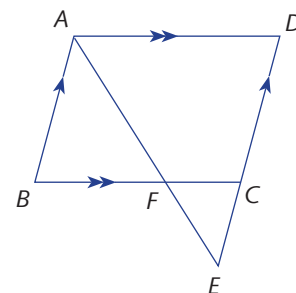
- Prove that $\triangle ABX$ is similar to $\triangle CDX$.
- Given that $AB = 4$ cm, $BX = 2.5$ cm and $DC = 8$ cm, calculate:
 - AX
 - DX .

- 10 a Prove that $\triangle AFB$ is similar to $\triangle EFC$.

- Given that $AD = 10$ cm, $CD = 4$ cm and $CE = 1$ cm, calculate FC .

- Name two other pairs of similar triangles.

- 11 AB and DC are the parallel sides of a trapezium and are of such length that when diagonal DB is constructed, $\angle DAB = \angle DBC$. Prove that $AB \times BC = DB \times AD$.



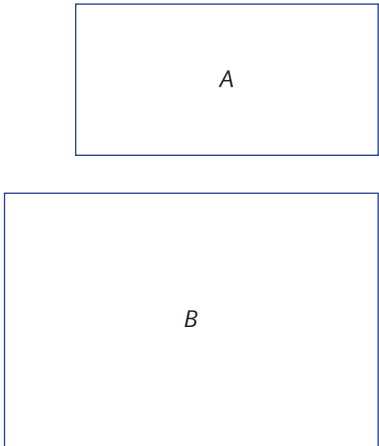
- 12 $ABCD$ is a parallelogram with E on BC such that $BE = 2EC$. The line AE and the line DC intersect at F .

- Prove that $AB = 2CF$.
- Prove that $AF = 3EF$.

10B Problem-solving

- 1 Mahmoud wants to buy a new car, which is valued at \$45 000. He could buy the car through an agreement with the car dealer or by borrowing the money from a bank.
 - a At the bank, Mahmoud asks for a loan of \$45 000. The bank offers a loan at 7% p.a. simple interest to be repaid in equal monthly instalments for 5 years. Calculate:
 - i the amount of interest Mahmoud would pay.
 - ii the total amount to be paid by Mahmoud.
 - iii the amount to be paid each month.
 - b At 'BJ's Car Sales', Mahmoud is offered the following deal: pay \$5000 in cash now and then 60 monthly instalments of \$934, starting in a months time.
Calculate:
 - i the total amount it would cost Mahmoud to buy the car this way.
 - ii the rate of interest charged per annum, at simple interest.
 - c Who offers the better deal, the car dealer or the bank?
- 2 The workers at a production factory are negotiating a pay rise with the management. The union wants a pay increase of 2% for the first year followed by a further pay increase of 4% in the second year, with no further increase for two more years. The management offers a 5% pay increase for the first year with no further increase for three more years. Riley is currently paid \$30 000 a year.
 - a Under the union's claim:
 - i what would Riley's salary be in the first year?
 - ii what would Riley's salary be in the second year?
 - iii how much would Riley earn during the four years?
 - b Under the management's offer:
 - i what would Riley's salary be in the first year?
 - ii how much would Riley earn during the four years?
 - c Which package would be better for Riley?
- 3 A container holds 1.25 litres of a fruit juice drink. This drink is 25% fruit juice and the remainder is water.
 - a If 0.5 litres of the drink is poured out and replaced by 50% fruit juice drink, find the percentage of the contents of the container that is now fruit juice.
 - b If x litres of the drink is poured out and replaced by 50% fruit juice drink, find the percentage of the drink in the container that is now fruit juice.
 - c How many litres of the fruit juice drink has to be poured out and replaced by the 50% fruit juice drink for the percentage to increase to 41%?

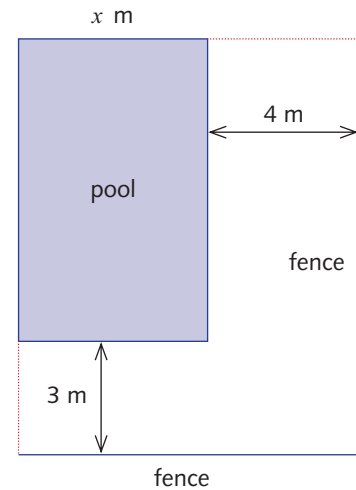
- 4 Two cars leave Roma on the Warrego Highway travelling in the same direction. One car leaves the town 30 minutes before the other. The first car averages 80 km/h while the second car averages 96 km/h.
- How far apart are the cars when the first car is:
 - 50 km from Roma?
 - x km from Roma, assuming that the first car is still in front of the second car, and that the second car has left Roma?
 - How far from Roma does the second car catch up to the first?
- 5 The pilot of an ultra-light aircraft has determined that the plane averages 75 km/h in still air.
- If she flies out for 20 km and then returns by the same route, how long will she take?
 - If she flies out for 1 hr 42 mins and back for 1 hr 42 mins, how far does she travel in total?
 - If there is a 40 km/h headwind as the pilot flies out and a 40 km/h tailwind on return, what is the longest time the pilot can fly out to be able to return in 4 hours? (Answer in hours and minutes.)
- 6 The cash price of a car is advertised as \$32 000. Alternatively you can buy the car with a deposit of 40% and repayments of \$377.60 per month for 5 years to a finance company.
- What is the price paid for the car under the alternative scheme?
 - What is the amount of interest paid to the finance company under the alternative scheme?
 - What is the rate of simple interest p.a. that will apply in the alternative scheme?
 - The car depreciates in value by 12% per year.
 - Find the value of the car at the end of the 5 years.
 - Calculate your average total cost per month over the 5-year period if you use the Finance Company option. Include interest and depreciation, and assume that it costs \$1000 per year for comprehensive insurance and registration, \$200 per month for petrol and \$800 per year for servicing.
- 7
- To fill my tank with petrol costs \$55.00 at \$1.10 per litre. What is the price in dollars per litre if, after a price rise, it costs \$62.50 to fill the tank with the same amount of petrol?
 - A person purchases 50 litres at \$1.12 per litre. Some time later the price has increased to \$1.40 per litre. How many litres can now be purchased for the same amount of money?
 - Let $\$C$ be the cost of buying V litres of petrol at R dollars per litre. Find the formula relating C , V and R .

- 8 George runs twice as fast as he walks. When going to school one day, he walks for twice the time that he runs and takes a total of 20 minutes to get to school.
- If x km/h is George's walking speed, find the distance that he travels to school in terms of x .
 - The next day George runs for twice the time that he walks.
 - If t hours is the time it takes George to get to school, find the distance to school in terms of x and t .
 - Hence find how many minutes it takes him to get to school on the second day.
- 9 A car can travel 100 km on 11 litres of petrol when its speed is 50 km/h, but when its speed is 90 km/h it consumes 15 litres per 100 km.
- If the car travels 28 km at 50 km/h and an additional 130 km at 90 km/h, how many litres of petrol will it consume?
 - The car travels x km at 50 km/h and $5x$ km at 90 km/h. Find a formula for y , the number of litres of petrol it consumes on the journey.
- 10 In rectangle A , the length is 5 m more than the width.
- Rectangle B is obtained from rectangle A by increasing the width of A by 50% and increasing the length of A by 20%.
- 
- If the width of A is 4 m, calculate:
 - the length of A
 - the length and width of B
 - the perimeter of A
 - the perimeter of B
 - the difference between the perimeters of A and B .
 - If the width of A is x m, find, in terms of x :
 - the length of A
 - the length and width of B
 - the perimeter of A , expressed in simplest form
 - the perimeter of B , expressed in simplest form
 - the difference between the perimeters of A and B , expressed in simplest form.
 - If the difference between the perimeters of A and B is 23 m, what is the value of x ?
- 11 Andrew has been contracted by a family to build a fence around two sides of their swimming pool. The swimming pool is in the shape of a rectangle and it is twice as long as it is wide. Council regulations require the fence to be a certain distance from the pool, as shown in the diagram.

- a** If the width of the pool is x m, express each of the following in terms of x .

- i** The length of the pool
- ii** The length of the fence that is parallel to the width of the pool
- iii** The length of the fence that is parallel to the length of the pool
- iv** The total length of the fence

Andrew gives the family a quote for the job. He says that it will cost \$6.00 per metre to build the section of the fence that is parallel to the pool's width, and the section parallel to the pool's length will cost \$8.00 per metre.



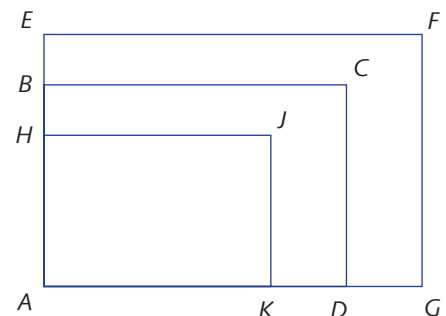
- b** Express the total cost $\$C$ of the fence in terms of x .

The family intend to pave the area between the pool and the fence. They need to calculate the area between the pool edge and the fence to determine the cost of the paving.

- c i** Express the area covered by the pool and the fence enclosure in terms of x .
- ii** Show that the area to be paved is equal to $(11x + 12) \text{ m}^2$.
- iii** If the width of the pool is 4 m, find the area to be paved.
- iv** If the area to be paved is 61.5 m^2 , calculate the width and length of the pool.

- 12** In the rectangle $ABCD$, $AB = 3$ cm and $BC = 4$ cm.

- a** Find the area of $ABCD$.
- b** The rectangle $AEFG$ was formed by increasing the dimensions of $ABCD$ by 20%. Find:
 - i** the area of $AEFG$
 - ii** the difference in area between $ABCD$ and $AEFG$
 - iii** the percentage change in the area from $ABCD$.



- c** The rectangle $AHJK$ was formed by decreasing the dimensions of $ABCD$ by 15%. Find:
 - i** the area of $AHJK$
 - ii** the difference in area between $ABCD$ and $AHJK$
 - iii** the percentage change in the area from $ABCD$.

d If each of the side lengths of $ABCD$ are increased by $x\%$ to form $ALMN$:

i find AL in terms of x

ii find LM in terms of x

iii find the area $ALMN$ in terms of x

iv find x , to three significant figures, if the area of $ALMN$ is 15 cm^2

v show that the percentage change in the area from $ABCD$ is $2x + \frac{x^2}{100} \%$.

13 Two trains travel between towns A and B . They leave at the same time with one train travelling from A to B and the other from B to A . They arrive at their destinations one hour and four hours respectively, after passing one another at point P . The slower train travels at 35 km/h .

a How far does the slower train travel after they pass?

b If the faster train travels at $x \text{ km/h}$, how far, in terms of x , does the faster train travel after they pass?

c Hence find the number of hours, in terms of x , that the faster train takes to reach point P . Repeat for the slower train to find a different expression, in terms of x , for the same time.

d Hence find the speed of the faster train.

14 Cam purchased a number of identical items, each costing $\$x$. The total cost was $\$40$.

a How many items were purchased?

b How many items could be purchased for $\$40$ if the price of each individual item increased by 40 cents? (Calculate your answer in terms of x .)

c If the number of items obtained for $\$40$ at the cheaper price was five more than the number obtained at the dearer price, write down an equation involving x .

d Hence find x and state the original cost of each item.

15 $ABCD$ is a square of side length 21 cm . $AM = x \text{ cm}$ and $MB = y \text{ cm}$.

a Find y in terms of x .

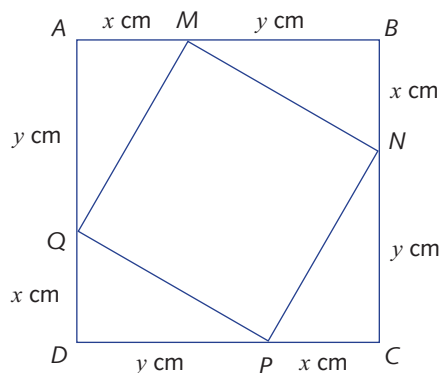
Triangles AMQ , BNM , CPN and DQP are congruent, right-angled triangles.

b Find the area of $MNPQ$ in terms of x .

c i If the area of $MNPQ$ is 225 cm^2 , find x .

ii In this case, find the dimensions of the triangles.

d If the area of the square $MNPQ$ is half the area of the square $ABCD$, find the value of x .

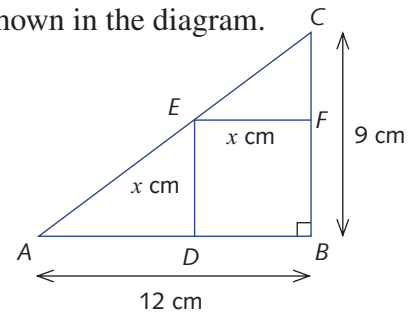


- 16 a In the right-angled triangle ABC is a square $BDEF$ as shown in the diagram.

i Why is $\triangle EFC$ similar to $\triangle ABC$?

ii Hence, find x .

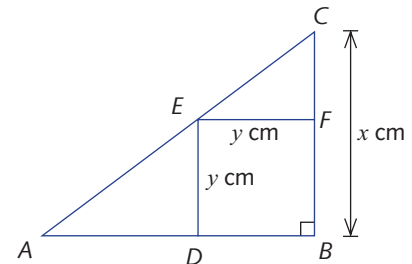
iii Hence, find the area of the square $BDEF$ as a percentage of the area of $\triangle ABC$, correct to one decimal place.



- b In this diagram, the square $BDEF$ is inside $\triangle ABC$ as shown. If $BC = x$ cm, $EF = y$ cm and $AB = 2BC$:

i find the relationship between x and y

ii hence find the area of the square $BDEF$ as a percentage of the area of $\triangle ABC$, correct to one decimal place.



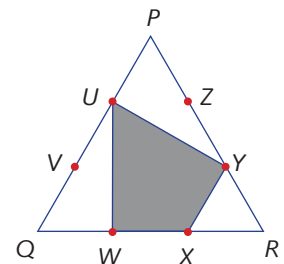
- 17 Let PQR be an equilateral triangle with sides of length 3 units. U, V, W, X, Y and Z divide the sides into unit lengths.

a Prove that $\triangle QUW$ is congruent to $\triangle PYU$.

b Show area $\triangle UQW = \frac{2}{9} \times \text{area } \triangle PQR$.

c Show area $\triangle XYR = \frac{1}{9} \times \text{area } \triangle PQR$.

d Find the area of the shaded region in terms of the area of $\triangle PQR$.



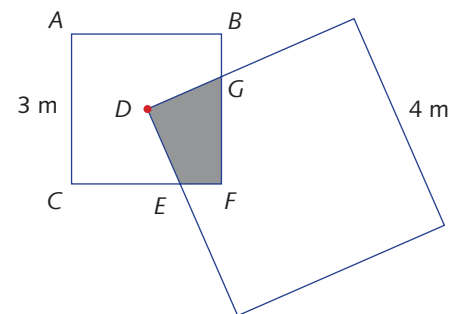
- 18 A 3-metre square and a 4-metre square overlap as shown in the diagram. D is the centre of the 3-metre square.

a Draw the interval DY where Y is on GF and $DY \perp GF$. Draw the interval DX where X is on CF and $DX \perp CF$.

b Show that $\triangle DXE \equiv \triangle DYG$.

c Explain why area $DEFG = \text{area } DXFY$.

d Find the area of the shaded region.



- 19 $ABCD$ is a parallelogram where $AP = 12$, $DQ = 16$, $CS = 10$, $PQ = 5$ and $QR = 2$.

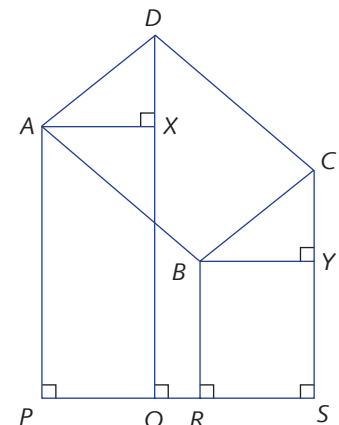
a Prove that $\triangle ADX \equiv \triangle BCY$.

b Explain why $BY = 5$.

c Find the length of the interval BR .

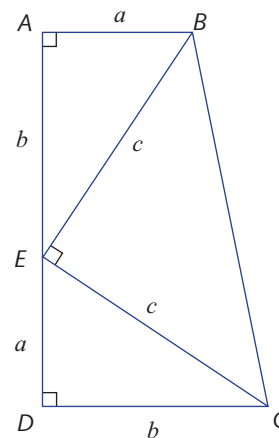
d Find the area of $BRSC$.

e Find the area of the parallelogram $ABCD$.



- 20** Triangles ABE and DEC are congruent, right-angled triangles with side lengths as shown.

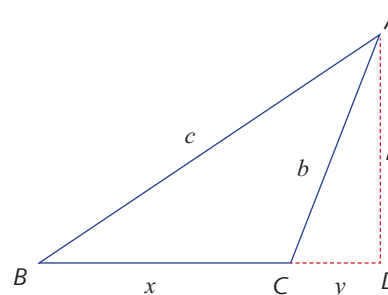
- a i** Find the area of triangle ABE .
ii Find the area of triangle BEC .
iii Use the results of **i** and **ii** to find the area of the trapezium $ABCD$.
b Use the formula to find the area of trapezium $ABCD$.
c Use your results from parts **a** and **b** to show that $a^2 + b^2 = c^2$, hence proving Pythagoras' theorem.



- 21 a** ABC is an obtuse-angled triangle. AD is an altitude from A to the line through B and C .

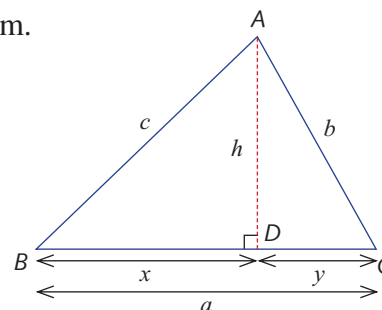
- i** In triangle ABD , show that $c^2 = x^2 + 2xy + y^2 + h^2$.
ii In triangle ACD , we know that $b^2 = y^2 + h^2$.

Use the result of **i** to show that $c^2 - (x^2 + b^2) = 2xy$ – that is, show that for an obtuse-angled triangle ABC with obtuse angle at C , $AB^2 = (BC^2 + AC^2) + 2BC \times CD$.



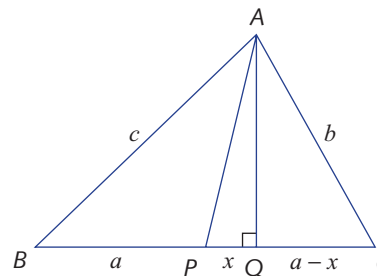
- b** ABC is a triangle with $\angle C$ acute, as shown in the diagram.

- i** Use Pythagoras' theorem in triangles ABD and ACD to show that $h^2 = c^2 - x^2$ and $h^2 = b^2 - y^2$ and $c^2 - b^2 = x^2 - y^2$.
ii Show that $(b^2 + a^2) - c^2 = 2ya$. That is, in a triangle ABC with acute angle at C , $AB^2 = BC^2 + AC^2 - 2BC \times CD$.

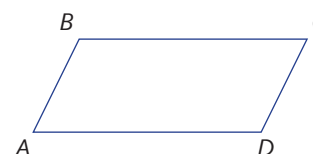


- 22 a** ABC is a triangle and AQ is an altitude where Q lies between B and C . P is the midpoint of BC .

Prove that $AB^2 + AC^2 = 2BP^2 + 2AP^2$



- b** Use part **a** to prove that for any parallelogram $ABCD$, $2AB^2 + 2BC^2 = AC^2 + BD^2$



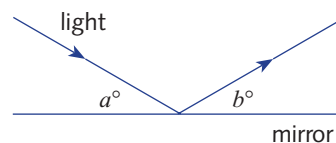
How does a sextant work?

A sextant is a navigational device used by sailors to determine the angle between the horizon and a reference point, usually a star. Knowing this angle allows the sailor to determine the latitude of the vessel's position.

A sextant uses the idea of double reflection to measure angles. The concept was first conceived by Sir Isaac Newton in 1699, but the first sextants were not commercially produced for about another twenty years.

In this investigation, you will discover how a sextant works and how it can be used to measure the angle between the horizon and an object in the sky. To do this, the following important result about reflection is needed.

When light is reflected by a mirror, the angle of the incoming ray of light is equal to the outgoing angle of the ray of light. In the diagram opposite, $a = b$.



A sextant uses double reflection to measure angles.

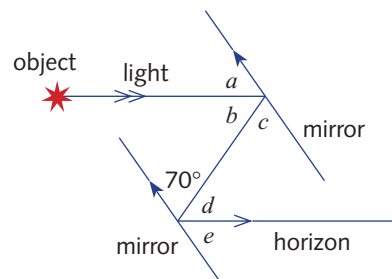
a In the diagram opposite, the two mirrors are parallel.

i Find the value of each pronumeral.

ii Does $b = d$?

iii By changing the angle of size 70° to other angles and finding b and d in each case, is it always true that $b = d$?

iv What does the fact that $b = d$ tell you about the line from the object and the horizon?



b Suppose that the two mirrors are by longer parallel. The angle measured by the sextant is the angle between the two mirrors ($\angle ADB$ in this diagram).

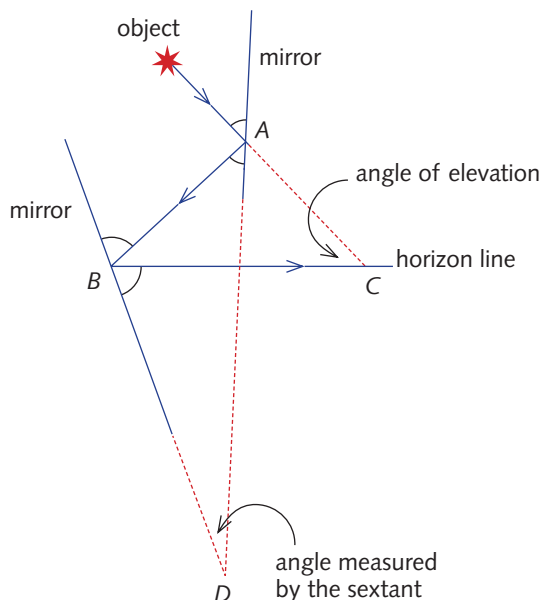
i Find the angle of elevation, given that $\angle CBD = 70^\circ$ and $\angle ADB = 25^\circ$.

ii Find the angle of elevation, given that $\angle CBD = 64^\circ$ and $\angle ADB = 39^\circ$.

iii By investigating other values of $\angle CBD$ and $\angle ADB$, can you guess a connection between the angle measured on the sextant and the angle of elevation?

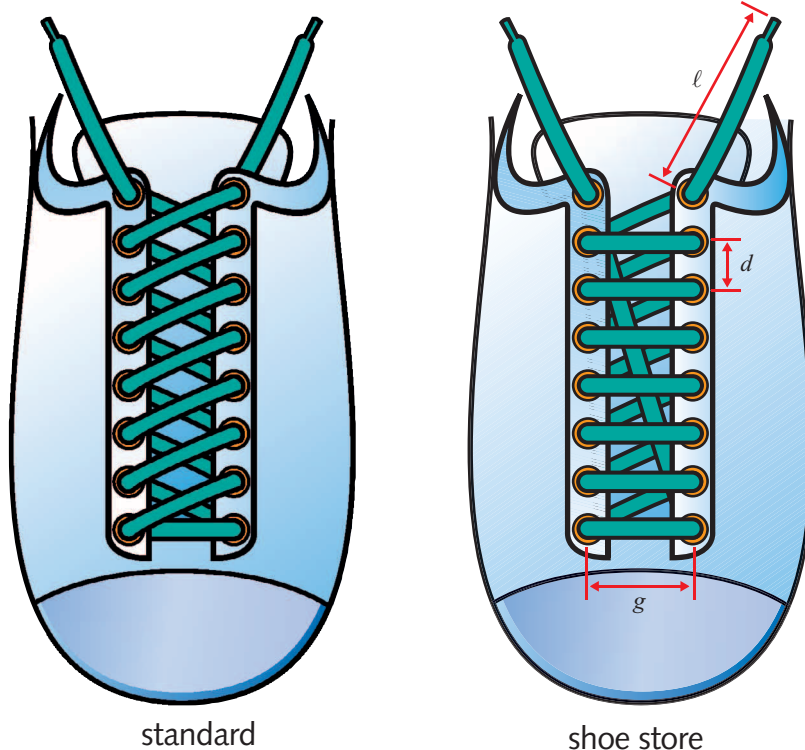
iv Find the angle of elevation if $\angle CBD = y^\circ$ and $\angle ADB = x^\circ$.

v Does this general result agree with your conjecture of part **iv**?



How long is a piece of string?

A shoe manufacturer is trying to calculate how long a shoe lace needs to be for a pair of shoes. There are at least two ways to lace shoes (as shown below).



The length of shoe lace required will depend on:

- the number n of pairs of eyelets
- the distance d between successive eyelets
- the length g of the gap between corresponding left and right eyelets
- the length ℓ of the end of each shoe lace.

1 Consider the standard lacing pattern.

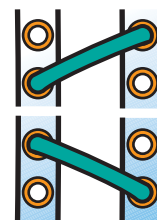
a How many diagonal segments (as shown in these diagrams) are there if n is equal to:

- i 3 ii 4 iii 5?

b How is the number of diagonal segments related to n ?

c How long is each diagonal segment? Calculate your answer in terms of d and g .

d What is the total length of lacing required for the standard lacing pattern? Calculate your answer in terms of n , d , ℓ and g .



2 Consider the shoe store lacing pattern.

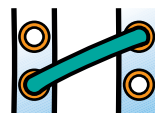
a How many horizontal segments (as shown here) are there if n is equal to:



- i 3 ii 4 iii 5?

b How is the number of horizontal segments related to n ?

c How many diagonal segments like the one shown in this diagram are there, if n is equal to:



- i 3 ii 4 iii 5?

d How is the number of diagonal segments related to n ?

e What is the length of each horizontal segment?

f What is the length of each diagonal segment? Give your answer in terms of d and g .

g How far is it from the bottom pair of eyelets to the top pair of eyelets if n equals:

- i 3 ii 4 iii 5?

h How is the distance from the bottom pair of eyelets to the top pair of eyelets related to n ?

i What is the length of the long diagonal segment? Give your answer in terms of n , d , ℓ and g .

j What is the total length of lacing required for the shoe store lacing pattern? Give your answer in terms of n , d , ℓ and g .

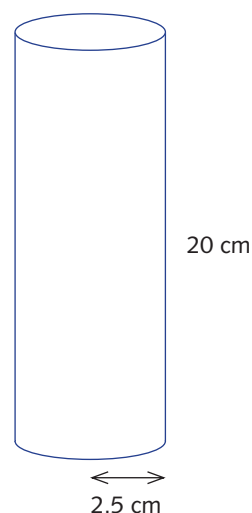
3 By investigating the total length of lace required for each lacing pattern for different values of n , g , ℓ and d , determine which pattern requires less shoe lace.

Packaging

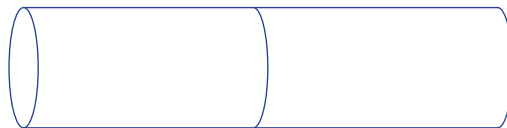
In a factory, balls are packed in canisters, which are then wrapped in bundles of six with shrink-wrap film. To make the bundle as secure as possible, the *total volume* of the pack, including any air space, needs to be as small as possible. If the volume was not as small as possible, the shrink-wrap film could be pulled tighter, thus decreasing the volume and making the package more secure.

In this project, you will investigate how cylindrical canisters can be packed so as to minimise the volume of the pack.

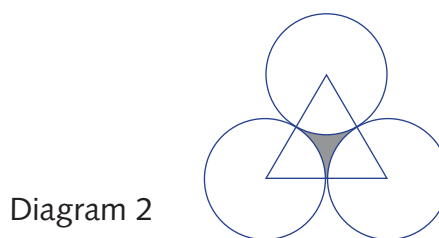
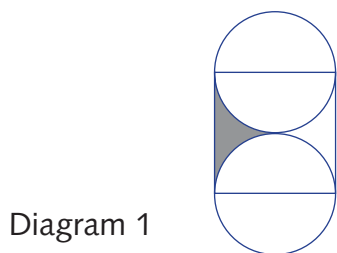
You will need the following information. The canisters are cylindrical, with a base radius of 2.5 cm and a height of 20 cm.



The final package can be no longer than 20 cm – that is, the cans cannot be stacked end-to-end like this.



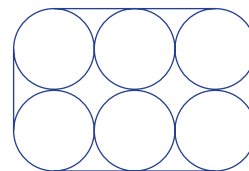
In the following, we consider cross-sections of the bundles of canisters. When canisters are bundled, two types of region are important. These are shown below by the shading. The radius of each circle is 2.5 cm.



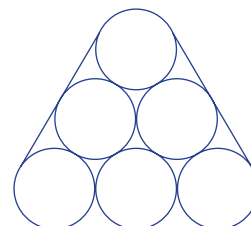
- 1
 - a Calculate the exact area of the shaded region in Diagram 1.
 - b
 - i What is the side length of the triangle shown in Diagram 2?
 - ii What is the height of the triangle? Calculate an exact answer.
 - iii What is the exact area of the shaded region in Diagram 2?
- 2
 - a Suppose that the six canisters are packed in a row. What is the total volume of the package? Calculate your answer to 3 decimal places.



- b Suppose that the six canisters are packed in two rows of three. What is the total volume of the package? Calculate your answer to 3 decimal places.



- c Suppose that the six canisters are packed as shown in the diagram opposite. What is the total volume of the package? Calculate your answer to 3 decimal places.



- d Which is the best packaging, **a**, **b** or **c**?
- 3 Instead of being packaged in bundles of six, the company now decides to package the canisters in bundles of seven. How should the canisters be packed so as to minimise the volume?

Answers to exercises

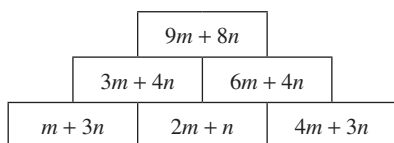
Chapter 1 answers

Exercise 1A

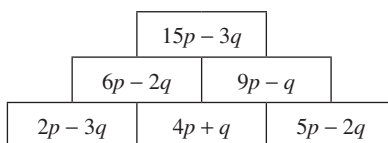
- | | | | | | | | |
|----|-------------------|-------------------|--------------------|-------------------|-------------------|--------------------|--|
| 1 | 21 | 2 | -60 | | | | |
| 3 | a 1 | b 34 | c -15 | d -13 | | | |
| 4 | a 69 | b 21 | c -21 | d -69 | | | |
| 5 | a 3 | b 11 | c -11 | d -3 | | | |
| 6 | a 7 | b -13 | c 1 | d 19 | e -19 | f -1 | |
| 7 | a $-1\frac{1}{4}$ | b $-1\frac{1}{6}$ | c $1\frac{11}{20}$ | d $\frac{11}{12}$ | | | |
| 8 | a 0.7 | b -3.3 | c 0.1 | d 14.2 | | | |
| 9 | a 5 | b 53 | c 45 | d 53 | e $-1\frac{5}{9}$ | f $-\frac{17}{50}$ | |
| 10 | a -72 | b 68 | c 14 | d -18 | e $-\frac{1}{2}$ | f $-\frac{7}{9}$ | |
| 11 | a 9 | b -14 | c -50 | d 34 | e $\frac{1}{12}$ | f $-\frac{1}{4}$ | |
| 12 | a $3\frac{2}{3}$ | b $-\frac{1}{3}$ | c $\frac{1}{3}$ | d $-3\frac{2}{3}$ | | | |
| 13 | a $2\frac{1}{2}$ | b $\frac{1}{2}$ | c $\frac{1}{12}$ | d $-1\frac{5}{6}$ | | | |

Exercise 1B

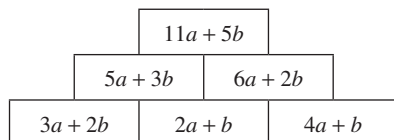
- 1 a, b, c, d, g, h, k, l, o, p, s, t
- | | | | | | | |
|---|---|--------------------------------------|--------------------------|----------------------|--------------------|--------------------|
| 2 | a 9a | b 11b | c 5c | d 7d | e 10x ² | f 3a ² |
| | g 3a | h 13f | i -7m | j p | k 2a ² | l 11m ² |
| | m 3ab | n 6mn | o -13xy | p -2a ² d | | |
| 3 | a 5a | b 3b | c 4mn | d 5pq | e 3x ² | |
| | f 5m ² | g 10ab | h 11m ² n | i 8a ² b | j 7lm | |
| 4 | a 5a + 11b | b 13c + 4d | c 12m + n | | | |
| | d 11p + 4 | e 14 + 5m | f m - 8n | | | |
| | g 13ab - b | h 6a ² + 7ab - 11b | i -5x ² + 12x | | | |
| | j 7a ² + 3a | k 11m ² - 5m | l p ² - 11p | | | |
| 5 | a 93xy ² | b 21xy | c 16xy ² | | | |
| | d -x ² - 10y | e 13v ² z - 25z | f 16yz + 4x | | | |
| | g 7x ³ - 4y ³ + 5x ² | h 13y ² - 4x ² | i 3x ² + 2xy | | | |
| | j 5a ² b - 8ab ² | | | | | |
| 6 | a $\frac{7x}{12}$ | b $\frac{7a}{10}$ | c $\frac{10b}{21}$ | d $\frac{8z}{15}$ | e $\frac{13c}{42}$ | |
| | f $\frac{13z}{40}$ | g $\frac{4z}{3}$ | h $\frac{3c}{10}$ | i $\frac{5x}{4}$ | | |
| 7 | a $\frac{z}{6}$ | b $\frac{2z}{15}$ | c $\frac{x}{56}$ | d $\frac{x}{10}$ | e $\frac{5x}{14}$ | |
| | f $\frac{4x}{21}$ | g $\frac{2x}{3}$ | h $\frac{x}{8}$ | i $\frac{6c}{7}$ | | |
| 8 | a $\frac{11x}{12}$ | b $\frac{22x}{21}$ | c $\frac{5x}{4}$ | d $\frac{13x}{6}$ | e $\frac{3x}{22}$ | |
| | f $\frac{x}{6}$ | g $-\frac{7x}{33}$ | h $\frac{43x}{12}$ | i $\frac{x}{4}$ | | |
- 9 b



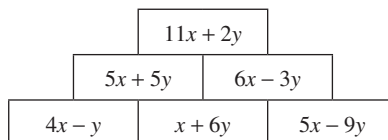
c



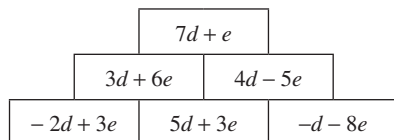
d



e



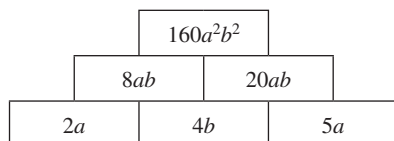
f



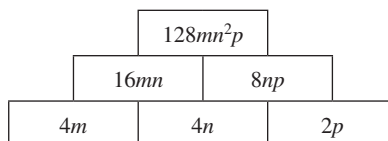
Exercise 1C

- | | | | | | |
|--------------------|-----------------|-------------------|------------------|--------------------|----------|
| 1 a $6a$ | b $20b$ | c $21c$ | d $-72d$ | e $12ab$ | |
| f $10cd$ | g $-20fg$ | h $-12mn$ | i $6pq$ | j $30\ell n$ | |
| k a^2 | l m^2 | m $8a^2$ | n $30b^2$ | o $-15a^2$ | |
| p $8m^2$ | q $56a^2b$ | r $6p^2q$ | s $-10mn^2$ | t $12cd^2e$ | |
| 2 a 2 | b 2 | c $2b$ | d $3n$ | e $4a$ | f $5p$ |
| g $-2b$ | h -8ℓ | i $3mn$ | j $7de$ | k $-5bc$ | l $-5df$ |
| 3 a $3x$ | b $9y$ | c $3a^2$ | d $2m$ | e $4t$ | f $5p$ |
| g $7t$ | h $9x$ | i $6y$ | j $-2x^2$ | k $4a^2$ | l $3x^2$ |
| 4 a $\frac{2x}{3}$ | b $\frac{a}{3}$ | c $-\frac{2m}{3}$ | d $\frac{2p}{3}$ | e $\frac{2x^2}{3}$ | |
| f $\frac{3xy}{4}$ | g $-3a$ | h $3b$ | i $\frac{1}{3}$ | j $-\frac{y}{2}$ | |

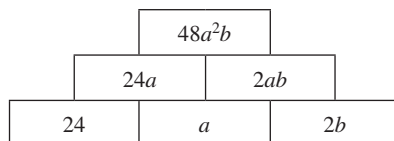
5 a



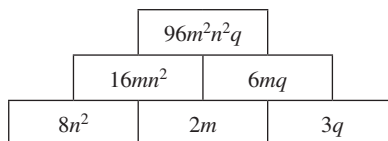
b



c



d



- | | | | | | |
|-----------------------|-------------------|--------------------|---------------------|------------------------------------|----------------------|
| 6 a $\frac{3b}{2}$ | b $\frac{2x}{5}$ | c $\frac{y^2}{10}$ | d $\frac{b^2}{6}$ | e $\frac{1}{10a^2}$ | |
| f $\frac{3p^2}{8q^2}$ | g $\frac{8}{y^2}$ | h $\frac{2mn}{3}$ | i $\frac{3y}{10z}$ | | |
| 7 a $\frac{3b}{2}$ | b $\frac{4x}{5}$ | c $\frac{8}{5}$ | d $\frac{2}{27}$ | e $3\frac{1}{3}$ or $\frac{10}{3}$ | f $\frac{3}{8}$ |
| g $\frac{2x}{5}$ | h $\frac{y}{4}$ | i $\frac{3}{10}$ | j $\frac{7}{15n^2}$ | k $-\frac{1}{4}$ | l $-\frac{12y}{x^2}$ |

Exercise 1D

- | | | | |
|---------------|---------------|---------------|---------------|
| 1 a $5x + 15$ | b $2b + 14$ | c $4a + 24$ | d $7d + 21$ |
| e $6a + 3$ | f $15d + 35$ | g $12b + 8$ | h $12d + 30$ |
| i $4a - 28$ | j $3b - 15$ | k $6d - 12$ | l $8f - 32$ |
| m $8f - 10$ | n $6g - 18$ | o $15p - 10$ | p $30q - 6$ |
| 2 a $-2a - 8$ | b $-3b - 18$ | c $-4m - 20$ | d $-5p - 35$ |
| e $-12a - 42$ | f $-6p - 3$ | g $-12b - 27$ | h $-20m - 25$ |
| i $-6a + 2$ | j $-24b + 42$ | k $-10b + 35$ | l $-21b + 14$ |
| m $-12b + 20$ | n $-20b + 35$ | o $-27x - 18$ | p $-48y + 60$ |

- 3 a $x+3$ b $3x+6$ c $4a+2$ d $5a+8$
e $8p+4$ f $15q+30$ g $4t-3$ h $3x-12$
i $-5\ell+3$ j $-4p+5$ k $-6m+10$ l $-20n+80$
- 4 a $4x+\frac{1}{2}$ b $\frac{5y}{2}+\frac{1}{5}$ c $\frac{x}{10}+\frac{1}{5}$
d $\frac{3x}{14}+\frac{5}{21}$ e $\frac{y}{6}-\frac{1}{4}$ f $-\frac{a}{5}+\frac{2}{5}$
- 5 a a^2+4a b b^2+7b c c^2-5c d d^2-9d
e $3e^2+e$ f $5f^2+6f$ g $6g^2-10g$ h $20h^2-28h$
i $10i^2+14i$ j $12j^2+21j$ k $-5k^2+4k$ l $-3\ell^2+\ell$
m $-10m^2+8m$ n $-15n^2-21n$ o $-12x^2+20x$
- 6 a $6a^2+3ab$ b $8c^2-4cd$ c $10d^2-20de$ d $6pq-10pr$
e $-6x^2-15xy$ f $-6z^2+8yz$ g $6a+8a^2b$ h $10m^2-20mn$
i $10x^2y+15x$ j $6p-15p^2q$ k $-12xy+18y^2$ l $-30ab+70b^2$
- 7 a Did not multiply the 6 by 4; $4(a+6)=4a+24$
b Added 1 and 5 instead of multiplying; $5(a+1)=5a+5$
c Added the 2nd term instead of subtracting; $8(p-7)=8p-56$
d $-3 \times -5 = 15$ not -15 ; $-3(p-5) = -3p+15$
e $a \times a = a^2$ not $2a$; $a(a+b) = a^2+ab$
f $2m \times 3m = 6m^2$ not $6m$; $2m(3m+5) = 6m^2+10m$
g $4a \times 3a = 12a^2$ not $7a^2$; $4a(3a+5) = 12a^2+20a$
h Did not multiply the -7 by $3a$; $3a(4a-7) = 12a^2-21a$
i $-6 \times x = -6x$ not $6x$; $-6(x-5) = -6x+30$
j $3x \times -7y = -21xy$ not $-21y$; $3x(2x-7y) = 6x^2-21xy$
- 8 a $8a+23$ b $5b+25$ c $2c+5$ d $3d+3$
e $2e+5$ f $3f-18$ g $13g+10$ h $7h+4$
i $3i-30$ j $j-6$ k $8a^2+13a$ l $10b^2-9b$
m $15a^2+6a$ n $2b^2-15b$ o 10 p $4+3q$
q $4ab$ r $28m^2+9mn$
- 9 a $\frac{5x+12}{6}$ b $\frac{7x+6}{12}$ c $\frac{19x-9}{15}$ d $\frac{19x-40}{12}$ e $\frac{34x+45}{21}$
f $\frac{24x-5}{10}$ g $\frac{14x+9}{12}$ h $\frac{-19x-10}{10}$ i $\frac{24-35x}{12}$
- 10 a $5y+14$ b $7x+9$ c $7a-12$ d $16b-9$
e $a+1$ f $b-22$ g -5 h x^2+x-6
i $2p^2-3p-5$ j $12y^2-29y+15$ k $4x^2-5x-7$ l $6p^2-6p-4$
m $10z$ n $4y^2-16y$ o $15z^2-10z$ p $6y^2+36y$
- 11 a i $12 \times 99 = 12 \times (100-1) = 12 \times 100 - 12 \times 1 = 1200 - 12 = 1188$
ii $14 \times 53 = 14 \times (50+3) = 14 \times 50 + 14 \times 3 = 700 + 42 = 742$
b i $14 \times 21 = 14 \times (20+1) = 14 \times 20 + 14 \times 1 = 280 + 14 = 294$
ii $17 \times 101 = 17 \times (100+1) = 17 \times 100 + 17 \times 1 = 1700 + 17 = 1717$
iii $70 \times 29 = 70 \times (30-1) = 70 \times 30 - 70 \times 1 = 2100 - 70 = 2030$
iv $8 \times 121 = 8 \times (100+21) = 8 \times 100 + 8 \times 21 = 800 + 168 = 968$
v $13 \times 72 = 13 \times (70+2) = 13 \times 70 + 13 \times 2 = 910 + 26 = 936$
vi $17 \times 201 = 17 \times (200+1) = 17 \times 200 + 17 \times 1 = 3400 + 17 = 3417$
- 12 a x^3+3x b x^3+2x^2+x c $2x^3-6x^2$
d $6x^2-2x^3$ e $15a^2+5a$ f $6a^2+12a^3-6a^4$

Exercise 1E

- 1 a $x^2 + 7x + 12$
d $x^2 + 10x + 21$
g $x^2 + 8x + 15$
- 2 a $x^2 - 6x + 5$
d $a^2 - 14a + 45$
g $x^2 - x - 42$
j $x^2 - 3x - 54$
- 3 a $6x^2 + 17x + 12$
d $6x^2 + 7x - 5$
g $2a^2 - 11a + 15$
j $6p^2 + 11p - 10$
m $15x^2 - 46x + 16$
p $7x^2 - 26x - 45$
- 4 a $2a^2 + 7ab + 3b^2$
d $6m^2 - 7mn - 3n^2$
g $8x^2 + 6xy - 5y^2$
j $6c^2 + 35cd - 6d^2$
m $3b^2 + 5ab - 2a^2$
- 5 a $x^2 + 7x + 10$
d $2x^2 + 7x + 3$
g $6x^2 - 19x + 15$
j $12x^2 - x - 35$
- 6 a $x + 1$
f $3x - 2$
- 7 a 12, 35
e 3, 13
- 8 a $x^2 - 9$
e $x^2 + 12x + 36$
i $49x^2 + 14x + 1$
- 9 a $\frac{a^2}{6} + \frac{7a}{6} + 2$
d $\frac{y^2}{12} + \frac{13y}{16} - \frac{9}{4}$
- b $x^2 + 9x + 20$
e $a^2 + 13a + 40$
h $x^2 + 3x + 2$
- b $x^2 - 5x + 6$
e $m^2 - m - 20$
h $x^2 + 8x - 33$
k $x^2 - 18x + 77$
- b $15x^2 + 22x + 8$
e $20x^2 - 11x - 3$
h $6ab - 2b + 15a - 5$
k $12x^2 + 13x - 14$
n $6x^2 - 23x + 7$
q $3a^2 - 11a - 20$
- b $2m^2 + 7mn + 3n^2$
e $6p^2 + pq - 5q^2$
h $3x^2 + 13ax - 10a^2$
k $4m^2 + 9mn - 9n^2$
n $15q^2 - 4pq - 4p^2$
- b $x^2 - 4x - 21$
e $3x^2 + 17x + 10$
h $15x^2 + 29x - 14$
k $4x^2 - 1$
- c $x^2 + 10x + 24$
f $a^2 + 12a + 27$
i $a^2 + 17a + 72$
- c $a^2 - 11a + 28$
f $p^2 - 2p - 24$
i $x^2 - 5x - 24$
l $x^2 + 3x - 28$
- c $5x^2 + 11x + 2$
f $6x^2 - 13x + 5$
i $8m^2 + 2m - 3$
l $8x^2 + 26x - 7$
o $6x^2 + 7x - 10$
r $8b^2 + 8b - 6$
- c $8c^2 - 10cd - 3d^2$
f $5\ell^2 - 7k\ell - 6k^2$
i $6x^2 + 13xy - 5y^2$
l $8p^2 + 2pq - 3q^2$
o $16pq - 4p^2 - 15q^2$
- c $x^2 - 10x + 24$
f $12x^2 - x - 1$
i $8x^2 + 2x - 3$
l $2x^2 - 3x - 20$
- d $x - 5$
e $2x + 1$
- i $2x - 7$
j $5x - 6$
- c 6, 24
g 2, 1, 5
- c $x^2 - 10x + 25$
g $49x^2 - 1$
k $4x^2 + 12x + 9$
- d $9x^2 - 1$
h $2x^2 + x - 15$
l $25a^2 - 1$
- c $\frac{2x^2}{25} - \frac{7x}{10} - 1$
- f $\frac{b^2}{4} - \frac{117b}{200} - \frac{1}{10}$

Exercise 1F

- 1 a $x^2 + 2x + 1$
d $x^2 + 12x + 36$
g $x^2 + 14x + 49$
j $x^2 + 4x + 4$
- 2 a $x^2 - 8x + 16$
d $x^2 - 2x + 1$
g $x^2 - 22x + 121$
- 3 a $9x^2 + 12x + 4$
d $9a^2 + 24ab + 16b^2$
g $9x^2 + 6ax + a^2$
- 4 a $9x^2 - 12x + 4$
d $4a^2 - 12ab + 9b^2$
g $9c^2 - 6bc + b^2$
- 5 a $\frac{x^2}{4} + 3x + 9$
- 6 a no
- 7 $(6 + 4)^2$ does not equal $6^2 + 4^2$; $(6 + 4)^2 = 6^2 + 2 \times 4 \times 6 + 4^2$.
- b $x^2 + 10x + 25$
e $x^2 + 20x + 100$
h $x^2 + 40x + 400$
k $x^2 + 18x + 81$
- b $x^2 - 14x + 49$
e $x^2 - 10x + 25$
h $x^2 - 2ax + a^2$
- b $4a^2 + 4ab + b^2$
e $4x^2 + 12xy + 9y^2$
h $25x^2 + 40xy + 16y^2$
- b $16x^2 - 24x + 9$
e $9a^2 - 24ab + 16b^2$
h $16x^2 - 40x + 25$
- c $x^2 + 6x + 9$
f $x^2 + 8x + 16$
i $a^2 + 16a + 64$
l $x^2 + 2ax + a^2$
- c $x^2 - 12x + 36$
f $x^2 - 40x + 400$
- c $4a^2 + 12ab + 9b^2$
f $4a^2 + 12a + 9$
- c $4a^2 - 4ab + b^2$
f $4x^2 - 12xy + 9y^2$
- c $\frac{4x^2}{25} - \frac{4x}{5} + 1$
- d $\frac{9x^2}{16} + x + \frac{4}{9}$
- c two 6 cm $\times$ 10 cm rectangles of paper

- 8 a The large square has side length $(a + b)$, so its area is $(a + b)^2$. The large square consists of four rectangles and a smaller square. Each rectangle has an area of ab and the smaller square has an area of $(a - b)^2$. So the area of the large square may also be written as $(a - b)^2 + 4ab$.
- b $(a - b)^2 + 4ab = a^2 - 2ab + b^2 + 4ab = a^2 + 2ab + b^2$ and $(a + b)^2 = a^2 + 2ab + b^2$
- 9 a 961 b 361 c 1764 d 324 e 2601
f 10201 g 9801 h 40401 i 90601 j 39601
- 10 a 1.0201 b 0.9801 c 16.0801 d 16.1604
e 0.998001 f 1.0404 g 9.0601 h 0.9604
- 11 a $2x^2 - 12x + 20$ b $2x^2 + 8$ c $5x^2 - 14x + 10$
d $8x^2 + 50$ e $2x^2 + 2x + 41$ f $5x^2 - 22x + 25$
- 12 a $4x^2 + 12x + 14$ b $4x^2 - 12x + 14$ c $4x - 6$
- 13 a $\frac{x^2}{2} + 2$ b $\frac{5x^2}{9} + \frac{10x}{3} + 10$ c $\frac{13x^2}{16} + 6x + 13$ d $\frac{x^2}{5} + \frac{5}{16}$

Exercise 1G

- 1 a $x^2 - 16$ b $x^2 - 49$ c $a^2 - 1$ d $a^2 - 81$ e $c^2 - 9$
f $d^2 - 4$ g $z^2 - 49$ h $100 - x^2$ i $x^2 - 25$
- 2 a $4x^2 - 1$ b $9x^2 - 4$ c $16a^2 - 25$
d $9x^2 - 25$ e $4x^2 - 49$ f $25a^2 - 4$
- 3 a $64 - 9a^2$ b $9a^2 - 4b^2$ c $81x^2 - y^2$
d $4r^2 - 9s^2$ e $4x^2 - 9y^2$ f $25a^2 - 4b^2$
- 4 a $\frac{4x^2}{9} - 1$ b $\frac{x^2}{4} - 9$ c $\frac{x^2}{9} - \frac{1}{4}$ d $\frac{4x^2}{25} - \frac{9}{16}$
- 5 a, c, d are difference of squares expansions; b, e, f are perfect square expansions.
- 6 a 399 b 899 c 396 d 896
e 391 f 3599 g 9999 h 9996
- 7 a 0.9999 b 24.9999 c 63.9996 d 0.9975 e 99.9999 f 399.99
- 8 a $a - b$ b $a + b$ c $(a - b)(a + b)$
e $(a - b)(a + b) = a^2 + ab - ab - b^2 = a^2 - b^2$

Exercise 1H

- 1 a $-4x - 12$ b $2x + 11$ c $32 - 8a$ d -24 e $2x^2 - 4$
f $7b^2 + 20$ g $a^2 + ab - 3b^2$ h $2y^2 - 4xy$ i $5m^2 + 21m + 1$
- 2 a 6 b -242 c $-\frac{3}{2}$ d $-\frac{35}{9}$
- 3 a $\frac{11}{8}$ b $\frac{8}{9}$ c 1 d $\frac{1}{2}$
- 4 a $\frac{7}{8}$ b $-\frac{3}{8}$ c $\frac{41}{12}$ d $\frac{77}{18}$
- 5 a $\frac{13}{192}$ b -15 c -8
- 6 a 4 b 4 c $\frac{196}{81}$ d 0 e 1
- 7 a $\frac{23a^2}{21}$ b $\frac{11a}{2b}$ c $\frac{13x}{3y^3}$ d $\frac{6m}{n^2}$
- 8 a $\frac{a^2b}{10}$ b $\frac{x^2}{3y}$ c $\frac{2m}{n^2}$ d $\frac{3p^3}{2n^2}$
- 9 a 5 b $9a^2$ c $4b^2$
d $a^2 + 2ab + b^2$ e $9m^2 - 12mn + 4n^2$
- 10 a $9a^2 + 9a$ b $4b^2 - 6b$
c $a^2 - 2ab + b^2 + 3a - 3b$ d $4m^2 + 4mn + n^2 + 6m + 3n$
- 11 a $9a^2 - 6a + 1$ b $4b^2 + 4b + 1$ c $4a^2$ d $16b^2 - 8b + 1$

- 12 a $2x^3 + 3x^2 + 4x$ b $3a^3 - 12a^2 + 3a$ c $m^3 + 5m^2 + 8m + 4$
d $2p^3 - p^2 - 12p - 9$ e $m^3 - 2m^2 - 21m + 12$ f $m^3 - 5m^2 + 2m - 10$
g $x^3 - 1$ h $x^3 + 1$
- 13 $3 + \sqrt{15}$
- 14 a $x^2 - y^2 + 2yz - z^2$ b $a^2 - b^2 + 2bc - c^2$ c $4x^2 - y^2 + 2yz - z^2$
d $x^2 + 2xz + z^2 + 2x + 2z + 1$ e $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$ f $x^2 + y^2 + z^2 + 2xy - 2xz - 2yz$
- 15 a $\frac{12}{5}$ b $\frac{50}{3}$ c $\frac{2}{5}$ d $\frac{2}{7}$ e $\frac{12}{17}$ f $\frac{2}{x+y}$
- 16 a i $\frac{a}{c} = 3, \frac{a-b}{b-c} = 3$ ii $\frac{a}{c} = \frac{3}{2}, \frac{a-b}{b-c} = \frac{3}{2}$ iii $\frac{a}{c} = 3, \frac{a-b}{b-c} = 3$
- 17 a $a + b - c$ b $b - a - 6c$ c a d $a - b - c$ e $3x + 4y$
- 18 a $6a^2 - 2bc$ b $2a^3$ c $\frac{1}{a^4} - \frac{2}{a^2b^2} + \frac{1}{b^4}$ d $a^4 - 2 + \frac{1}{a^4}$
e $3a^2 + 6$ f $ab - \frac{1}{ab}$ g 0

Review exercise

- 1 a 9 b 31 c -14 d -18
- 2 a 85 b 25 c -25 d -85
- 3 a 15 b -15 c 3 d 27 e -27 f -3
- 4 a 3 b 8 c 63 d $-\frac{8}{9}$ e $\frac{16}{9}$ f 3.41
- 5 a $9a$ b $11b$ c $10x$ d $3a$ e $6a$
f $13x$ g $3a$ h $8m$ i $-2a^2$
- 6 a $9a$ b $2b$ c $13mn$ d $6pq$ e $13ab$
f $11m^2n$ g $13a^2b$ h $11\ell m$ i $21xy^2$
- 7 a $3x + 6$ b $4b + 24$ c $15b - 10$ d $-12d - 30$
e $8x - 30$ f $12g - 18$ g $15p - 60$ h $80q - 32$
i $-18x - 48$ j $-6p - 6$ k $-15b + 25$ l $-20b + 35$
m $7g + 30$ n $11h + 5$ o $3x - 16$ p $y - 2$
q $10a^2 + 8a$ r $-10b^2 + 31b$ s $6z - 8y + 2$ t $z^2 + 2z - 6$
u $2y^2 - 3y - 5$ v $12y^2 - 40y + 25$
- 8 a $x^2 + 6x + 8$ b $x^2 + 11x + 30$ c $a^2 - 15a + 44$
d $a^2 - 10a + 25$ e $m^2 + m - 30$ f $p^2 - 3p - 28$
g $10x^2 + 14x + 4$ h $8x^2 - 14x + 5$ i $44x^2 - 62x - 6$
j $12x^2 - 32x + 5$ k $8x^2 + 2x - 3$ l $49a^2 + 14a - 3$
- 9 a $x^2 + 22x + 121$ b $x^2 + 12x + 36$ c $x^2 - 30x + 225$
d $x^2 - 20x + 100$ e $x^2 - 4xy + 4y^2$ f $4a^2 + 20ab + 25b^2$
g $25x^2 + 20x + 4$ h $25x^2 - 60x + 36$
- 10 a $x^2 - 36$ b $z^2 - 49$ c $p^2 - 1$ d $a^2 - 121$
e $25x^2 - 1$ f $49x^2 - 25$ g $36a^2 - 25$ h $100 - 9a^2$
i $25a^2 - 4b^2$ j $144x^2 - y^2$ k $16x^2 - 9y^2$ l $64x^2 - 9a^2$
- 11 a $6x^2 - xy - y^2$ b $9x^2 - 6ax - 8a^2$ c $15c^2 + 32bc + 16b^2$
d $9x^2 + 30xy + 25y^2$ e $a^2 - 4ab + 4b^2$ f $25\ell^2 + 20\ell m + 4m^2$
g $9x^2 - y^2$ h $25m^2 - 4n^2$ i $9x^2 + 30ax + 25a^2$
- 12 a $\frac{a^2}{6} - \frac{2a}{3} - 2$ b $\frac{2x^2}{3} + \frac{10x}{3} - 24$ c $\frac{a^2}{4} - 1$
d $\frac{6x^2}{25} + \frac{7x}{5} - 3$ e $\frac{5b^2}{6} - \frac{4b}{3} - 6$ f $\frac{a^2}{3} + 2a - 24$

Challenge exercise

- 1 a $2(x) + 2(x + 3) = 4x + 6$ b 14 cm c 7.5
d $(x^2 + 3x) \text{ cm}^2$ e 18 cm^2

2 a $(3x+9)$ cm b $\left(\frac{3}{2}x+6\right)$ cm c $\left(\frac{9x^2}{2}+\frac{63x}{2}+54\right)$ cm²

d $\left(\frac{9x^2}{2}+\frac{63x}{2}+54\right)-(2x+6)(x+4)=\frac{5}{2}x^2+\frac{35}{2}x+30$ e $\frac{5}{4}$

3 $ad+ae+af+bd+be+cd+ce+cf$

	a	b	c
d	da	db	dc
e	ea	eb	ec
f	fa	fb	fc

4 a i 88 ii 90 b product of inner pair – product of outer pair = 2

c $n+1, n+2, n+3$

d i n^2+3n

ii n^2+3n+2

iii $n^2+3n+2-(n^2+3n)=2$

5 a i 72

ii 80

b $n+2, n+4, n+6$

c i n^2+6n

ii n^2+6n+8

iii 8

7 a x^4+x^2+1

b x^3+1

c $x^{10}-1$

8 a x^3-1

b x^5-1

c $x^{10}-1$

9 13^2+11^2

10 a i 25

ii 121

iii 361

iv 841

b $n^4+2n^3-n^2-2n+1$

Chapter 2 answers

Exercise 2A

1 a $x=10$

b $a=13$

c $b=2$

d $a=58$

2 a $b=4$

b $x=8$

c $y=27$

d $y=45$

e $b=24$

f $x=14$

3 a 4.36

b 6.08

c 7.81

d 27.06

4 a $x=\sqrt{34}\approx 5.83$

b $y=\sqrt{80}\approx 8.94$

c $x=\sqrt{45}\approx 6.71$

d $d=\sqrt{55}\approx 7.42$

e $y=\sqrt{72}\approx 8.49$

f $y=\sqrt{56}\approx 7.48$

g $a=\sqrt{340}\approx 18.44$

h $b=\sqrt{51}\approx 7.14$

i $x=\sqrt{58}\approx 7.62$

j $y=2\sqrt{28}\approx 10.58$

5 a yes b yes

c no

d no

e no

f yes

6 a 5 cm

b $\sqrt{61}$ cm

c $\sqrt{65}$ cm

d $\sqrt{136}$ cm

e $\sqrt{51}$ cm

7 no

8 2.332 m

9 1.2 m

10 a 5.14 m

b 5.35 m

c 15.63 m

11 a 2.8 cm

b 4.2 cm

12 70.7 cm

13 25.3 m

14 3.8 m

15 b $A=\frac{1}{2}\times 24\times 10=120=\frac{1}{2}\times 26\times h=13h$ c $x=\frac{50}{13}$

Exercise 2B

1 a 7 b 3 c 11 d 231

2 a 12 b 48 c 50 d 45 e 28 f 147 g 242 h 44

3 a $\sqrt{15}$ b $\sqrt{12}$ c $\sqrt{30}$ d $\sqrt{33}$ e $\sqrt{42}$ f $\sqrt{30}$

g $\sqrt{14}$ h $\sqrt{26}$ i $\sqrt{51}$ j $\sqrt{45}$ k $\sqrt{42}$ l $\sqrt{190}$

4 a $\sqrt{3}$ b $\sqrt{2}$ c $\sqrt{5}$ d $\sqrt{11}$ e $\sqrt{7}$ f $\sqrt{13}$

g $\sqrt{2}$ h $\sqrt{23}$ i $\sqrt{7}$ j $\sqrt{\frac{20}{9}}=\frac{2\sqrt{5}}{3}$ k $\sqrt{\frac{1}{7}}$ l $\sqrt{\frac{2}{3}}$

- 5 a $6\sqrt{2}$ b $12\sqrt{5}$ c $33\sqrt{7}$ d $30\sqrt{7}$
e $45\sqrt{2}$ f $35\sqrt{6}$ g $12\sqrt{11}$ h $28\sqrt{3}$
- 6 a $2\sqrt{2}$ b $5\sqrt{5}$ c 3 d 4 e 13
f 16 g 6 h 6 i 8
- 7 a $2\sqrt{2}$ b $2\sqrt{3}$ c $3\sqrt{5}$ d $2\sqrt{6}$ e $3\sqrt{3}$ f $2\sqrt{11}$
g $5\sqrt{2}$ h $3\sqrt{6}$ i $2\sqrt{5}$ j $7\sqrt{2}$ k $3\sqrt{7}$ l $2\sqrt{15}$
m $3\sqrt{14}$ n $2\sqrt{17}$ o $5\sqrt{3}$ p $3\sqrt{11}$ q $2\sqrt{7}$ r $11\sqrt{2}$
- 8 a $6\sqrt{2}$ b $4\sqrt{2}$ c $4\sqrt{5}$ d $12\sqrt{2}$ e $4\sqrt{3}$ f $6\sqrt{5}$
g $4\sqrt{7}$ h $6\sqrt{6}$ i $4\sqrt{6}$ j $6\sqrt{7}$ k $4\sqrt{10}$ l $8\sqrt{2}$
m $8\sqrt{5}$ n $4\sqrt{11}$ o $8\sqrt{3}$ p $10\sqrt{2}$ q $9\sqrt{2}$ r $9\sqrt{3}$
- 9 a $\sqrt{12}$ b $\sqrt{112}$ c $\sqrt{275}$ d $\sqrt{108}$ e $\sqrt{98}$ f $\sqrt{54}$
g $\sqrt{80}$ h $\sqrt{175}$ i $\sqrt{48}$ j $\sqrt{405}$ k $\sqrt{192}$ l $\sqrt{52}$
m $\sqrt{396}$ n $\sqrt{600}$ o $\sqrt{180}$ p $\sqrt{1440}$ q $\sqrt{700}$ r $\sqrt{176}$
- 10 a $\frac{2}{3}$ b $\frac{2}{5}$ c $\frac{4}{5}$ d $\frac{5}{6}$ e $\frac{5}{11}$ f $\frac{5}{11}$ g $\frac{7}{11}$ h $\frac{12}{13}$
- 11 a $2\sqrt{2}$ b 3 c $2\sqrt{5}$ d $6\sqrt{2}$ e $2\sqrt{13}$ f $2\sqrt{10}$
g $3\sqrt{10}$ h $\sqrt{34}$ i $5\sqrt{3}$ j 3 k $2\sqrt{41}$ l $10\sqrt{21}$
- 12 a 6 b $2\sqrt{3}$ 13 a 12 b $2\sqrt{6}$
- 14 $3\sqrt{5}$ 15 $8\sqrt{2}$ 16 $\sqrt{3}$

Exercise 2C

- 1 a 0 b $10\sqrt{2}$ c $5\sqrt{3}$ d $7\sqrt{2}$ e $5\sqrt{7}$ f $43\sqrt{5}$
g $\sqrt{3}$ h $-2\sqrt{11}$ i $7\sqrt{2}$ j $10\sqrt{3}$ k $5\sqrt{5}$ l $8\sqrt{7}$
m $12\sqrt{2}$ n $7\sqrt{3}$ o $5\sqrt{2}$
- 2 a $6\sqrt{2}$ b $-15\sqrt{6}$ c $3\sqrt{5}$ d $9\sqrt{13}$ e $-2\sqrt{7}$ f $4\sqrt{19}$
g 0 h $13\sqrt{23}$ i $11\sqrt{3}$ j $-10\sqrt{2}$ k $\sqrt{6} + 7\sqrt{2}$ l $2\sqrt{5} + 3\sqrt{3}$
- 3 a $7 - 3\sqrt{3}$ b $3\sqrt{3} - \sqrt{2}$ c $-11\sqrt{6} + 20$ d $6\sqrt{14} + 7\sqrt{6}$ e $-\sqrt{7} + 2\sqrt{14}$ f $-3\sqrt{5} + 4\sqrt{2}$
- 4 a $3\sqrt{2}$ b $\sqrt{2}$ c $8\sqrt{2}$ d $5\sqrt{2}$ e $5\sqrt{3}$
f $6\sqrt{3}$ g $6\sqrt{5}$ h $5\sqrt{7}$ i $12\sqrt{2}$
- 5 a $\sqrt{2}$ b $6\sqrt{3}$ c $6\sqrt{2}$ d $\sqrt{3}$ e $9\sqrt{2}$ f $-7\sqrt{5}$
g $5\sqrt{6}$ h $4\sqrt{3}$ i $2\sqrt{5}$ j $11\sqrt{2}$ k $5\sqrt{10}$ l $28\sqrt{2}$
- 6 a $\frac{5\sqrt{2}}{6}$ b $\frac{9\sqrt{3}}{10}$ c $\frac{3\sqrt{7}}{10}$ d $\frac{11\sqrt{11}}{21}$ e $\frac{5\sqrt{2}}{6}$ f $\frac{7\sqrt{5}}{8}$
g $\frac{\sqrt{2}}{6}$ h $\frac{6\sqrt{6}}{5}$ i $\frac{\sqrt{3}}{10}$ j $\frac{6\sqrt{2}}{35}$
- 7 a 7 b 5 c 6
- 8 a $\sqrt{13}; 5 + \sqrt{13}$ b $2\sqrt{10}; 2\sqrt{10} + 8$ c $\sqrt{21}; 7 + \sqrt{21}$ d $\sqrt{15}; \sqrt{15} + 3\sqrt{3}$

9 a $3\sqrt{3}$ b $4\sqrt{2}$ c $\frac{2\sqrt{5}}{3}$ d $\frac{8\sqrt{2}}{3}$

10 a $\sqrt{7}$ b $3\sqrt{7}$ c $4\sqrt{7}$

11 $BA = 2\sqrt{7}$; perimeter $= 12\sqrt{7}$ 12 $28\sqrt{3}$

Exercise 2D

1 a $\sqrt{6}$ b $\sqrt{10}$ c $2\sqrt{21}$ d $4\sqrt{14}$ e $6\sqrt{6}$ f $8\sqrt{10}$ g $15\sqrt{14}$ h $24\sqrt{21}$

2 a 1 b 8 c 3 d $\sqrt{7}$ e $4\sqrt{3}$ f $4\sqrt{11}$

g $\sqrt{5}$ h 3 i $9\sqrt{5}$ j $4\sqrt{2}$ k $10\sqrt{5}$ l $\frac{7}{2}$

3 a $3\sqrt{2}$ b 6 c 14 d $20\sqrt{2}$ e $8\sqrt{3}$ f $12\sqrt{3}$ g $42\sqrt{5}$ h $42\sqrt{2}$

4 a $3\sqrt{2} + 3$ b $4 - 2\sqrt{3}$ c $2\sqrt{10} - 10$ d $18 - 3\sqrt{3}$

e $20\sqrt{10} - 15\sqrt{5}$ f $14 - 7\sqrt{2}$ g $40\sqrt{3} - 12\sqrt{15}$ h $18 - 10\sqrt{3}$

i $9\sqrt{2} + 12\sqrt{6}$ j $6\sqrt{10} + 6\sqrt{5}$ k $6\sqrt{7} - 21\sqrt{2}$ l $15\sqrt{3} + 9\sqrt{5}$

5 a $62 + 11\sqrt{5}$ b $68 + 24\sqrt{6}$ c $16 + 3\sqrt{2}$ d $-23 + \sqrt{5}$

e $-2 - 2\sqrt{3}$ f $107 - 11\sqrt{7}$ g $123 + 70\sqrt{2}$ h $52 - 16\sqrt{3}$

6 a $3\sqrt{10} + 2\sqrt{2} + 9\sqrt{5} + 6$ b $6\sqrt{6} - 9\sqrt{2} + 2\sqrt{3} - 3$ c $8\sqrt{7} - 20 + 4\sqrt{21} - 10\sqrt{3}$

d $16 - 8\sqrt{7} - 8\sqrt{6} + 4\sqrt{42}$ e 2 f $4\sqrt{6} + 2$ g $7\sqrt{35} + 18$ h $7\sqrt{10} - 2$

7 a $\sqrt{6} - \sqrt{3} + \sqrt{2} - 1$ b $\sqrt{2} + \sqrt{3}$ c 1 d $-\sqrt{3} - \sqrt{2}$

e $2\sqrt{6} + 2\sqrt{2} - 2\sqrt{3} - 2$ f $\sqrt{2} + 3\sqrt{3} + 2$ g $\sqrt{6} + 2\sqrt{2}$ h $6\sqrt{5}$

i $2\sqrt{2}$ j 2 k $2\sqrt{2}$

Exercise 2E

1 a $10\sqrt{3} + 28$ b $12\sqrt{2} + 38$ c $16\sqrt{5} + 36$

d $6\sqrt{3} + 28$ e $2\sqrt{35} + 12$ f $4\sqrt{6} + 14$

g $30\sqrt{6} + 77$ h $6\sqrt{10} + 47$ i $16\sqrt{15} + 68$

j $x + 2\sqrt{xy} + y$ k $a^2x + 2ab\sqrt{xy} + b^2y$ l $xy + 2\sqrt{xy} + 1$

2 a $11 - 4\sqrt{7}$ b $19 - 8\sqrt{3}$ c $21 - 4\sqrt{5}$ d $31 - 12\sqrt{3}$

e $8 - 2\sqrt{15}$ f $14 - 4\sqrt{6}$ g $82 - 8\sqrt{10}$ h $95 - 24\sqrt{14}$

i $64 - 12\sqrt{15}$ j $168 - 112\sqrt{2}$ k $x - 2\sqrt{xy} + y$ l $51 - 36\sqrt{2}$

3 a $80\sqrt{2} + 120$ b $60\sqrt{2} + 162$ c $6\sqrt{7} + 24$ d $20\sqrt{7} + 145$

e $42 - 8\sqrt{5}$ f $28 - 12\sqrt{5}$ g $160 - 60\sqrt{7}$ h $95 - 30\sqrt{10}$

i $99 - 44\sqrt{2}$ j $15 - 10\sqrt{2}$ k $18 - 12\sqrt{2}$ l $539 - 210\sqrt{6}$

4 a 4 b 5 c 1 d -9 e 89

f 1 g 2 h 10 i -43 j -13

k 44 l -138 m -470 n $x - y$

- 5 a $3 - 2\sqrt{2}$ b $3 + 2\sqrt{2}$ c 6 d $-4\sqrt{2}$
 e 1 f $\sqrt{2} - 1$ g $\sqrt{2} + 1$ h $2\sqrt{2}$
 6 a 16 b $4\sqrt{15}$ c 10 d $4\sqrt{6}$ e 18 f $8\sqrt{6}$
 7 $a^2b - c^2d$
 8 a i $\sqrt{14}$ ii $\frac{1}{2}$ iii $4 + \sqrt{14}$ b i 4 ii 2 iii $4 + 2\sqrt{6}$
 c i 6 ii 6 iii $2\sqrt{15} + 6$
 9 a i $16 + 4\sqrt{3}$ ii $19 + 8\sqrt{3}$ b i 16 ii 13
 10 a $3 + 3\sqrt{3}$ b $\sqrt{3} + \frac{3}{2}$

Exercise 2F

- 1 a $\frac{\sqrt{5}}{5}$ b $\frac{5\sqrt{6}}{6}$ c $\sqrt{3}$ d $2\sqrt{7}$ e $\frac{\sqrt{21}}{7}$ f $\sqrt{3}$ g $\frac{\sqrt{15}}{3}$ h $\sqrt{2}$ i $\sqrt{2}$ j $\frac{2\sqrt{21}}{3}$
 2 a $\frac{\sqrt{3}}{15}$ b $\frac{7\sqrt{2}}{6}$ c $\frac{2\sqrt{2}}{7}$ d $\frac{\sqrt{35}}{21}$ e $\frac{\sqrt{5}}{15}$ f $\frac{\sqrt{10}}{3}$ g $\frac{4\sqrt{14}}{35}$ h $4\sqrt{6}$ i $\frac{\sqrt{2}}{8}$ j $\frac{\sqrt{3}}{3}$
 3 a 0.71 b 0.58 c 2.12 d 2.89 e 3.46 f 4.24 g 0.29 h 0.24
 4 a $\frac{3\sqrt{2}}{2}$ b $\frac{7\sqrt{3}}{6}$ c $\frac{5\sqrt{2} - \sqrt{3}}{3}$ d $\sqrt{10}$ e $2\sqrt{6} + \sqrt{2}$ f $\frac{13\sqrt{3}}{6}$
 5 a $\frac{\sqrt{7}}{2}$ b $\frac{2\sqrt{7}}{7}$ c $\sqrt{7}$ d $\frac{\sqrt{42}}{6}$
 6 a $\frac{13\sqrt{3}}{6}$ b $\frac{11\sqrt{3}}{6}$ c $\frac{169}{12}$ d $\frac{121}{12}$
 7 a $\frac{5\sqrt{2}}{2}$ b $2\sqrt{5}$ c $\frac{30\sqrt{13}}{13}$

Exercise 2G

- 1 a i 12 cm ii 90° b 13 cm c 14.3 cm
 2 a 17 cm b $\sqrt{233} \approx 15.3$ cm c $\sqrt{165} \approx 12.8$ cm
 d $\sqrt{116} \approx 10.8$ cm e $\sqrt{62} \approx 7.9$ cm f $\sqrt{a^2 + b^2 + c^2}$ cm
 3 $\sqrt{241} \approx 15.5$ cm 4 2 cm 5 $\frac{\sqrt{1969}}{5} \approx 8.87$ m 6 $\sqrt{99} \approx 9.95$ cm
 7 a $2\sqrt{5}$ cm b $\sqrt{5}$ cm c i 3 cm ii 5 cm
 d right-angled triangle e $BE = \frac{3a}{2}$ cm, $EH = 2a = \frac{4a}{2}$ cm, $BH = \frac{5a}{2}$ cm

Exercise 2H

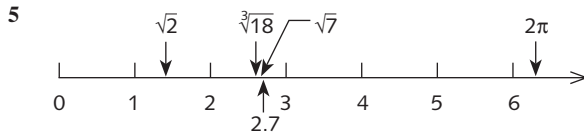
- 1 a 2 b 17 c 50 d 13
 2 a $\frac{\sqrt{6} - 1}{5}$ b $3\sqrt{2} + 3$ c $\frac{3 - \sqrt{5}}{2}$ d $\frac{\sqrt{5} + 3}{2}$ e $\frac{4\sqrt{5} - 4\sqrt{2}}{3}$
 f $\frac{\sqrt{7} + \sqrt{5}}{2}$ g $3\sqrt{2} - \sqrt{15}$ h $-2 - \sqrt{6}$ i $\frac{\sqrt{10} - 1}{9}$ j $\frac{\sqrt{15} + 10}{17}$
 k $\frac{4\sqrt{15} - 3\sqrt{10}}{30}$ l $\frac{-2 - 3\sqrt{2}}{14}$ m $\frac{-6\sqrt{42} - 16\sqrt{3}}{31}$ n $\frac{33\sqrt{10} - 18\sqrt{6}}{497}$ o $\frac{10 - \sqrt{6}}{47}$

- 3 a 2.41 b 0.27 c 0.32 d 3.15
 4 $p = 5, q = 2$ 5 a $5\sqrt{5} + 2$ b $\frac{7}{3}$
 6 a 8 b $-2\sqrt{15}$ c $\frac{9+2\sqrt{14}}{5}$

Exercise 21

- 1 a $\frac{5}{9}$ b $\frac{7}{9}$ c 1 d $\frac{13}{90}$ e $\frac{7}{30}$ f $\frac{28}{45}$
 2 a $\frac{13}{99}$ b $\frac{7}{99}$ c $\frac{91}{99}$ d $\frac{241}{999}$ e $\frac{613}{999}$ f $\frac{8}{495}$ g $\frac{107}{330}$ h $\frac{203}{396}$ i $\frac{1}{825}$
 3 a $\frac{11}{3}$ b $\frac{103}{20}$ c $\frac{4}{9}$ d $\frac{5}{33}$ e $\frac{36}{7}$ f $\frac{13}{10}$

- 4 a yes b no c no d yes



To 2 decimal places: $\sqrt{7} \approx 2.65$, $\sqrt[3]{18} \approx 2.62$, $\sqrt{2} \approx 1.41$, $2\pi \approx 6.28$

- 6 a no b yes c no d no e yes f no
 g yes h no i yes j yes k no l no
 8 a A triangle with sides 8, 3, $\sqrt{73}$ b A triangle with sides 2, 4, $\sqrt{12}$
 c A triangle with sides 2, $\sqrt{21}$, 5
 9 b $13 + \frac{1}{1 + \frac{1}{10}}$

Review exercise

- 1 a $h = 10$ b $h = 6.5$
 2 a $x = 7.5$ b $y = 1.75$ 3 28.3 cm 4 25 km
 5 a 8.49 cm b 4.24 cm c 21.73 cm d 15.36 cm
 6 a $5\sqrt{2}$ b $\sqrt{2}$ 7 a $30\sqrt{2}$ b $30\sqrt{2}$
 8 a $6\sqrt{3} + 6$ b $30 - 10\sqrt{2}$ 9 a $10 - \sqrt{2}$ b $32 - 9\sqrt{3}$
 10 a $4\sqrt{5}$ b $6\sqrt{3}$ c $5\sqrt{5}$ d $6\sqrt{2}$ e $32\sqrt{2}$ f $8\sqrt{7}$ g $20\sqrt{2}$ h $4\sqrt{7}$
 11 a $\sqrt{5}$ b $3\sqrt{7}$ c $3\sqrt{11}$ d $11\sqrt{2}$
 12 a $\frac{3\sqrt{11}}{11}$ b $\frac{\sqrt{15}}{75}$ c $\frac{4\sqrt{7}}{49}$ d $\frac{3\sqrt{17}}{17}$ e $\sqrt{3}$ f $\frac{\sqrt{15}}{25}$ g $2\sqrt{7}$ h $\frac{11\sqrt{3}}{3}$
 13 a $\frac{\sqrt{3}+3}{2}$ b $\frac{22(3\sqrt{5}-2)}{41}$ c $-4\sqrt{7} - 4$ d $\frac{3\sqrt{17}-3}{2}$
 e $3\sqrt{3} + 6$ f $-3\sqrt{11} - 3$ g $-5\sqrt{7} - 10$ h $20 + 10\sqrt{3}$
 14 a $23 - 4\sqrt{15}$ b $-\frac{4\sqrt{3}}{3} - 4$

- 15 a $\frac{\sqrt{89}}{2}$ m b $\frac{3\sqrt{21}}{2}$ m 16 $5\sqrt{3}$ cm
- 17 a $\sqrt{64+x^2}$ b $20-x$ c 8.4 km
- 18 a 6 cm b i $(12-r)$ cm ii $(6+r)$ cm c $r=4$
- 19 a equilateral b i $5\sqrt{3}$ cm ii $5\sqrt{2}$ cm
- 20 a $\frac{8}{9}$ b $\frac{7}{45}$ c $\frac{13}{18}$ d $\frac{1}{30}$ e $\frac{9}{11}$ f $\frac{32}{33}$ g $\frac{101}{999}$ h $\frac{1}{660}$
- 21 $2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{2}}}$ 22 $\frac{479281}{999999} = \frac{43571}{90909}$ 24 a 7 b 8
- 25 a $\sqrt{2}-1$ b $\sqrt{2}+\sqrt{5}+1$ c $12+4\sqrt{5}-2\sqrt{2}$ d $\sqrt{2}-\sqrt{5}-3$

Challenge exercise

- 2 $\theta < 90^\circ$
- 3 a i $m-p$ ii $m+p$
 b $a^2 = h^2 + (m-p)^2$, $b^2 = h^2 + (m+p)^2$, $d^2 = h^2 + p^2$
 c $a^2 + b^2 = 2h^2 + 2m^2 + 2p^2$
 e By Pythagoras' theorem, $4m^2 = a^2 + b^2$. Hence by Apollonius' theorem $m = d$.
- 4 $2\sqrt{74-20\sqrt{6}}$ cm
- 7 Suppose $\sqrt{6} = \frac{p}{q}$ where $\frac{p}{q}$ is a fraction in simplest form. Then $6 = \frac{p^2}{q^2}$, $p^2 = 6q^2$. So p^2 is even. Hence p is even. So $p = 2k$ where k is an integer. Therefore $(2k)^2 = 6q^2$, $4k^2 = 6q^2$. Therefore $3q^2$ is an even number. So q^2 is even and q is even. This is a contradiction.
- 8 a $17+12\sqrt{2}$ b $28-16\sqrt{3}$
- 11 a $\frac{3+2\sqrt{2}-\sqrt{5}-\sqrt{10}}{2}$ b $\frac{2\sqrt{3}+\sqrt{21}-3}{12}$
- 12 Raise to a suitable integer power 13 $2\sqrt{rR}$

Chapter 3 answers

Exercise 3A

- 1 a 0.72 b 0.076 c 0.98 d 0.16 e 0.08 f 0.0625
 g 1.75 h 0.006 i 0.7775 j 0.001 k 1.426 l 0.0025
- 2 a $\frac{7}{20}$ b $\frac{14}{25}$ c $\frac{3}{4}$ d $\frac{3}{8}$ e $\frac{1}{3}$ f $\frac{1}{6}$
 g $\frac{29}{400}$ h $\frac{8}{125}$ i $2\frac{1}{10}$ j $1\frac{1}{4}$ k $1\frac{1}{8}$ l $1\frac{9}{25}$
- 3 a 60% b $37\frac{1}{2}\%$ c $56\frac{1}{4}\%$ d 225% e 35%
 f $66\frac{2}{3}\%$ g $133\frac{1}{3}\%$ h 0.75% i 43% j $22\frac{1}{2}\%$
 k 4% l $1\frac{1}{2}\%$ m 120% n 203% o $117\frac{1}{2}\%$ p 0.75%

- 4 a $\frac{27}{50}$, 0.54 b 40%, 0.4 c $32\%, \frac{8}{25}$
 d $\frac{37}{200}$, 0.185 e $6\%, \frac{3}{50}$ f 87.5%, 0.875
 g 102%, $1\frac{1}{50}$ h $1\frac{43}{500}$, 1.086 i 140%, 1.4
- 5 a 6 b 570 c 68.64 d 645.6 e 153.58 f 900.25 g 8.90 h 6 i 1269
 6 a \$6.20 b \$227.52 c \$48 d \$1408 e \$2 f \$13 g \$77.50 h \$2.10 i \$116
 7 a 14% b 2% c 3.5% d 140% e 25% f 400%
 8 a 13.6% b 1.7% c 0.23% d 2.38% e 1.37% f 6.25% g 1600% h 0.27% i 36500%
 j 0.63% k 0.02% l 2.60%
 9 a 0.0048% b 0.0057% c 0.013% d 0.0025%
 10 37
 11 a 0.768 g b 0.00368 g
 12 50 minutes 36 seconds
 13 a 14560 b 27560 14 2950 megatonnes
 15 a i 14.4% ii 0.4% iii 0.42% b 0.8 g
 16 a 397.713% b 0.138%
 17 a 24.7% b 0.000664% 18 a 56% b 180%

Exercise 3B

- 1 a \$240 b \$12 2 a 1290.91 kg b 284 kg
 3 a 85 weeks 5 days b 6 weeks
 4 a \$600 b 12.5 kg c 2 h d 60 cm e 52 ha f \$98
 5 a \$1200 b \$1120 c \$1600 d \$1392
 6 a 434.782 kg b 2 m or 200 cm c 202.17 ha d 60 cm
 e \$721.65 f 4 hours 25 minutes
 7 a mortgage: 40%, groceries: 16%, car/transport: 22%, savings: 11% b \$127, 11%
 8 a 6% b 12% c 15% d 8.5%
 9 a \$32, \$368 b \$104, \$1196 c \$46, \$529 d \$11.40, \$131.10
 10 a \$20000 b \$44900 c \$256500 d \$24340
 11 a \$36200 b \$95000 c \$42560 d \$242480
 12 a 14% b 9% c 8.5% d 3.8%
 13 a \$500 b \$19000 c \$318980 d \$30865
 14 a \$560, \$1040 b \$980, \$1820 c \$163.80, \$304.20 d \$270.20, \$501.80
 15 a 12.5% profit b 7.4% loss c 10.4% profit d 7.3% loss
 16 a \$47000 b 9.02% c \$10000, \$240000
 17 a costs: \$750000 total: \$768000 b costs: \$14600000, total: \$13943000
 c costs: \$6600000, total: \$6349200
 18 a i \$0 ii \$300 iii \$4200 iv \$57700
 b i 0% ii 2.14% iii 11.67% iv 28.85%
 c i \$20400 ii \$32880 iii \$75000 iv \$77000

Exercise 3C

- 1 a \$840 b \$4200 2 a \$135 b \$405
 3 a \$72 b \$229.50 c \$4725 d \$15120
 4 a 5 years b 4 years 5 a 7.5% b 6.8%
 6 a \$5000 b \$14166.67
 7 a 4 years b 12 years c 4.5% d 4% e \$3000 f \$800

- 8 11 years 9 7% 10 \$384 000
11 \$3800 12 \$790 697.67

Exercise 3D

- | | | | |
|-----------------------|---------------------|---------------------|---------------------|
| 1 a 10 800 per day | b 86 400 per day | c 6804 per day | d 159 840 per day |
| 2 a \$207.60 | b \$1.74 | c \$358 110 | d \$9788.34 |
| 3 a 760 mm | b 190 mm | c 520 mm | d 110 mm |
| 4 a 36% increase | b 42.5% increase | c 16.3% decrease | d 62.5% increase |
| 5 a 18% decrease | b 4.3% increase | c 10.7% increase | d 26.2% increase |
| 6 a 600 | b 4300 | c 893 | d 35 893 |
| 7 a \$68 | b \$40.80 | c \$578 | d \$1.36 |
| 8 a \$2800 | b \$430 | c \$2.40 | d \$32 |
| 9 a 13 330 megalitres | b 33 330 megalitres | c 10 000 megalitres | d 25 000 megalitres |
| 10 a \$15 600 | b \$10 898.12 | c \$20 498.73 | d \$60 000 |
| e \$33 333.33 | f \$25 083.99 | g \$184 849.19 | h 35% |
| i -85.44% | j -83.81% | | |
| 11 a i \$5600 | ii \$392 000 | iii \$2 653 000 000 | iv \$47 771.50 |
| b i \$40 000 | ii \$857 142.86 | iii \$3 314 285 714 | iv \$660 |
| c i \$6000 | ii \$82 600 | iii \$1 902 630 710 | iv \$5 584 560 |
| 12 a i \$187 | ii \$5086.40 | iii \$75 812 | iv \$7.48 |
| b i \$500 | ii \$7110 | iii \$175 290 | iv \$4.80 |
| c i \$660 | ii \$7460.20 | iii \$594 000 | iv \$10.23 |
| 13 a 50% | b 19.36% | c 53.85% | d 72.41% |
| 14 a i 9.09% | ii 18.03% | iii 70.59% | iv 2.25% |
| b i 11.11% | ii 28.21% | iii 300% | iv 2.35% |
| 15 a 5% | b 11% | c 13% | d 17% |

Exercise 3E

- | | | | |
|---------------------|------------------|---|------------------|
| 1 \$3.62 | | | |
| 2 a \$1487.64 | b \$2496.26 | c \$42.34 | d \$761.98 |
| 3 a \$1605.78 | b \$555.60 | c \$4121.82 | d \$1265.79 |
| 4 612 kg | | | |
| 5 a 20 days | b 1248 days | c 6 days | d 214 days |
| 6 a \$270.68 | b \$32 481.41 | c \$6945.61 | d \$1 044 818.69 |
| 7 a 25% decrease | b 25% decrease | c same since $\left(\times \frac{3}{2} \times \frac{1}{2}\right)$ is the same as $\left(\times \frac{1}{2} \times \frac{3}{2}\right)$ | |
| 8 a 58% | b \$25 | | |
| 9 a 4% decrease | b 64% decrease | c 1% decrease | d 9% decrease |
| 10 a \$17.78 per kg | b \$15.46 per kg | c \$14.05 per kg | |

- 11 a 855 b 489 c 151 d 92
- 12 17369
- 13 a \$15400 b \$53900 c \$7100 d \$118200
- 14 a 22.3% b 9.0% c 11.9% d 9.3%
- 15 a 30.41% b 41.95% c 51.57% d 59.60%
- 16 a 99.144% b 99.921% c 99.993% d 99.999%

Exercise 3F

- 1 a i \$105000 ii \$110250 iii \$134009.56
iv \$34.01% v \$34009.56 b \$30000
- 2 a i \$216000 ii \$233280 ii \$317374.86
iv 58.69% v \$117374.86 b \$96000
- 3 \$1127566.42, 252% 4 24582, 17.06%
- 5 a \$17291580.82 b \$10000 6 a \$1983676.51 b \$630000
- 7 a \$16105.10 b \$6105.10 8 a 36.05% b \$73502.99
- 9 \$100511.06
- 10 a \$46319.35 b \$21454.82 c \$14601.79 d \$45.46
- 11 a 63743 b 60248 c 50873 d 38376
- 12 a \$120754.72 b \$113919.54 c \$95649.05 d \$71474.53
- 13 a \$48306.09 b \$42645.03 c \$35981.63 d \$27841.90
- 14 a \$12828.54, \$8828.54 b \$32614.36, \$22614.36
c \$3196265.32, \$1196265.32 d \$1247.22, \$2752.78
e \$3066.13, \$6933.87 f \$1251460.57, \$748539.43
- 15 \$3000, \$3180, \$3370.80, \$3573.05 16 \$24000, \$25920, \$27993.60, \$30233.09
- 17 a i 32.25% ii 33.1% iii 33.82% iv 34.01% v 34.39% vi 34.59%
b The investments have similar outcomes compared to the simple interest of 30% for one year.
- 18 76.98% 19 49.61% for both loans since multiplication is commutative
- 20 \$65518.99

Exercise 3G

- 1 a \$280000 b \$196000 c \$137200
d \$96040 e 75.99% f \$75990 p.a.
- 2 a \$200000 b \$160000 c \$65536
d 73.79% e \$30744 p.a.
- 3 a \$183500.80, 67.23% b \$792796.92
- 4 6537 5 better by \$20469.94 6 worse by \$64639.28
- 7 year 1: \$2076000 depreciation: \$1384000, year 2: \$1245600 depreciation: \$830400,
year 3: \$747360 depreciation: \$498240, year 4: \$448416 depreciation: \$298944
- 8 year 1: \$2550 depreciation: \$850, year 2: \$1912.50 depreciation: \$637.50, year 3: \$1434.38 depreciation: \$478.13
- 9 Lara: \$135089.81, Kate: \$32768
- 10 a taxi: 99.2%, car: 83.2% b taxi: \$7813, car: \$167924, difference \$160111
- 11 a \$5161 b \$6660 c \$51173 d 92.2% e \$4717 p.a.

- 12 a \$96 000 b \$128 000 c \$404 543 d 82.2% e \$55 424 p.a.
 13 a \$15 523 b \$33 348 c 83.2% d \$3964 per year
 e \$2481 per year f \$5942 per year
 14 a 12.16% b 164.5 mL
 15 0.0064%

Review exercise

- 1 a \$33.60 b 45% 2 a 10.7% b 64.3%
 3 a \$48 060, \$491 940 b \$95 000, \$86 545
 4 a \$157 500 b \$3 657 500 c 4.2%
 5 \$7000 6 a \$2800 b no, salary should be \$80 640
 7 a 6.25% b 375 8 \$2516 129; \$2 672 129
 9 a \$196.80 b \$60 10 a 14% increase b 489 961
 11 a 40.5% b \$4 500 000 12 a 43% b 564 mm
 13 a \$53 150 b \$56 498.45 c \$72 138.91 d 44.3%
 e \$22 138.91 f \$18 900
 14 a \$87 000 b \$19 200 p.a.

Challenge exercise

- 1 17.65 2 2.35% increase 3 16.64% increase 4 $\approx 3.9\%$
 5 \$40 per hour 6 78% 7 $\approx 58.9\%$ gain

Chapter 4 answers

Exercise 4A

- 1 a x b $2a$ c $4b$ d $3a$ e y f $2y$
 g $4a$ h $-3b$ i $4a$ j $2xy$ k $4m$ l $5ab$
 2 a $3b$ b ab c $2a$ d 3
 e $2a + 3$ f $3p - 2$ g $4mn - 3n$ h $4m - 3$
 i a j $b - 10$ k $2yz - 6y$ l $6z - 18$
 m $6 - 2z$ n $z - 3$
 3 a $6(x + 4)$ b $5(a + 3)$ c $c(a + 5)$ d $a(a + 1)$
 e $y(y + x)$ f $3(x + 9)$ g $4(x + 6)$ h $7(a - 9)$
 i $9(a + 4)$ j $y(y + 6)$ k $y(y - 3)$ l $6(2x + 3)$
 m $12(2y - 3)$ n $-7(2a + 3)$ o $-3(2y + 3)$ p $-4(1 + 3b)$
 4 a $4a(b + 4)$ b $4a(3a + 2)$ c $9mn(2m + n)$
 d $5ab^2(3a + 2)$ e $2a(2a + 3)$ f $4a(2a + 3b)$
 5 a $3b(1 - 2b)$ b $2x(2x - 3y)$ c $3mn(3 - 4m)$
 d $9y(2 - y)$ e $2a(2 - 3b^2)$ f $2x(3y - 2x)$
 g $7mn(2n - 3m)$ h $3pq(2q - 7p)$ i $5ab(2b - 5a)$
 6 a $5b(1 - 2b)$ b $-8ab(2a + 1)$ c $-xy(x + 3)$ d $4p(4p - q)$ e $5x(6 - xy)$
 f $2p(9p - 2q)$ g $-2ab(4ab + 1)$ h $3xy(4y - x)$ i $-5mn^2(5m + 2)$

- 7 a 4 b $2b + 3$ c $2b + 3$ d $2m + 3p$
 e $a + 2$ f $2a + b$ g $2y + x$ h $8p^2$
- 8 a $2ab(2a - 1 + 4b)$ b $4n(m^2 - m + 4n)$ c $7(ab + 2a^2 + 3b)$
 d $2(m^2 + 2mn + 3n)$ e $ab(5a + 3 + 4b)$ f $2a(3a + 4b + 5b^2)$
 g $5pq(pq + 2q + 3p)$ h $5(\ell^2 - 3\ell m - 4m^2)$

Exercise 4B

- 1 a $(x - 4)(x + 4)$ b $(x - 7)(x + 7)$ c $(x - 10)(x + 10)$
 d $(x - 12)(x + 12)$ e $(a - 11)(a + 11)$ f $(m - 8)(m + 8)$
 g $(b - 5)(b + 5)$ h $(d - 20)(d + 20)$ i $(2x - 5)(2x + 5)$
 j $(3x - 4)(3x + 4)$ k $(4x - 1)(4x + 1)$ l $(5m - 3)(5m + 3)$
 m $(3x - 2)(3x + 2)$ n $(4y - 7)(4y + 7)$ o $(10a - 7b)(10a + 7b)$
 p $(8m - 9p)(8m + 9p)$ q $(1 - 2a)(1 + 2a)$ r $(3 - 4y)(3 + 4y)$
 s $(5a - 10b)(5a + 10b) = 25(a - 2b)(a + 2b)$ t $(x - 3)(x + 3)$
- 2 a $3(x - 4)(x + 4)$ b $4(x - 5)(x + 5)$ c $5(x - 3)(x + 3)$
 d $6(x - 2)(x + 2)$ e $10(x - 10)(x + 10)$ f $7(x - 3)(x + 3)$
 g $2(2x - 5)(2x + 5)$ h $3(2m - 5)(2m + 5)$ i $3(1 - 2b)(1 + 2b)$
 j $5(2 - y)(2 + y)$ k $3(3a - 2b)(3a + 2b)$ l $4(2x - 5y)(2x + 5y)$
 m $5(3m - 5n)(3m + 5n)$ n $3(3a - 8\ell)(3a + 8\ell)$ o $8(2y - x)(2y + x)$
 p $8(2q - 5p)(2q + 5p)$
- 3 a $(x - 3)(x + 3)$ b $(x - 5)(x + 5)$ c $(3x - 2)(3x + 2)$
 d $(x - 6)(x + 6)$ e $(2 - 3x)(2 + 3x)$ f $(4 - 9x)(4 + 9x)$
 g $(3 - 10x)(3 + 10x)$ h $(4x - 7)(4x + 7)$ i $(11 - 5x)(11 + 5x)$
 j $2(5x - 3)(5x + 3)$ k $3(3 - 2x)(3 + 2x)$ l $5(5x - 4)(5x + 4)$
 m $4(10 - 3x)(10 + 3x)$ n $7(2x - 5)(2x + 5)$ o $10(3 - 2x)(3 + 2x)$
- 4 a 480 b 520 c 1260 d 8800 e 3.2
 f 3 g 52 h 8520 i 5.428
- 5 a i 7 ii 9 iii 11 iv 13
 v 15 vi 17 vii 19 viii 201
- b When considering the difference of the squares of consecutive whole numbers, take the sum of the two numbers.
 c $(n + 1)^2 - n^2 = (n + 1 - n)(n + 1 + n) = n + n + 1 = 2n + 1$
- 6 a $a^2 - b^2$ b $a - b$ c $a - b$
 d i $(a - b)(a + b)$ ii $a^2 - b^2 = (a - b)(a + b)$

Exercise 4C

- 1 a $(x + 3)(x + 2)$ b $(x + 9)(x + 2)$ c $(x + 5)(x + 2)$ d $(x + 6)(x + 5)$
 e $(x + 7)(x + 2)$ f $(x + 10)(x + 9)$ g $(x + 4)(x + 5)$ h $(x + 11)(x + 2)$
 i $(x + 3)(x + 4)$ j $(x + 8)(x + 4)$ k $(x + 8)(x + 5)$ l $(x + 15)(x + 5)$
 m $(x + 7)(x + 5)$ n $(x + 27)(x + 1)$ o $(x + 7)(x + 8)$ p $(x + 14)(x + 4)$
- 2 a $(x - 3)(x - 2)$ b $(x - 11)(x - 3)$ c $(x - 15)(x - 2)$ d $(x - 6)(x - 7)$
 e $(x - 7)(x - 2)$ f $(x - 45)(x - 2)$ g $(x - 11)(x - 4)$ h $(x - 20)(x - 5)$
 i $(x - 8)(x - 10)$ j $(x - 15)(x - 6)$ k $(x - 10)(x - 4)$ l $(x - 8)(x - 3)$
 m $(x - 6)(x - 7)$ n $(x - 3)(x - 5)$ o $(x - 28)(x - 2)$ p $(x - 12)(x - 2)$
- 3 a $(x + 3)(x - 2)$ b $(x - 11)(x + 3)$ c $(x + 6)(x - 5)$ d $(x - 21)(x + 2)$
 e $(x - 7)(x + 2)$ f $(x - 15)(x + 6)$ g $(x - 11)(x + 4)$ h $(x + 20)(x - 5)$
 i $(x + 10)(x - 8)$ j $(x - 12)(x + 5)$ k $(x + 8)(x - 5)$ l $(x - 12)(x + 2)$
 m $(x + 7)(x - 3)$ n $(x + 5)(x - 3)$ o $(x - 8)(x + 7)$ p $(x + 8)(x - 3)$
- 4 a $(x - 2)(x - 1)$ b $(x + 6)(x + 2)$ c $(x - 5)(x + 2)$ d $(x + 6)(x + 5)$
 e $(x - 7)(x + 2)$ f $(x - 15)(x + 6)$ g $(x - 1)(x - 4)$ h $(x - 9)(x + 2)$
 i $(x - 4)(x + 3)$ j $(x - 7)(x - 4)$ k $(x + 5)(x - 2)$ l $(x + 10)(x - 9)$

- 5 a $(x+3)^2$ b $(x+7)^2$ c $(x+10)^2$ d $(x-5)^2$
 e $(x-9)^2$ f $(x+8)^2$ g $(x+5)^2$ h $(x+6)^2$
 i $(x+15)^2$ j $(x-8)^2$ k $(x-10)^2$ l $(x-4)^2$

Exercise 4D

- 1 a $2(x+3)(x+4)$ b $3(x+2)(x+6)$ c $3(x-1)(x-8)$
 d $4(x-3)^2$ e $7(x+1)^2$ f $5(x+2)(x-3)$
 g $4(x-4)(x+3)$ h $2(x-3)(x-6)$ i $5(x+1)(x+7)$
 j $3(x-5)(x+8)$ k $3(x+5)(x-6)$ l $5(x+6)^2$
 m $2(x+6)(x-8)$ n $5(x+4)(x+9)$ o $3(x+12)(x-2)$
 p $2(x+10)^2$ q $3(x-3)^2$ r $5(x-2)^2$
 s $5(x-4)^2$ t $2(x+6)^2$ u $3(x-2)(x-6)$
- 2 a $4(x+2)(x-2)$ b $2(x+3)(x-3)$ c $3(x+4)(x-4)$
 d $3(a+3)(a-3)$ e $6(x+10)(x-10)$ f $3(a+3b)(a-3b)$
 g $5(3x-2)(3x+2)$ h $2(3t-5s)(3t+5s)$ i $5(a-2)(a+2)$
 j $3(3x+y)(3x-y)$ k $5(3+b)(3-b)$ l $3(2+m)(2-m)$
 m $2(8+x)(8-x)$ n $\frac{1}{2}(a+2b)(a-2b)$ o $\frac{1}{3}(9x+y)(9x-y)$
 p $\frac{1}{2}(a+8)(a-8)$ q $7(x+5)(x-5)$ r $\frac{1}{3}(y+6)(y-6)$
 s $\frac{1}{4}(a+6)(a-6)$ t $\frac{1}{5}(x+10)(x-10)$ u $\frac{1}{4}(x+2y)(x-2y)$
- 3 a $-(x+6)(x+2)$ b $-(x-1)(x+12)$ c $-(x-1)(x+7)$
 d $-(x-9)(x+1)$ e $-(x+2)^2$ f $-(x+9)(x+5)$
 g $-(x+1)(x-48)$ h $-(x+3)^2$ i $-2(x-6)^2$
 j $-(x+5)(x-8)$ k $-(x-7)(x+6)$ l $-(x-4)(x-9)$
 m $-(x-3)(x+8)$ n $-(x-2)(x-20)$ o $-(x-7)(x+4)$
 p $-(x-3)(x-8)$ q $-3(x-2)(x+12)$ r $-(x+7)(x+8)$
 s $-(x+7)(x+9)$ t $-(x-7)(x-5)$ u $-(x+2)(x-9)$

Exercise 4E

- 1 a $\frac{1}{x+3}$ b $\frac{x}{x+2}$ c $\frac{x}{3}$ d $\frac{3}{x-1}$ e $\frac{x}{x+3}$ f $\frac{x}{2}$
- 2 a $\frac{x-1}{x-2}$ b $\frac{x-2}{x-5}$ c $\frac{x+1}{x+2}$ d $\frac{x+5}{x-1}$ e $\frac{x-2}{x-1}$
 f $\frac{x+4}{x}$ g $\frac{x-3}{x-8}$ h $\frac{x+1}{x}$ i $\frac{x+2}{x+3}$ j $\frac{x-5}{x-3}$
 k $\frac{x+3}{x-2}$ l $\frac{2(x-3)}{3(x-2)}$ m $\frac{x-3}{x-2}$ n $\frac{x+3}{x+4}$ o $\frac{3(x-3)}{x+4}$
- 3 a -1 b $-k$ c $\frac{6x}{y-x}$ d $3k+2\ell$
 e $\frac{2p+3q}{p}$ f $\frac{1}{pq+1}$ g $\frac{m}{n}$ h $-(s+t)$
 i $-\frac{3}{x}$ j $3(p-q)$ k $\frac{x(x-y)}{2}$ l $-m^2$
- 4 a $\frac{x^2}{2}$ b $\frac{5(x+1)}{x+3}$ c $\frac{4}{3x}$ d $\frac{x+2}{x(x+1)}$
 e $\frac{1}{x(x-1)}$ f 2 g $\frac{3(2x-1)}{x+5}$ h $\frac{5(x+1)}{x+3}$

$$\begin{array}{lllll}
5 \quad \mathbf{a} \quad \frac{4}{3x} & \mathbf{b} \quad \frac{x+1}{2x(x-1)} & \mathbf{c} \quad \frac{3x}{2(x-1)} & \mathbf{d} \quad \frac{x}{x-1} & \mathbf{e} \quad \frac{x-3}{2x-1} \\
\mathbf{f} \quad \frac{6x^2(2x-1)}{(x-3)(x+3)^2} & \mathbf{g} \quad -3x^2 & \mathbf{h} \quad \frac{x+2}{2x} & \mathbf{i} \quad \frac{x-7}{x-3} & \mathbf{j} \quad \frac{3mn}{2(m-n)} \\
6 \quad \mathbf{a} \quad \frac{x+1}{x+2} & \mathbf{b} \quad 1 & \mathbf{c} \quad 1 & \mathbf{d} \quad 5\ell & \mathbf{e} \quad \frac{q}{p^2} & \mathbf{f} \quad \frac{3a^2}{2(a-1)(a+3)^2}
\end{array}$$

Review exercise

$$\begin{array}{llll}
1 \quad \mathbf{a} \quad 4(x+4) & \mathbf{b} \quad 7(x-3) & \mathbf{c} \quad 3(2a-3) & \mathbf{d} \quad 5(2m+5) \\
\mathbf{e} \quad a(4b+7) & \mathbf{f} \quad p(6q-11) & \mathbf{g} \quad 5n(m-2) & \mathbf{h} \quad 4v(u-2) \\
\mathbf{i} \quad a(a+9) & \mathbf{j} \quad b(b-6) & \mathbf{k} \quad 3ab(a+2) & \mathbf{l} \quad 4mn(m-3) \\
\mathbf{m} \quad ab(a-4b) & \mathbf{n} \quad 3p(q-2p) & \mathbf{o} \quad 6pq(p^2-3) & \\
2 \quad \mathbf{a} \quad (x-3)(x+3) & \mathbf{b} \quad (x-4)(x+4) & \mathbf{c} \quad (y-12)(y+12) & \\
\mathbf{d} \quad (a-10)(a+10) & \mathbf{e} \quad (3a-5)(3a+5) & \mathbf{f} \quad (4m-1)(4m+1) & \\
\mathbf{g} \quad (3-2b)(3+2b) & \mathbf{h} \quad (10-9b)(10+9b) & \mathbf{i} \quad (4x-y)(4x+y) & \\
\mathbf{j} \quad (3a-b)(3a+b) & \mathbf{k} \quad 2(m-5)(m+5) & \mathbf{l} \quad 3(a-3)(a+3) & \\
\mathbf{m} \quad (1-6b)(1+6b) & \mathbf{n} \quad \frac{1}{4}(4y-1)(4y+1) & \mathbf{o} \quad (pq-1)(pq+1) & \\
3 \quad \mathbf{a} \quad (x+2)(x+6) & \mathbf{b} \quad (x+3)(x+6) & \mathbf{c} \quad (x+5)(x+6) & \\
\mathbf{d} \quad (x+4)(x+7) & \mathbf{e} \quad (x-8)(x-3) & \mathbf{f} \quad (x-6)(x-4) & \\
\mathbf{g} \quad (x-2)(x-12) & \mathbf{h} \quad (x-1)(x-24) & \mathbf{i} \quad (x+5)(x-4) & \\
\mathbf{j} \quad (x-8)(x+6) & \mathbf{k} \quad (x-6)(x+2) & \mathbf{l} \quad (x+8)(x-5) & \\
\mathbf{m} \quad (x-8)(x+1) & \mathbf{n} \quad (x-12)(x+11) & \mathbf{o} \quad (x+3)(x+4) & \\
\mathbf{p} \quad (x+7)(x+5) & \mathbf{q} \quad (a+4)(a+15) & \mathbf{r} \quad (x-2)(x-48) & \\
4 \quad \mathbf{a} \quad (a-11)^2 & \mathbf{b} \quad (m-7)^2 & \mathbf{c} \quad (s+4)^2 & \\
\mathbf{d} \quad (a+12)^2 & \mathbf{e} \quad (a-6)^2 & \mathbf{f} \quad (z-20)^2 & \\
5 \quad \mathbf{a} \quad 2(x+4)(x+5) & \mathbf{b} \quad 3(x-3)(x-7) & \mathbf{c} \quad 5(x-4)(x-6) & \\
\mathbf{d} \quad 2(x+9)(x-5) & \mathbf{e} \quad 3(x-7)(x+5) & \mathbf{f} \quad 2(x-13)(x+10) & \\
6 \quad \mathbf{a} \quad \frac{y}{y-3} & \mathbf{b} \quad \frac{a-2}{a+3} & \mathbf{c} \quad \frac{5p-8}{9p-2} & \mathbf{d} \quad \frac{m-1}{m+3} \\
\mathbf{e} \quad \frac{x(x-4)}{2} & \mathbf{f} \quad \frac{y(y+1)}{(y+9)(y-7)} & \mathbf{g} \quad \frac{5(x+2)^2}{x(x-2)} & \mathbf{h} \quad \frac{-50x^2}{x+3} \\
7 \quad \mathbf{a} \quad (5a-4b)(5a+4b) & \mathbf{b} \quad (a+7)^2 & \mathbf{c} \quad (a-5)(a+4) & \\
\mathbf{d} \quad (1-6m)(1+6m) & \mathbf{e} \quad (2-3xy)(2+3xy) & \mathbf{f} \quad \left(\frac{1}{3}-\frac{a}{5}\right)\left(\frac{1}{3}+\frac{a}{5}\right) & \\
\mathbf{g} \quad \left(m-\frac{1}{2}\right)\left(m+\frac{1}{2}\right) & \mathbf{h} \quad x(x-7y)(x+7y) & \mathbf{i} \quad 3(a-5)(a+5) & \\
\mathbf{j} \quad (b-8)(b-12) & \mathbf{k} \quad (n-25)(n-6) & \mathbf{l} \quad (m+7)(m+13) & \\
\mathbf{m} \quad (a-14b)(a+7b) & \mathbf{n} \quad (m-15)(m+11) & \mathbf{o} \quad (x+4y)(x-y) & \\
\mathbf{p} \quad 5n(2m-n)(2m+n) & \mathbf{q} \quad (a-9)(a+7) & \mathbf{r} \quad 5(q-3p)(q+2p) & \\
\mathbf{s} \quad (x-13)(x+10) & \mathbf{t} \quad -(x+7)(x-6) & \mathbf{u} \quad (x+9)(x-2) & \\
8 \quad \mathbf{a} \quad \frac{x+3}{3} & \mathbf{b} \quad \frac{x+1}{x} & \mathbf{c} \quad \frac{x-2}{x-1} & \mathbf{d} \quad \frac{x+5}{x-1} & \mathbf{e} \quad \frac{2p+3q}{p} \\
\mathbf{f} \quad \frac{2(x-y)}{3x} & \mathbf{g} \quad -\frac{3x}{2} & \mathbf{h} \quad \frac{x+2}{x+3} & \mathbf{i} \quad \frac{x}{x-3} & \\
9 \quad \mathbf{a} \quad \frac{1}{x-5} & \mathbf{b} \quad \frac{x-5}{y} & \mathbf{c} \quad -\frac{y}{x} & \mathbf{d} \quad \frac{x+1}{x^2} & \mathbf{e} \quad \frac{(x-2)(x-1)}{(x+2)(x+1)} \\
\mathbf{f} \quad \frac{3(x-4)(x-5)}{2(x-1)(x-3)} & \mathbf{g} \quad \frac{4(x-1)^2}{x+2} & \mathbf{h} \quad \frac{x-2}{5(x+3)} & \mathbf{i} \quad \frac{2(x+5)}{3(x-5)} &
\end{array}$$

Challenge exercise

- 1 a $(x^2 + 1)(x - 1)(x + 1)$ b $(x^2 + 4)(x - 2)(x + 2)$ c $(x - \sqrt{3})(x + \sqrt{3})$
d $(x - \sqrt{5})(x + \sqrt{5})$ e $(x - \sqrt{2})^2$ f $(x + \sqrt{2})^2$
g $\left(x + \frac{1}{2}\right)^2$ h $\left(x + \frac{3}{2}\right)^2$ i $\left(x - \frac{1}{2}\right)^2$
j $\left(x - \frac{3}{2}\right)^2$ k $(x - 2y)(x + y)$ l $(x + 2y)(x - y)$
- 2 a x b $\frac{x-2}{x+4}$ c $\frac{x+1}{x-1}$
- 3 a $(x-2)^2$ b $(x+3)^2$ c $(x-a)^2$ d $(x+a)^2$ 4 $\frac{ad+bc}{ad-bc}$
- 5 a $(x-1)(x+1)(x^2+3)$ b $x^2(x-2)(x+2)$
c $(x+2-y)(x+2+y)$ d $(x+4-a)(x+4+a)$
e $(m-1-n)(m-1+n)$ f $\left(p - \frac{5}{2} - q\right)\left(p - \frac{5}{2} + q\right)$
- 6 a $(x^2 + 2 - 2x)(x^2 + 2 + 2x)$ b $(x^2 + 2a^2 - 2ax)(x^2 + 2a^2 + 2ax)$
- 7 a i $(x+9)m$ ii $(x+7)m$
b $(10x+50)m$ c $80m$ d $(4x^2 + 48x + 144)m^2$
e i side length $(2x+12)m$ ii $(8x+48)m$
- 9 $\sqrt{a+b}$ 10 $\frac{x(x+y+z)}{z(x-y+z)}$

Chapter 5 answers

Exercise 5A

- 1 $(h+5)\text{ cm}$ 2 $(w+5)\text{ kg}$ 3 $2x-3$ 4 a $(w+5)m$ b $(\ell-5)m$
5 a $a+18$ b $2a-5$ 6 $(3x-20)m$ 7 a $5x\text{ km/h}$ b $(20x+3)\text{ km/h}$
8 a xii b x c vii d vi e ii f i
g iii h v i xi j ix k viii l iv
9 a $(x+6)\text{ cm}$ 2 $(x-4)\text{ cm}$

Exercise 5B

- 1 a $a=3$ b $b=12$ c $c=17$ d $d=18$ e $a=3$ f $b=3$
g $c=7$ h $d=7$ i $m=-2$ j $n=-4$ k $p=-3$ l $q=-3$
m $b=22\frac{1}{2}$ n $a=12\frac{1}{4}$ o $a=\frac{2}{9}$ p $x=11\frac{1}{3}$ q $y=-\frac{2}{9}$ r $x=-\frac{5}{12}$
- 2 a $a=1$ b $b=5$ c $c=7$ d $d=6$ e $e=6$ f $f=4$
g $g=-4$ h $h=-2$ i $a=-3$ j $a=-8$ k $a=-5$ l $b=-7$
m $b=1\frac{1}{4}$ n $x=-1\frac{7}{8}$ o $x=9\frac{1}{3}$ p $a=-4\frac{1}{6}$ q $m=-1\frac{1}{5}$ r $b=-2\frac{5}{8}$
- 3 a $a=-2$ b $b=-3$ c $c=9$ d $d=5$ e $e=-3$ f $f=-6$
4 a -2 b 1 c 4 d 5 e 1 f -3
g 2 h 3 i $\frac{10}{3}$ j 3 k $\frac{5}{4}$ l $\frac{2}{5}$

Exercise 5C

- 1 a 1 b 7 c 4 d 1 e 9 f 3
g 1 h 2 i -1 j 1 k 3 l -7

- 2 a $4\frac{1}{2}$ b $5\frac{1}{5}$ c $\frac{1}{7}$ d $5\frac{1}{4}$ e $1\frac{1}{2}$
- f $\frac{1}{6}$ g $3\frac{2}{15}$ h $3\frac{1}{10}$ i $-\frac{5}{12}$
- 3 a $a=2$ b $b=4$ c $c=3$ d $d=1$ e $x=-2$ f $y=3$
- g $a=1$ h $a=12$ i $a=-9$ j $a=\frac{9}{4}$ k $x=30$ l $x=-4$
- m $x=2$ n $x=4\frac{1}{8}$

Exercise 5D

- 1 a $x=\frac{3}{8}$ b $x=\frac{1}{18}$ c $y=\frac{5}{12}$ d $y=\frac{17}{18}$ e $x=-\frac{1}{60}$
- f $y=-\frac{23}{100}$ g $x=\frac{1}{15}$ h $x=\frac{13}{45}$ i $y=-\frac{13}{60}$
- 2 a $a=-6$ b $a=\frac{88}{45}$ c $b=24$ d $b=-\frac{28}{3}$ e $x=-6$
- f $x=-4$ g $m=\frac{9}{2}$ h $m=-\frac{20}{3}$ i $x=-14$
- 3 a $y=4$ b $x=11$ c $p=6$ d $y=5$ e $x=9$
- f $a=3$ g $x=8$ h $x=4$ i $y=3$
- 4 a $e=1.1$ b $f=3.9$ c $g=6.6$ d $h=10.3$
- e $u=13$ f $r=2.5$ g $x=-10.5$ h $y=-8.4$
- i $x=5$ j $x=4$ k $x=6$ l $x=12$
- 5 a 16 b -1 c -90 d -72
- e $\frac{60}{7}$ f $\frac{150}{11}$ g $\frac{2}{5}$ h 1
- 6 a $\frac{20}{7}$ b 12 c 2.4 d -7 e 1.4 f 4
- 7 a $-\frac{13}{3}$ b -10 c 1 d $-\frac{17}{3}$ e 10
- f $\frac{21}{8}$ g -8 h $\frac{37}{5}$ i 1 j 1
- 8 a $a=3$ b $b=13$ c $c=18$ d $d=-6$ e $e=\frac{9}{8}$ f $f=\frac{15}{7}$
- g $g=\frac{11}{5}$ h $h=1$ i $i=-\frac{32}{3}$ j $j=8$ k $k=1$ l $\ell=\frac{6}{7}$
- m $m=-\frac{11}{2}$ n $n=-4$ o $q=\frac{1}{6}$ p $r=31$

Exercise 5E

- 1 $x=15$ 2 14 3 56 kg 4 \$2.40 5 $p=4$ 6 $q=-24$
- 7 a i $x+20$ ii $x+12$ iii $x+32$
- b Alana is 8 years old and Derek is 28 years old.
- 8 a i $x+5$ ii $2x$ iii $4x+5$
- b Alan has 8 toys, Brendan has 13 toys and Calum has 16 toys.
- 9 a $(4x+2)$ m b length 50 m, width 12 m
- 10 a i $$(x+3600)$ ii $$(x-2000)$
- b Ms Minas earns \$53 600, Mr Brown earns \$50 000 and Ms Lee earns \$48 000.
- 11 a 20 of each b 15 20-cent coins and 30 10-cent coins
- c 24 20-cent coins and 12 10-cent coins
- 12 a $4\frac{3}{8}$ b 100 c $\frac{17}{6}$

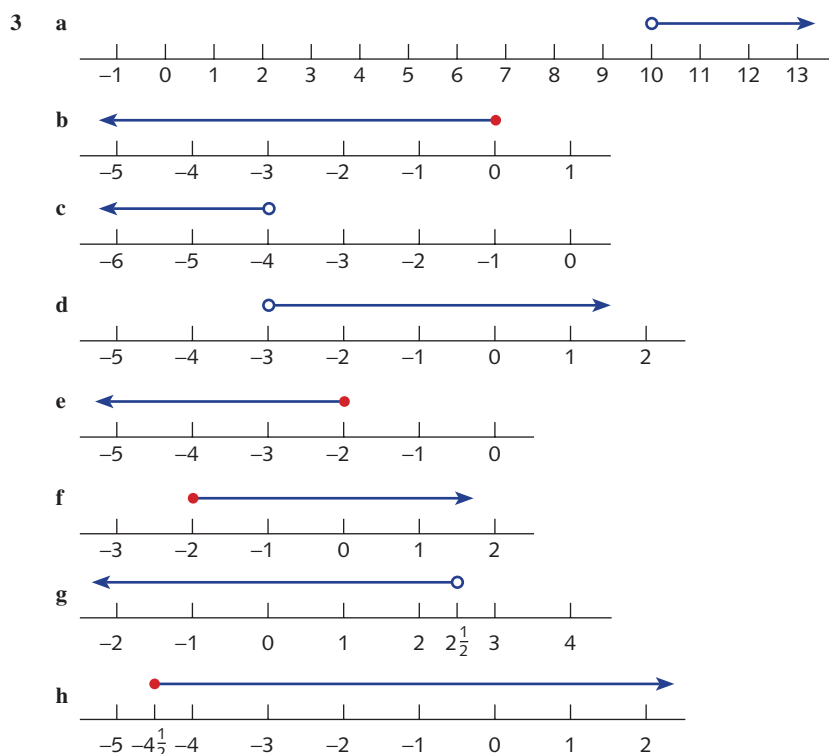
- 13 length 4.5 m, width 10.5 m 14 \$600 15 90
 16 10 km 17 40 18 7.5 L 19 600 g of A and 400 g of B
 20 4 10-cent coins, 8 20-cent coins, 5 50-cent coins 21 $6\frac{2}{3}$ L of water
 22 a $(w + 60)$ cm b $150 \text{ cm} \times 90 \text{ cm}$ c $160 \text{ cm} \times 100 \text{ cm}$
 d $(w + 160)$ cm ii $(w + 100)$ cm
 e $(200w + 16000) \text{ cm}^2$ f 40000 cm^2 g 200 cm
 23 a 12 b $22 - x$ c 10x cents d $15(22 - x)$ cents
 e $10x - 15(22 - x) = 20$, that is, $25x - 330 = 20$ f 14
 24 a 7.5 cm, 8 cm b 5 cm, 6 cm
 c $10 - 2.5t$ cm, $10 - 2t$ cm d 10:20 p.m.

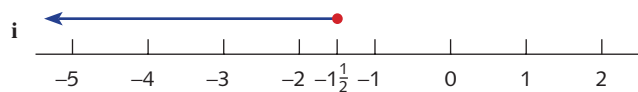
Exercise 5F

- 1 a $c - b$ b $d + e$ c $p - q$ d $m - n$ e $\frac{b}{c}$ f $\frac{c - e}{b}$
 g $\frac{c - ab}{a}$ h $\frac{n - mp}{mn}$ i ab j $bc - a$ k $a(c - b)$ l $\frac{np}{m}$
 m $\frac{b(d - c)}{a}$ n $\frac{cd - b}{a}$ o $\frac{mn + n}{m}$ p $\frac{fhk - fg}{h}$ q $\frac{a^2 + ab^2}{b}$ r $a + b^2$
 2 a $\frac{d - b}{a - c}$ or $\frac{b - d}{c - a}$ b $\frac{m + n}{n - m}$ c $\frac{d - ab}{a - c}$ or $\frac{ab - d}{c - a}$ d $\frac{ab - cd}{a - c}$ or $\frac{cd - ab}{c - a}$

Exercise 5G

- 1 a $7 > 2$ b $3 > -4$ c $-4 < -2$ d $-54 > -500$ e $-6 < 0$
 f $-13 > -45$ g $21 < 40$ h $-2 < 5$ i $99 > -100$
 2 a $-7 \leq -2$ b $5 \geq -7$ c $\geq$ or $\leq$ d $-10 \geq -50$ e $\geq$ or $\leq$
 f $-23 \geq -45$ g $12 \leq 26$ h $\geq$ or $\leq$ i $98 \geq 89$

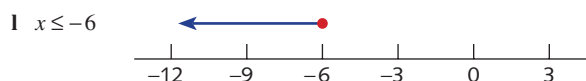
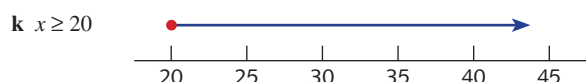
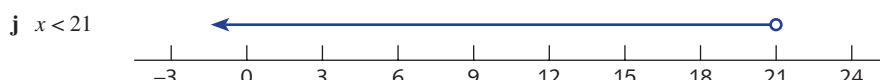
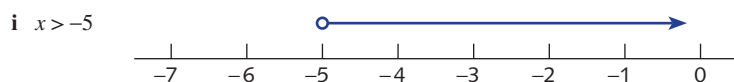
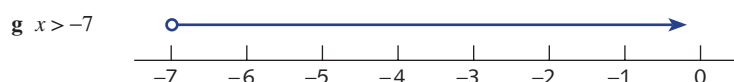
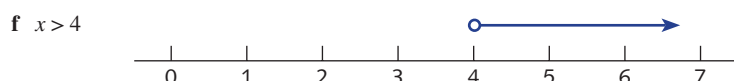
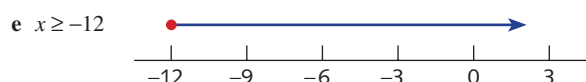
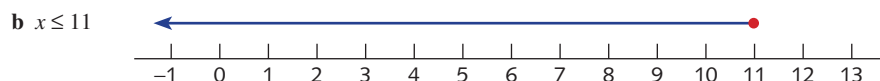
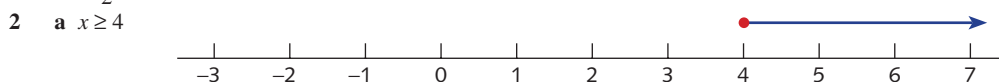




- 4 a $\{x: x > -2\}$ b $\{x: x \leq 4\}$ c $\{x: x < -2\}$ d $\{x: x \geq 1\}$ e $\{x: x \leq -2\frac{1}{2}\}$
 f $\{x: x \geq -\frac{1}{2}\}$ g $\{x: x < 1\frac{1}{2}\}$ h $\{x: x > \frac{1}{4}\}$ i $\{x: x > -1\frac{2}{3}\}$

Exercise 5H

- 1 a infinitely many b infinitely many
 c $-\frac{1}{2}, 0, 2.6$, etc. d $2.7, 11.8, 14\frac{1}{2}$, etc.

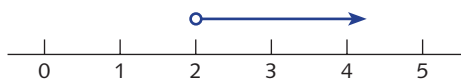


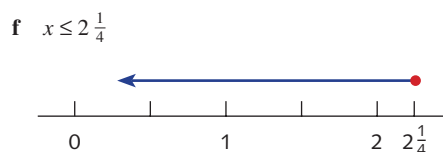
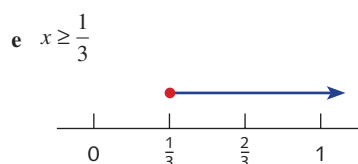
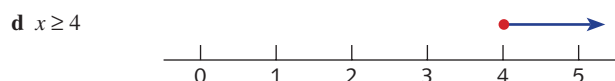
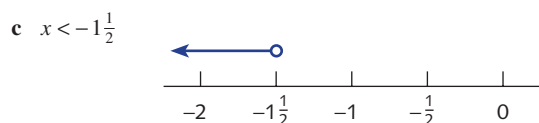
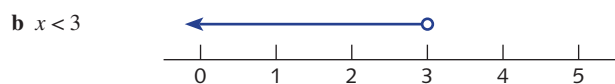
- 3 a $x \geq 2$ b $x \leq 4$ c $x \leq 1$ d $x > 7\frac{1}{2}$ e $x \geq 3\frac{1}{3}$ f $x \leq -3\frac{1}{2}$
 g $x \geq -2$ h $x > \frac{9}{10}$ i $x \leq 10\frac{1}{2}$ j $x \geq 4\frac{1}{2}$ k $x > 9\frac{3}{8}$ l $x \leq 5\frac{1}{2}$
 4 a $x \geq -5$ b $x \leq -13$ c $x < 3\frac{1}{2}$ d $x \geq -10$ e $x \leq -28$ f $x < -20$
 g $x > -48$ h $x \leq 16$ i $x < -27$

- 5 a $x < -1$ b $x \geq 2$ c $x > 5\frac{3}{4}$ d $x > 3$ e $x \leq -5$ f $x \leq 14$
- 6 a $x \leq 0$ b $x < -46$
- 7 a $x \leq 7$ b $x > 8.5$ c $x < -6$ d $x \geq -15$ e $x \leq -157$ f $x > 10.5$
- 8 a $x \leq 11$ b $x \leq -15$ c $x > -9\frac{1}{2}$ d $x \geq -23$ e $x \leq -42$ f $x > -5\frac{1}{3}$
- 9 $p > 6$ 10 $q < 68$ 11 $p \leq 3$ 12 $d > 6$ 13 $a > -6$
- 14 a i $\$(25 + 0.06x)$ ii $\$(20 + 0.08x)$ b $x > 250$ km

Review exercise

- 1 (w - 5) kg 2 $2x - 5$ 3 a $(w + 10)$ m b $(\ell - 10)$ m
- 4 a $a = -2$ b $b = -3$ c $m = -6$ d $n = -8$ e $p = -8$ f $q = -\frac{27}{4}$
- g $a = -\frac{11}{3}$ h $a = -\frac{31}{5}$ i $a = -\frac{9}{7}$ j $b = \frac{11}{7}$ k $e = -\frac{15}{4}$ l $f = -\frac{29}{7}$
- m $x = \frac{5}{2}$ n $x = \frac{33}{10}$ o $x = \frac{3}{2}$ p $x = \frac{1}{11}$ q $x = 2$ r $p = -1$
- 5 a $x = \frac{3}{5}$ b $x = 5$ c $x = 9$ d $x = \frac{3}{7}$ e $x = -\frac{6}{7}$
- f $x = \frac{136}{5}$ g $x = \frac{1}{7}$ h $x = \frac{49}{6}$ i $a = 13$ j $b = -\frac{19}{3}$
- k $a = -31$ l $a = \frac{29}{33}$ m $x = \frac{8}{9}$ n $m = -\frac{12}{35}$
- 6 a $x = -6.2$ b $a = -8$ c $c = 2$ d $\ell = 4$ e $x = 4$
- f $x = 10$ g $x = 7.5$ h $x = 1.5$ i $x = -\frac{3}{7}$
- 7 a $x = \frac{26}{5}$ b $x = \frac{52}{3}$ c $x = -15$ d $x = -18$ e $p = -\frac{8}{5}$
- f $p = \frac{8}{7}$ g $x = \frac{17}{2}$ h $x = \frac{17}{7}$ i $z = -4$ j $y = -12$
- k $k = -\frac{25}{9}$ l $\ell = 28$ m $m = 3$ n $n = -10$ o $y = \frac{11}{2}$
- p $x = -\frac{21}{4}$ q $s = \frac{1}{9}$ r $x = 91$ s $a = \frac{19}{8}$ t $x = \frac{13}{11}$
- 8 a $(4x + 15)$ m b width 23.5 m, length 109 m
- 9 a i $\$(x + 14800)$ ii $\$(x - 22000)$
b Mr Jersey's income \$119 600, Mr Guersney's income \$134 400,
Mrs Mann's income \$97 600
- 10 a 60 of each coin b 45 20-cent coins and 90 10-cent coins
c 72 20-cent coins and 36 10-cent coins
- 11 a 5 b $85\frac{5}{7}$ c 3
- 12 24 m by 16 m
- 13 a $\frac{ab - n}{m}$ b $\frac{p + mn}{m}$ c $\frac{ac + d}{ab}$ d $\frac{n + q}{m - p}$ e $\frac{acd - ab}{c}$ f $\frac{ab - mn}{m}$
- 14 a $\geq$ b $\geq$ c $\leq$ d $\geq$ e $\geq$ f $\leq$ g $\geq$ h $\leq$ i $\geq$
- 15 a $x > 2$





- 16** **a** $\{x : x > -1\}$ **b** $\{x : x < 2\}$ **c** $\{x : x \geq -3\frac{1}{2}\}$ **d** $\{x : x \leq 1\frac{1}{2}\}$
- 17** **a** $x \geq 10$ **b** $x \leq 3$ **c** $x \leq -8$ **d** $x < -15$ **e** $x \geq 2$ **f** $x \leq \frac{19}{3}$
- g** $x \geq \frac{25}{2}$ **h** $x > 18.7$ **i** $x \leq \frac{11}{2}$ **j** $x \geq -2$ **k** $x > \frac{109}{105}$ **l** $x > -2$
- m** $x \geq \frac{19}{4}$ **n** $x < -\frac{7}{3}$ **o** $x \leq 54$ **p** $x > 18$ **q** $x \geq 23$ **r** $x < -6$
- s** $x < \frac{203}{55}$ **t** $x \leq \frac{25}{19}$
- 18** **a** $x \geq -15$ **b** $x > -\frac{5}{4}$ **c** $x \geq 9$ **d** $x > \frac{7}{3}$ **e** $x < 18$ **f** $x \geq -\frac{1}{10}$
- g** $x \leq \frac{17}{5}$ **h** $x \leq \frac{17}{14}$ **i** $x \leq -\frac{25}{6}$ **j** $x > \frac{13}{3}$ **k** $x > \frac{7}{13}$ **l** $x \leq 2$

Challenge exercise

- 1** 320 km/h, 640 km/h **2** 33
- 3** A receives \$150, B receives \$130, C receives \$140.
- 4** 1900 and 2100 leaflets/h
- 5** 1 cm³ of silver weighs 10.5 g and 1 cm³ of copper weighs 9 g
- 6** 15 km/h **7** $4\frac{1}{8}\%$ **8** $\frac{600}{3781}$ m, $\frac{3}{199}$ m, $3\frac{657}{3781}$ m **9** 54 km

Chapter 6 answers

Exercise 6A

- 1** **a** 40 **b** 20 **c** 35 **d** 50 **e** 42 **f** 64
- 2** **a** 44.721 **b** 9047.787 **c** 58 **d** 18.850 **e** 430 **f** 18771.37
- 3** **a** 21 **b** 30 **c** 12 **d** $13\frac{1}{3}$
- 4** **a** 40 **b** 6 **c** 34 **d** 4000 **e** 12 **f** 9
- g** 72 **h** 18
- 5** **a** 10 **b** 12.5 **c** 20.9 **d** 10.4
- 6** **a** $1\frac{1}{3}$ **b** $3\frac{3}{5}$ **c** $5\frac{17}{23}$ **d** $1\frac{13}{17}$

- 7 a -19.5 b 5 8 a 4 b 12
- 9 a 3 b $\frac{43}{9}$ c $\frac{251}{133}$ d $\frac{23}{7}$
- 10 a $2\sqrt{5}$ b 4.47
- 11 13.5 cm 12 210 13 a 32°F b 212°F c 77°F
- 14 25.6 cm 15 a 24 cm² b 30 cm² c $16\sqrt{6}$ cm² d 84 cm²
- 16 a 31.30 m/s b 28.35 m/s c 26.66 m/s d 32.86 m/s
- 17 a 30 m b 34.125 m c 38.5 m d 48 m e 70 m f 96 m

Exercise 6B

- 1 a \$150 b $n = \frac{P+150}{5}$ c i 85 ii 110 iii 10 iv 30
- 2 a $n = \frac{C-250}{12}$ b i 40 ii 70 iii 80 iv 120
- 3 a $t = \frac{C-50}{1.2}$ b i 5 hours ii 8 hours iii 20 hours iv 50 hours
- 4 a $u = v - at$ b i 10 ii 64 iii -31.85 c $a = \frac{v-u}{t}$
- d i 2.5 ii -6.2 iii -0.2 e $t = \frac{v-u}{a}, \frac{1}{3}$
- 5 a $a = t - (n-1)d$ b i 2 ii 20 iii 3.6
- c $d = \frac{t-a}{n-1}$ d i 5 ii 3 iii 1.5 e $n = \frac{t-a}{d} + 1, 30$
- 6 a $c = y - mx$ b $x = \frac{y-c}{m}$ c $b = \frac{2A}{h}$ d $r = \frac{C}{2\pi}$
- e $h = \frac{2A}{(a+b)}$ f $\ell = \frac{P-A}{2h}$ g $u = \frac{2s-at^2}{2t}$ h $a = \frac{2(s-ut)}{t^2}$
- i $h = \frac{A-2\pi r^2}{2\pi r}$ j $h = \frac{3V}{\pi r^2}$ k $a = \frac{2S}{n} - \ell$
- l $n = \frac{2S}{a+\ell}$ m $s = \frac{V-\pi r^2}{\pi r}$ n $h = \frac{2E-mv^2}{2mg}$
- 7 $n = \frac{S}{180} + 2$ a 8 b 12 c 20
- 8 $m = \frac{2E}{v^2}$ a 8 kg b 3.5 kg c 20 kg
- 9 $C = \frac{5(F-32)}{9}$ a 20°C b -5°C c 100°C d 180°C e 36.9°C f -26°C
- 10 30 m/s
- 11 a $a = \sqrt{c-b^2}$ b $\ell = \sqrt{\frac{K-4m}{3}}$ c $b = \frac{x^2}{a}$ d $h = \frac{5d^2}{64}$
- e $n = \frac{2\pi}{T}$ f $v = \frac{m}{D}$ g $r = \sqrt{\frac{m}{2E}}$ h $r = \frac{1}{2}\sqrt{\frac{a}{T}}$
- 12 a 82.4°F b -15 $\frac{5}{9}$ °C c -40°F = -40°C
- d i 430°F ii 185°C iii 390°F iv 250°C

Exercise 6C

- 1 **a** $S = C + P$ **b** $R = W - L$ **c** $D = 90n$
d $c = 100D$ **e** $m = 60h$ **f** $d = 7m$
- 2 **a** $n = 100p$ **b** $x = 1000y$ **c** $s = 1000t$
d $q = 500p$ **e** $x = 500y$
- 3 **a** $z = 100x + y$ **b** $x = \frac{60y + z}{3600}$ **c** $x = y + \frac{z}{60}$ **d** $m = \frac{c}{20}$
e $n = \frac{m}{5}$ **f** $m = \frac{y}{x}$ **g** $p = \frac{nk}{4}$ **h** $q = \frac{xb}{8}$
- 4 **a** $y = x - 3$ **b** $y = x^2 + 4$ **c** $y = 8\sqrt{\frac{x}{5}}$
d $y = 180 - x$ **e** $y = \frac{80}{x}$ **f** $y = \frac{5x}{4}$
- 5 **a** $x = 10000y$ **b** $S = 0.8m$ **c** $A = \pi r^2$
d $c = 2\pi r$ **e** $S = 180(n - 2)$ **f** $c^2 = a^2 + b^2$
g $A = \frac{\pi r^2 \theta}{360}$ **h** $d = 75t$ **i** $h = \frac{w}{25}$
- 6 **a** **i** $C = 18x$ **ii** $C = 154 + 7x$ **b** $C = 20 + 0.4x$ **c** $y = 50 - 4x$
d $t = 2^n$ **e** $T = 20 + 45w$ **f** $A = \frac{x^2}{4\pi}$ **g** $p = 90x$
- 7 $n = 4(x + y + 1)$

Review exercise

- 1 **a** 195 **b** -20.5 **c** 8
- 2 **a** 20 **b** $\frac{49}{16}$ **3 a** $-2 + 2\sqrt{7}$ **b** -48
- 4 **a** $h = \frac{V}{\pi r^2} + \frac{r}{3}$ **b** 1290 cm
- 5 **a** $\ell = \frac{A}{w}$ **b** $r = \frac{C}{2\pi}$ **c** $h = \frac{V}{\pi r^2}$ **d** $b = \frac{A}{2h} - \ell$
e $r = \sqrt{\frac{A}{4\pi}}$ **f** $x = \frac{aw^2}{100}$ **g** $V = \frac{w^2 \pi h}{3}$ **h** $y = \frac{xz}{2x - z}$
- 6 **a** $d = 4.9t^2$ **b** 11.025 m
- 7 **a** $m = M - \frac{M}{t^2} = \frac{M(t^2 - 1)}{t^2}$ **b** $M = \frac{t^2 m}{t^2 - 1}$ **c** 6
- 8 **a** $h = \frac{S}{2\pi r} - r$ **b** 4.37 cm **9 a** 2870 **b** 3281
- 10 **a** $F = \left(\frac{2\pi}{T}\right)^2 \frac{W}{g}$ **b** $x = \frac{180(P - 2r)}{\pi r}$ **c** $x = \frac{f(D^2 - 1)}{D^2 + 1}$
- 11 **a** 10 **b** 15 **c** $\frac{n(n+1)}{2}$ **d** 630
- 12 **a** $t = \frac{ab}{M^2 - a^2}$ **b** $\ell = \sqrt{\frac{9V^2}{a^4} + \frac{a^2}{2}}$ **c** $a = \frac{Eb^3(w^2 + m)}{w^2}$
d $r = g\left(\frac{T}{2\pi}\right)^2 - \ell$ **e** $h = \frac{ka^2 - yx^2}{px^2}$ **f** $u = \sqrt{v^2 - 2as}$
- 13 **a** $V = \frac{975\pi}{8}$ **b** $r = \frac{1}{2t}\left(\frac{V}{\pi\ell} + t^2\right)$

- 14 a 20.1 m b 30.4 m
- 15 a \$1050 b $P = \frac{C}{\ell w}$
- 16 a 120 b $n = \frac{360}{180 - I}, 5$
- 17 a $y = 2x^2 + 4$ b $y = 90 - x$ c $y = \frac{x}{100}$ d $y = \frac{100}{x}$

Challenge exercise

- 1 a $M = \frac{m(P+1)}{(P-1)}$ b $M = \frac{F\ell}{P-F}$ c $R = \frac{ST}{T-S}$ d $i = \frac{R}{E-P}$ e $\ell = \frac{h}{t-T}$
- f $L = \sqrt{\frac{12I}{M} - 3R^2}$ g $L = \frac{c}{\ell} \sqrt{L^2 - \ell^2}$ h $e = \frac{T-2\pi}{T+2\pi}$ i $r = \frac{cs}{vs+c}$ j $k = g\left(\frac{P}{2\pi}\right)^2 - h$
- k $\ell = \frac{p^2}{P^2 - p^2}$ l $R = \sqrt{\frac{A}{\pi} + r^2}$ m $a = \frac{d^2 - b^2 - c^2}{2(b+d)}$ n $M = \frac{m^2 + p^2}{2m}$
- 2 a i 7 ii 17 iii 52 iv $5n + 2$
b i 10 ii 17 iii 73 iv $7n + 3$
c $2mn + n + m$ d i 10 ii 12
e i $(n+1)m$ ii $n(m+1)$ iii $m(n+1) + n(m+1)$
f Yes g 67 h 7 i 33×1
j $m(n+1)(p+1) + n(m+1)(p+1) + p(m+1)(n+1)$
- 3 a i d cm ii $(r-h)$ cm c i 65 cm ii 25 cm d 80 cm
- 4 $H = \frac{n}{2}(n-1)$
- 5 a $2\sqrt{210}$ b $10\sqrt{3}$ c $2\sqrt{2310}$ d 1764 e 3234
- 6 a $Q = 0.98^n P$ b $a = 100(1 - 0.95^n)$

Chapter 7 answers

Exercise 7A

- 1 a i obtuse angle ii revolution iii acute angle
iv straight angle v right angle vi reflex angle
- b i 70° ii 18° iii 45° c i 160° ii 8° iii 90°
- 2 a $\theta = 135^\circ$ (revolution at A)
b $\beta = 110^\circ$ (vertically opposite angles at M), $\alpha = 70^\circ$ (supplementary angles)
c $\alpha = 22\frac{1}{2}^\circ$ (supplementary angles)
d $\theta = 150^\circ$ (vertically opposite angles)
e $\theta = 72^\circ$ ($5\theta = 360^\circ$, angles in a revolution)
f $\alpha = 15^\circ$, $6\theta = 90^\circ$
- 3 a $\alpha = 140^\circ$ (alternate angles, $AB \parallel CD$)
b $\beta = 65^\circ$ (corresponding angles, $AB \parallel CD$), $\gamma = 65^\circ$ (corresponding angles, $PQ \parallel RS$)
c $\alpha = 80^\circ$ (co-interior angles, $VW \parallel XY$), $\beta = 80^\circ$ (alternate angles, $WX \parallel YZ$)
d $\beta = 112^\circ$ (co-interior angles, $KN \parallel LM$), $\alpha = 68^\circ$ (co-interior angles, $LK \parallel MN$),
 $\gamma = 112^\circ$ (co-interior angles, $NK \parallel LM$)
e $\theta = 60^\circ$ ($\theta + 2\theta = 180^\circ$, co-interior angles, $AB \parallel CD$)
f $\theta = 85^\circ$, $\angle BED = 130^\circ$ (co-interior angles, $AD \parallel BE$), $\angle BEF = 145^\circ$ (co-interior angles, $BE \parallel CF$), $\theta = 85^\circ$ (angle at a point E)

- 4 a $AB \parallel CD$ (alternate angles are equal)
 b $BE \parallel CD$ (corresponding angles are equal)
 c $QP \parallel RS$ (co-interior angles are supplementary)
 d no parallel lines
 e $NO \parallel KM$ (alternate angles are equal)
 f $AB \parallel DC$ (co-interior angles are supplementary)
- 5 a $\theta = 35^\circ$ (angle sum of $\triangle ABC$)
 b $\alpha = 65^\circ$ (angle sum of quadrilateral $KLMN$)
 c $\theta = 130^\circ$ (exterior angle of $\triangle ABC$)
 d $\alpha = 20^\circ$ (angle sum of $\triangle URS$), $\beta = 100^\circ$ (angle sum of $\triangle UST$)
 e $\theta = 50^\circ$ ($3\theta + 210^\circ = 360^\circ$, angle sum of quadrilateral $EFGH$)
 f $\beta = 135^\circ$ (exterior angle of $\triangle ABD$)
- 6 a $\alpha = 80^\circ$ (base angles of isosceles $\triangle ABC$), $\beta = 20^\circ$ (angle sum of $\triangle ABC$)
 b $\alpha = \beta = 70^\circ$ (base angles of isosceles $\triangle ABC$)
 c $x = 2$ (opposite sides of $\triangle LMN$ are equal)
 d $\alpha = 60^\circ$ ($\triangle PQR$ is equilateral), $\beta = 45^\circ$ (base angle of right isosceles $\triangle QRS$)
 e $\alpha = 60^\circ$, $x = 6$, $y = 2$ ($\triangle KLM$ is equilateral)
 f $\alpha = 50^\circ$ (base angles of isosceles $\triangle TRU$), $\beta = 70^\circ$ (base angles of isosceles $\triangle STU$), $\gamma = 40^\circ$ (angle sum of $\triangle STU$)
- 7 a Construct CX parallel to AB , $\angle XCB = 45^\circ$ (alternate angles, $CX \parallel AB$), $\angle XCD = 60^\circ$ (alternate angles, $CX \parallel DE$).
 $\theta = 45^\circ + 60^\circ = 105^\circ$
 b Construct RX parallel to QP , $\angle SRX = 40^\circ$ (co-interior angles, $ST \parallel RX$), $\angle QRX = 80^\circ$, $\alpha = 100^\circ$ (co-interior angles, $QP \parallel RX$)
 c Construct CX parallel to DQ , $\angle DCX = 30^\circ$ (corresponding angles, $CX \parallel DQ$), $\angle XCB = 60^\circ$ (complementary angles),
 $\theta = 60^\circ$ (corresponding angles, $BP \parallel CX$)
- 8 a $\alpha = 40^\circ$ (co-interior angles, $PQ \parallel TS$),
 $\beta = 110^\circ$ (angle sum of quadrilateral $PQST$),
 $\gamma = 40^\circ$ (base angle of isosceles $\triangle QRS$, supplementary to $\angle PQS$)
 b $\theta = 60^\circ$, $\angle AMB = 70^\circ$ (vertically opposite), $\angle ABM = 60^\circ$ (angle sum of $\triangle ABM$), $\theta = 60^\circ$, (alternate angles, $AB \parallel CD$)
 c $\alpha = 55^\circ$, $\angle BOM = 35^\circ$ (BMO isosceles), $\angle OMA = 70^\circ$ (exterior angle of $\triangle BMO$), $\angle MOA = \alpha$ ($\triangle MOA$ isosceles),
 $\alpha + \alpha + 70^\circ = 180^\circ$ (angle sum $\triangle AOM$), $\alpha = 55^\circ$
 d $\angle OPQ = 65^\circ$ (corresponding angles $PQ \parallel XY$),
 $\theta = 105^\circ$ (exterior angle $\triangle OPQ$)
 e $\angle ACB = 40^\circ$ (alternate angles, $AF \parallel BG$), $\angle BAC = 40^\circ$ ($\triangle ABC$ isosceles), $\theta = 100^\circ$ (angle sum $\triangle ABC$)
 f $\alpha = 60^\circ$ ($\triangle ADC$ is equilateral), $\beta = 30^\circ$ (co-interior angles, $BA \parallel CD$),
 $x = -1$ ($\triangle ADC$ is equilateral)
 g $\alpha = 65^\circ$ (corresponding angles, $AB \parallel ED$), $\beta = 115^\circ$ (supplementary angles),
 $\gamma = 32\frac{1}{2}^\circ$ ($2\gamma = \alpha$, exterior angle)
 h $\alpha = 120^\circ$ (co-interior angles, $AD \parallel BC$), $\beta = 60^\circ$ (co-interior angles, $AB \parallel CD$), $\gamma = 60^\circ$ (corresponding angles, $AB \parallel CD$)

Exercise 7B

- 1 a $\angle AOB = 180^\circ - \theta$ (supplementary angles at O)
 b $\angle AOB = 35^\circ + \theta$ (exterior angle of $\triangle OBC$)
 c $\angle AOB = 80^\circ - \theta$ (exterior angle of $\triangle AOB$)
 d $\angle AOB = 180^\circ - \theta$ (co-interior angles are supplementary, $AP \parallel XY$)
 e $\angle AOB = 135^\circ - \theta$ (angle sum of $\triangle AOB$)
 f $\angle AOB = 270^\circ - 2\theta$ (angle sum of quadrilateral $AOBM$)
 g $\angle AOB = 180^\circ - 2\theta$ (angle sum of isosceles $\triangle AOB$)
 h $\angle AOB = 90^\circ - \frac{1}{2}\theta$ (angle sum of isosceles $\triangle AOB$)
 i $\angle AOB = 120^\circ - \theta$ (adjacent supplementary angles, $\angle BOC = 60^\circ$, $\triangle BOC$ is equilateral)

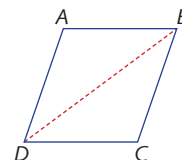
Exercise 7B (continued)

- 2 a $2\alpha = 180^\circ$ (supplementary angles) so $\alpha = 90^\circ$. Hence, $AB \perp PQ$.
b $2\alpha = 180^\circ$ (supplementary angles), so $\angle ABP = 120^\circ$ and $\angle ABQ = 60^\circ$.
Hence $\angle ABP = 2 \times \angle ABQ$.
c $2\alpha + 2\beta = 180^\circ$ (supplementary angles) so $\alpha + \beta = 90^\circ$. Hence, $PO \perp QO$.
d $2\alpha + 2\beta = 180^\circ$ (angle sum of $\triangle AOB$) so $\alpha + \beta = 90^\circ$. Hence, $\angle AOB = 90^\circ$.
e $\alpha + \beta = 3\alpha$ (exterior angle of $\triangle ABC$). Hence $\beta = 2\alpha$.
f $10\alpha = 360^\circ$ (angle sum of quadrilateral $ABCD$), so $\alpha = 36^\circ$.
 $\angle DAB + \angle ADC = 5\alpha = 180^\circ$ (or $\angle ABC + \angle BCD = 5\alpha = 180^\circ$).
Hence $AB \parallel DC$ (co-interior angles are supplementary)
- 3 a Construct OP parallel to AF (P to the left of O). Then $\angle AOP = \alpha$ (alternate angles) and $\angle BOP = \beta$ (alternate angles).
Hence $\angle AOB = 120^\circ = \alpha + \beta$.
b Construct AC parallel to OG . Then $\angle FAC = \alpha$ (corresponding angles) and $\angle PAC = \beta$ (corresponding angles).
 $\angle FAP = \angle FAC - \angle PAC = \alpha - \beta = 40^\circ$.
c $\angle CBE = 50^\circ$ (alternate angles, $FE \parallel BC$), $\alpha + \beta + 50^\circ = 180^\circ$ (supplementary angles, $AD \parallel BC$), $\alpha + \beta = 130^\circ$.
- 4 a i $\angle XAB = \beta$ because alternate angles are equal, $XY \parallel BC$.
 $\angle YAC = \gamma$ because alternate angles are equal, $XY \parallel BC$
ii $\alpha + \beta + \gamma = 180^\circ$ because of straight angle at A .
b i $\angle ACG = \alpha$ because alternate angles are equal, $AB \parallel GC$.
 $\angle PCG = \beta$ because corresponding angles are equal, $AB \parallel GC$.
ii $\angle ACP = \angle PCG + \angle ACG = \alpha + \beta$ (adjacent angles).
- 5 a $\angle APO = \alpha$ since $\triangle AOP$ is isosceles (OA and OP are radii)
b $\angle BPO = \beta$ since $\triangle BOP$ is isosceles (OB and OP are radii)
c $2\alpha + 2\beta = 180^\circ$ (angle sum of $\triangle APB$), hence $\alpha + \beta = 90^\circ$.
- 6 a i $\angle BAC = 180^\circ - \alpha$, $\angle ABC = 180^\circ - \beta$ and $\angle ACB = 180^\circ - \gamma$ (supplementary angles).
ii $\angle BAC + \angle ABC + \angle ACB = 180^\circ - \alpha + 180^\circ - \beta + 180^\circ - \gamma = 540^\circ - (\alpha + \beta + \gamma)$.
Hence $\alpha + \beta + \gamma = 360^\circ$.
b Join the diagonal AC . The interior angles of $\triangle ABC$ add to 180° and the interior angles of $\triangle ACD$ add to 180° . Since the sum of the interior angles of quadrilateral $ABCD$ is the sum of the angles of $\triangle ABC$ and $\triangle ACD$, then the interior angles of $ABCD$ add to 360° .
c i $\angle DAB = 180^\circ - \alpha$, $\angle ABC = 180^\circ - \beta$, $\angle BCD = 180^\circ - \gamma$ and $\angle CDA = 180^\circ - \theta$ (supplementary angles).
ii $\angle DAB + \angle ABC + \angle BCD + \angle CDA = 720^\circ - (\alpha + \beta + \gamma + \theta) = 360^\circ$. Hence $\alpha + \beta + \gamma + \theta = 360^\circ$.
- 7 a i 540° ii 1080° iii 1800°
b i 720° ii 900° iii 1260° iv 1440° c angle sum $= 180(n - 2)^\circ$
- 8 a i 1800° ii $1800^\circ - 360^\circ = 1440^\circ$ b i 1080° ii 720°
c angle sum $= (180n - 360)^\circ$
- 9 a i Each pair of interior and exterior angles is supplementary, and as there are ten pairs, the sum of the exterior angles plus the sum of the interior angles is $10 \times 180^\circ = 1800^\circ$.
ii The sum of the interior angles is equal to $(180^\circ \times 10) - 360^\circ = 1440^\circ$ so the sum of the exterior angles is equal to $1800^\circ - 1440^\circ = 360^\circ$.
b The sum of the exterior angles plus the interior angles is 1080° . The sum of the interior angles is 720° so the sum of the exterior angles is 360° .
c 360°
- 10 a i 144° ii 36° b i 120° ii 60°
c interior angle $= \frac{180^\circ n - 360^\circ}{n} = 180^\circ - \frac{360^\circ}{n}$; exterior angle $= \frac{360^\circ}{n}$
- 11 a interior angle $= \left(180^\circ - \frac{360^\circ}{5}\right) = 108^\circ$ b $\alpha = 72^\circ$, $\beta = 72^\circ$, $\gamma = 36^\circ$

Exercise 7C

- 1 a $\triangle ABP \equiv \triangle ABQ$ (SSS) b $\triangle FAG \equiv \triangle NTL$ (AAS) c $\triangle PQR \equiv \triangle RSP$ (RHS)
d $\triangle ACD \equiv \triangle BCD$ (SAS) e $\triangle ABC \equiv \triangle ADC$ (SSS) f $\triangle BAD \equiv \triangle BCD$ (SSS)
- 2 a $\triangle BAC \equiv \triangle EFD$ (SSS) b $\triangle ABC \equiv \triangle FDE$ (AAS) c $\triangle ABC \equiv \triangle EDF$ (RHS)
d $\triangle ACB \equiv \triangle EFD$ (SAS) e $\triangle ABC \equiv \triangle EFD$ (AAS)
- 3 $\triangle ABC \equiv \triangle OPN$ (SSS); $\triangle DEF \equiv \triangle RSQ$ (RHS); $\triangle GJH \equiv \triangle LMK$ (SAS)
- 4 a $\triangle ABC \equiv \triangle GJH$ (SSS) b $\triangle ABC \equiv \triangle ZYX$ (SAS) c $\triangle ABC \equiv \triangle LDP$ (AAS)
d $\triangle ABC \equiv \triangle IHJ$ (SAS) e $\triangle ABC \equiv \triangle RQP$ (SAS)
- 5 a i $\angle PQR$ ii $\angle RPQ$ iii PR iv BC
b i $\angle TVU$ ii $\angle MLN$ iii UV iv ML
- 6 a $\alpha = 30^\circ, \beta = 100^\circ$ b $\alpha = 40^\circ, \beta = 80^\circ$ c $a = 12, b = 12, \alpha = 67^\circ, \beta = 23^\circ$
d $x = 6, y = 4,$ e $\alpha = 83^\circ, \beta = 55^\circ, \gamma = 42^\circ, \delta = 83^\circ$
- 7 a i $AB = AC$ (given), $AM = AM$ (common), $BM = CM$ (given) so $\triangle ABM \equiv \triangle ACM$ (SSS).
ii Hence $\alpha = \beta$ (matching angles of congruent triangles)
b i $OA = OP$ (given), $\angle AOB = \angle POQ$ (vertically opposite angles),
 $\angle OAB = \angle OPQ$ (given) so $\triangle OAB \equiv \triangle OPQ$ (AAS).
ii Hence $x = y$ (matching sides of congruent triangles).
c i $CB = CD$ (given), $CA = CE$ (given), $\angle ACB = \angle ECD$ (vertically opposite angles)
so $\triangle ABC \equiv \triangle EDC$ (SAS).
ii Hence $\angle B = \angle D$ (matching angles of congruent triangles)
d i $TS = TU$ (given), $RT = RT$ (common),
 $\angle RST = \angle RUT$ (given) so $\triangle RTS \equiv \triangle RTU$ (RHS)
ii Hence $\angle SRT = \angle URT$ (matching angles of congruent triangles; $\alpha = \beta$)
e $OG = OG$ (common), $OF = OH$ (radii), $\angle FOG = \angle HOG$ (given, so $\triangle GOF \equiv \triangle GOH$ (SAS).
Hence $FG = GH$ (matching sides of congruent triangles).
f $BD = BD$ (common), $\angle ABD = \angle CBD = 70^\circ$ (given), $\angle DAB = \angle DCB = 80^\circ$ (given) so $\triangle DAB \equiv \triangle DCB$ (AAS).
Hence $AD = DC$ and $AB = BC$ (matching sides of congruent triangles).
- 8 $OA = OB$ (radii of a circle), $OM = OM$ (common side), $AM = BM$ (given)
 $\triangle OAM \equiv \triangle OBM$ (SSS), $\angle OMA = \angle OMB = 90^\circ$ (supplementary angles)
- 9 a i $AB = DC$ (given), $\angle ANB = \angle DNC = 90^\circ$ (vertically opposite angles) and $BN = CN$ (given) so $\triangle ABN \equiv \triangle DCN$ (RHS).
ii Hence $\angle A = \angle D$ (matching angles of congruent triangles)
iii Hence $AB \parallel CD$ (alternate angles are equal)
b i $PQ \parallel SR$ because $\angle QPR = \angle SRP$ (alternate angles are equal)
ii $PR = PR$ (common), $QR = SP$ (given), and $\angle QPR = \angle SRP$ so $\triangle PRS \equiv \triangle RPQ$ (RHS)
iii $\angle QRP = \angle SPR$ (matching angles in congruent triangles). Hence $PS \parallel QR$ (alternate angles are equal)
c $\angle AOB = \angle COD$ (vertically opposite angles), $OA = OB = OC = OD$ (radii). So $\triangle AOB \equiv \triangle DOC$ (SAS).
Hence $\angle ABO = \angle DCO$ (matching angles in congruent triangles). Hence $AB \parallel CD$ (alternate angles equal)
d $AB = AB$ (common), $AC = AD$ (given), $\angle BAC = \angle BAD$ (given) so $\triangle BAC \equiv \triangle BAD$ (SAS). $\angle ABC = \angle ABD$ (matching angles in congruent triangles), so $\angle ABC = \angle ABD = 90^\circ$ (supplementary angles). Hence $AB \perp CD$.
- 10 a $\angle BAR = \angle CAR$ (given), $AR = AR$ (common), $\angle ARC = \angle ARB$ (supplementary angles) so $\triangle ARB \equiv \triangle ARC$ (AAS).
 $AB = AC$ (matching sides in congruent triangles) and hence $\triangle ABC$ is isosceles.
b $BP = CQ$ (given), $\angle BQC = \angle CPB$ (given) and BC is common so $\triangle BQC \equiv \triangle CPB$ (RHS).
Hence $\angle QBC = \angle PCB$ (matching angles in congruent triangles) so $\triangle ABC$ is isosceles.
- 11 Let Z be the point where the bisector of $\angle ABC$ meets AC . Let W be the point where the bisector of $\angle ACB$ meets AB . $AB = AC$ ($\triangle ABC$ isosceles). $\angle ABZ = \angle ACW$ (bisector of base angles of $\triangle ABC$). $\angle BAZ = \angle CAW$, $\triangle ABZ \equiv \triangle ACW$ (AAS). $AZ = AW$ (matching sides of congruent triangles). Therefore, $WB = ZC$, $\triangle WXB \equiv \triangle ZXC$ (AAS). $BX = CX$ (matching sides).
- 12 a Construct OA and OB . $OA = OB$ (radii), ON is common, $\angle ANO = \angle BNO = 90^\circ$, so $\triangle AON \equiv \triangle BON$ (RHS), so $AN = BN$ (matching sides of congruent triangles), so PQ bisects AB .
b Construct OA, OB, OC and OD . $OA = OD$ (radii of larger circle), so $\angle DAO = \angle ADO$ (base angles of isosceles $\triangle ADO$).
 $OB = OC$ (radii of smaller circle), so $\angle CBO = \angle BCO$ (base angles of isosceles $\triangle BCO$). $\angle ABO = 180^\circ - \angle CBO$ (supplementary at B) $= 180^\circ - \angle BCO = \angle DCO$ (supplementary at B), so $\angle AOB = \angle DOC$. So $\triangle ABO \equiv \triangle DCO$ (AAS).
So $AB = CD$ (matching sides of congruent triangles).

- 13 a i $AB = AC$ (given), $\angle BAM = \angle CAM$ (given), AM is common. So $\triangle BAM \equiv \triangle CAM$ (SAS).
 ii Hence $\angle B = \angle C$ (matching angles of congruent triangles)
 b i $\angle ABM = \angle ACM$ (given), $\angle MAB = \angle MAC$ (given) and AM is common so $\triangle ABM \equiv \triangle ACM$ (AAS).
 ii Hence $AB = AC$ (matching sides of congruent triangles)
 c i $AC = BC$ so $\angle B = \theta$ (the base angles of an isosceles triangle are equal)
 ii $AB = BC$ so $\angle C = \theta$ (the base angles of an isosceles triangle are equal)
 iii The angle sum of a triangle is 180° . $\angle A + \angle B + \angle C = 3\theta = 180^\circ$ so $\theta = 60^\circ$
 d i $AB = AC$ (opposite angles are equal) ii $AC = BC$ (opposite angles are equal)
 14 a i $\angle B = 180^\circ - \alpha$ (co-interior angles are supplementary, $AD \parallel BC$),
 $\angle D = 180^\circ - \alpha$ (co-interior angles are supplementary, $AB \parallel DC$).
 ii $\angle C = \alpha$ (angle sum of quadrilateral $ABCD$)
 b i AC is common, $\angle DAC = \angle BCA$ (alternate angles are equal) $\angle BAC = \angle DCA$ (alternate angles are equal)
 so $\triangle ACD \equiv \triangle CAB$ (AAS).
 ii Hence $AB = DC$ and $AD = BC$ (matching sides of congruent triangles)
 c i $\angle AMB = \angle CMD$ (vertically opposite angles), $AB = CD$ (opposite sides of a parallelogram are equal),
 $\angle ABM = \angle CDM$ (alternate angles are equal) so $\triangle ABM \equiv \triangle CDM$ (AAS).
 ii Hence $AM = CM$ and $BM = DM$ (matching sides of congruent triangles)
 15 a $ABCD$ is a rhombus. Therefore $AB = BC = CD = DA$
 so $\triangle ABD \equiv \triangle CDB$ (SSS). Hence $\angle ABD = \angle CDB$ (matching angles of congruent triangles).
 Thus $AB \parallel CD$ (alternate angles are equal). Similarly $AD \parallel CB$. Thus $ABCD$ is a parallelogram.



- b i $AP = PB = AQ = BQ$ (given), AB is common. So $\triangle APB \equiv \triangle AQB$ (SSS).
 ii Hence $\angle PAM = \angle QAM$ (matching angles of congruent triangles)
 iii $AP = AQ$ (given), $\angle QAM = \angle PAM$ (from ii), AM is common, so $\triangle QAM \equiv \triangle PAM$ (SAS). Hence $\angle QMA = \angle PMA$
 (matching angles in congruent triangles) so $\angle QMA = \angle PMA = 90^\circ$ (supplementary angles). Hence $AM \perp PQ$.
 c i DC is common, $\angle ADC = \angle BCD$ (given), $AD = BC$ (opposite sides of a parallelogram). So $\triangle ADC \equiv \triangle BCD$ (SAS)
 ii Hence $AC = BD$ (matching sides of congruent triangles).
 16 a $OA = OB$ (radii), OM is common, $AM = BM$ (given). So $\triangle OMA \equiv \triangle OMB$ (SSS).
 b $\angle OMA = \angle OMB$ (matching angles of congruent triangles) and $\angle OMA$ and $\angle OMB$ are supplementary
 so $\angle OMA = \angle OMB = 90^\circ$. Hence $OM \perp AB$.
 17 a $AO = BO$ (radii), $AP = BP$ (radii), PO is common, so $\triangle AOP \equiv \triangle BOP$ (SSS).
 b $\angle AOP = \angle BOP$ (matching angles of congruent triangles), hence OP bisects $\angle AOB$.
 c OM is common, $\angle AOM = \angle BOM$ (from b), $OA = OB$ (radii) so $\triangle AOM \equiv \triangle BOM$ (SAS).
 d Hence $AM = BM$ (matching sides of congruent triangles). $\angle AMO = \angle BMO$ (matching angles of congruent triangles) and
 $\angle AMO$ and $\angle BMO$ are supplementary so $\angle AMO = \angle BMO = 90^\circ$. Hence $AB \perp OP$.

Exercise 7D

- 1 a i $2\alpha + 2\beta = 360^\circ$ (angle sum of quadrilateral $ABCD$) so $\alpha + \beta = 180^\circ$.
 ii $\alpha + \beta = 180^\circ$ so $AB \parallel DC$ (co-interior angles are supplementary) and $AD \parallel BC$ (co-interior angles are supplementary).
 b i $RS = UT$ (given), $RU = ST$ (given), $SU = SU$ (common), so $\triangle RSU \equiv \triangle TUS$ (SSS).
 ii Hence $\angle RSU = \angle TUS$ (matching angles of congruent triangles), so $RS \parallel UT$ (alternate angles are equal). Similarly
 $\angle RUS = \angle TSU$ (matching angles of congruent triangles), so $RU \parallel ST$ (alternate angles are equal).
 c i $KL = NM$ (given), $\angle LKM = \angle NMK$ (alternate angles, $KL \parallel NM$), $KM = KM$ (common), so $\triangle LKM \equiv \triangle NMK$ (SAS).
 ii Hence $\angle LMK = \angle NKM$ (matching angles of congruent triangles), so $KN \parallel LM$ (alternate angles are equal).
 d i $EO = OG$ (given), $OH = FO$ (given), $\angle EOH = \angle GOF$ (vertically opposite angles at O), so $\triangle EHO \equiv \triangle GFO$ (SAS).
 ii $EH = FG$ (matching sides of congruent triangles), $\angle HEO = \angle FGO$ (matching angles of congruent triangles),
 so $EH \parallel FG$ (alternate angles are equal).
 iii If one pair of opposite sides of a quadrilateral are equal and parallel, then the quadrilateral is a parallelogram.
 2 a Since its diagonals bisect each other, $APBQ$ is a parallelogram.
 b Since its opposite sides are equal, $ABQP$ is a parallelogram.
 c Since one pair of opposite sides, namely LM and ST , are equal and parallel, $LMST$ is a parallelogram.

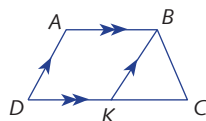
- 3 a Correct. Since $AM = OB$ and $OA = BM$, $AMBO$ is a parallelogram.
 b Correct. Since $FG = PM = GM = FP$, $FGMP$ is a parallelogram.
 c True. Since $AM = MB$ and $OM = MS$, $ASBO$ is a parallelogram.
- 4 a $\angle PAQ = \angle QD = 50^\circ$ (alternate angles, $CD \parallel AB$), $PC \parallel AQ$ (corresponding angles are equal, $\angle BPC = \angle PAQ = 50^\circ$), $AP \parallel QC$ (given), so $APCQ$ is a parallelogram. So $AP = QC$.
 b $XS \parallel UY$ (given), $XS = UY$ (given), so $SYUX$ is a parallelogram. So $UX \parallel YS$.
 c $AD = CB$ (opposite sides of a parallelogram), $BY = DX$ (given), $\angle ADX = \angle CBY$ (alternate angles, $AD \parallel BC$), $\triangle AXD \equiv \triangle CYB$ (SAS), $AX = CY$ (matching sides), similarly $AY = XC$ so $AXCY$ is a parallelogram.
 d $\angle ADQ = \angle CBP$ (alternate angles, $AD \parallel BC$), $AD = BC$ (opposite sides of a parallelogram), $DQ = BP$ (given), $\triangle BPC \equiv \triangle DQA$ (SAS). $AQ = CP$ (matching sides).
 e $AB = CD$ (opposite sides of a parallelogram), $\angle BAX = \angle DCY$ (alternate angles, $AB \parallel CD$), $\angle BXA = \angle CYC$ (given), $\triangle CYD \equiv \triangle AXB$ (AAS), $BX = DY$ (matching sides of congruent triangles).
 f $DA = BC$ (opposite sides of a parallelogram), $\angle ADQ = \angle PBC$ (opposite angles of a parallelogram), $BP = DQ$ (given), $\triangle ADQ \equiv \triangle CBP$ (SAS). $AQ = PC$ (matching sides of congruent triangles).
 Also $\angle BPC = \angle DQA$ (matching angles of congruent triangles) and $\angle BPC = \angle PCD$ (alternate angles, $AB \parallel DC$).
 Hence $\angle DQA = \angle PCD$. Thus $AQ \parallel PC$ and $ABCD$ is a parallelogram.

Exercise 7E

- 1 a Opposite sides of a parallelogram are equal. Therefore all sides are equal.
 b i Since the diagonals of the quadrilateral bisect each other, $ASBT$ is a parallelogram.
 ii $SM = TM$ (given), $\angle AMS = \angle TMT = 90^\circ$ (supplementary at M), $AM = TM$ (common), so $\triangle AMS \equiv \triangle TMT$ (SAS).
 iii $AS = TM$ (matching sides of congruent triangles)
 c i Since the diagonals of the quadrilateral bisect each other, $RSTU$ is a parallelogram.
 ii $US = RT$ (given), $RS = RS$ (common), $RU = ST$ (opposite sides of parallelogram equal), so $\triangle URS \equiv \triangle TSR$ (SSS).
 iii $\angle URS = \angle TSR$ (matching angles of congruent triangles)
 iv $\angle URS + \angle TSR = 180^\circ$ (co-interior angles, $UR \parallel ST$), $2\angle URS = 180^\circ$ ($\angle URS = \angle TSR$), $\angle URS = 90^\circ$ and $URST$ is a rectangle.
- 2 a Given parallelogram $ABCD$:
 if $\angle ABC = 90^\circ$, then $\angle BCD = 90^\circ$ (co-interior angles $AB \parallel CD$), and $\angle DAB = 90^\circ$ (co-interior angles $DA \parallel CB$), $\angle CDA = 90^\circ$ (angle sum of a quadrilateral is 360°). Since all interior angles are right angles, $ABCD$ is a rectangle.
 b Suppose parallelogram $ABCD$ has $\angle ABC = \angle BCD = \angle CDA = \angle DAC$. Since angle sum of a quadrilateral is 360° , and all angles are equal, then each interior angles is $360^\circ \div 4 = 90^\circ$, so all interior angles are right angles, and $ABCD$ is a rectangle.
- 3 a $\angle ABF = \angle GAB = \alpha$ (alternate angles, $AG \parallel BF$)
 b $AF = BF$ (equal sides of isosceles $\triangle AFB$). So 2 adjacent sides are equal in a parallelogram. Hence $AFBG$ is a rhombus.
- 4 a Since AB and FG bisect each other at right angles, $AFBG$ is a rhombus.
 b Since AB and FG are equal and bisect each other, $AFBG$ is a rectangle.
 c Since AB and FG are equal and bisect each other at right angles, $AFBG$ is a square.
- 5 Since AP and BQ are equal and bisect each other, $ABPQ$ is a rectangle.
- 6 a Since AB and FG bisect each other, $AFBG$ is a parallelogram. Since $\angle AFB$ is a right angle, $AFBG$ is a rectangle.
 b Since the diagonals of a rectangle are equal and bisect each other, $AM = BM = FM = radius$ of the circle.
- 7 a i $AP = AQ = BP = BQ$, so $APBQ$ is a rhombus.
 ii Since the diagonals of a rhombus bisect each other at right angles, PQ is the perpendicular bisector of AB .
 b i $OP = OQ = PM = QM$, so $OPMQ$ is a rhombus.
 ii Since the diagonals of a rhombus bisect the vertex angles through which they pass, OM is the bisector of $\angle AOB$.
 c i $PA = PB = AM = BM$, so $APBM$ is a rhombus.
 ii Since the diagonals of a rhombus bisect each other at right angles, PM is perpendicular to ℓ .
- 8 a i $\angle ABP = \angle CBP = \angle ADQ = \angle CDQ = 45^\circ$ (diagonals of square $ABCD$ meet each side at 45°), $BP = DQ$ (given), $AB = BC = CD = DA$ (equal sides of square $ABCD$), so $\triangle ABP \equiv \triangle CBP \equiv \triangle ADQ \equiv \triangle CDQ$ (SAS).
 ii $AP = CP = AQ = CQ$ (matching sides of congruent triangles), so $APCQ$ is a rhombus.
 b $\angle FMA = \angle GAM$ (alternate angles, $AG \parallel FM$). Also $AF = FM$ (in isosceles $\triangle AFM$).
 So 2 adjacent sides of parallelogram $AFMG$ are equal. So $AFMG$ is a rhombus.

Exercise 7E (continued)

- 9 a i $AF = AG$ (given), $BF = BG$ (given), $AB = AB$ (common), so $\triangle ABF \equiv \triangle ABG$ (SSS).
 ii $\angle FAM = \angle GAM$ (matching angles of congruent triangles), so AB bisects $\angle FAG$. $\angle FBM = \angle GBM$ (matching angles of congruent triangles), so AB bisects $\angle FBG$.
 iii $\angle AFB = \angle AGB$ (matching angles of congruent triangles).
 iv $AF = AG$ (given), $AM = AM$ (common), $\angle FAM = \angle GAM$ (matching angles of congruent triangles ABF and ABG), so $\triangle AMF \equiv \triangle AMG$ (SAS).
 v $\angle AMF = \angle AMG$ (matching angles of congruent triangles), and $\angle AMF + \angle AMG = 180^\circ$ (straight angle at M), $2\angle AMF = 180^\circ$ ($\angle AMF = \angle AMG$), $\angle AMF = 90^\circ$, so $AB \perp FG$.
 b $FM = GM$ (given), $\angle AMF = \angle AMG = 90^\circ$
 $AM = AM$ (common), so $\triangle AMF \equiv \triangle AMG$ (SAS), so $AF = AG$ (matching sides of congruent triangles).
 $\angle BMF = \angle BMG = 90^\circ$ (straight angle at M), $BM = BM$ (common), so $\triangle BMF \equiv \triangle BMG$ (SAS), so $BF = BG$ (matching sides of congruent triangles). Since $AF = AG$ and $BF = BG$, $AFBG$ is a kite.
 c $AP = AM$ (radii of circle centre A), $BP = BM$ (radii of circle centre B), so $APBM$ is a kite. Since the diagonals of a kite are perpendicular, $PM \perp \ell$.
- 10 Draw diagonals BP and CQ of rhombus $BCPQ$. $\angle AQB = \angle QAB$ ($\triangle ABQ$ isosceles), $\angle QBC = 2\angle QAB$ (exterior angle), $\angle QAB = \angle PBC$ (diagonal of rhombus bisects angle), $BP \parallel AQ$ (corresponding angles equal), similarly $QC \parallel PD$. QC and BP meet at right angles at X (diagonals of a rhombus are perpendicular). AR is perpendicular to DR .
- 11 $\angle XSP = \angle SXP$ ($\triangle SPX$ isosceles), $\angle RSX = \angle PXS$ (alternate angles, $SR \parallel PQ$), $\angle PSR = 2\angle PSX$, SX bisects $\angle RSP$
- 12 a $\angle A = 130^\circ$ and $\angle B = 110^\circ$
 b $BC = BC$ and $AB \parallel DC$. Draw BK parallel to AD .



$ABKD$ is a parallelogram. $AD = BK$ (opposite sides of a parallelogram)

Thus $BK = BC$ and hence $\triangle KBC$ is isosceles.

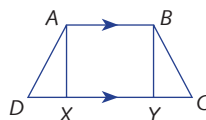
Therefore $\angle BKC = \angle BCK$.

$\angle BCK = \angle ADK$ (corresponding angles $AD \parallel BK$).

Therefore $\angle BCD = \angle ADC$

- c $\triangle DBC \equiv \triangle CAD$ (SAS) since $\angle BCD = \angle ADC$, $DC = CD$ (common) and $AD = BC$ (given). Thus $DB = CA$ (matching sides of congruent triangles).

d



Draw a perpendicular from A to meet line DC at X and from B to meet DC at Y . $ABCD$ is a rectangle and so $AX = BY$.

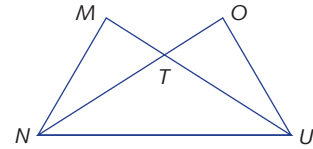
$\triangle DBY \equiv \triangle CAX$ (RHS). Thus $\angle BDY = \angle ACY$.

Therefore $\triangle DBC \equiv \triangle CAD$ (SAS) and $AD = BC$ (matching sides of congruent triangles)

Review exercise

- | | | | | | |
|----------------|------------------|---------------|----------------|--------------|----------------|
| 1 a acute | b straight angle | c right angle | d reflex angle | e revolution | f obtuse angle |
| 2 a 34° | b 23° | c 43° | d 75° | e 64° | f 44° |
| 3 a 26° | b 91° | c 146° | d 90° | e 67° | f 64° |
- 4 a $\alpha = 90^\circ$ (supplementary); $\beta = 115^\circ$ (supplementary)
 b $\alpha + 15^\circ = 35^\circ$ (vertically opposite), $\alpha = 20^\circ$, $\beta = 145^\circ$ (supplementary)
 c $\angle BCD = 34^\circ$ (revolution), $\alpha = 34^\circ$ (alternate angles, $AB \parallel CD$)
 d $\alpha = 60^\circ$ (angle sum of triangle), $\beta = 48^\circ$ (revolution), $\theta = 62^\circ$ (angle sum of triangle)
 e $\alpha = 48^\circ$ (complementary), $\beta = 270^\circ$ (revolution)
 f $\beta = 120^\circ$ (corresponding $AB \parallel FC$), $\alpha = 60^\circ$ (co-interior, $AE \parallel BD$)
 g $\angle LMK = 87^\circ$ (alternate angles, $CD \parallel AB$), $\beta = 93^\circ$, $\alpha = 90^\circ$ (alternate angles $AB \parallel CD$)

- 5 $MT = OT$ (given) and $\angle MNT = \angle OUT$ (given)
 $\triangle NMT \equiv \triangle UOT$ (AAS).
 Therefore $NT = UT$.



- 6 $\triangle ABC$ is isosceles with $AB = AC$. $BD \perp AC$, $CE \perp AB$.

a $\angle DCB = \angle ECB$ ($\triangle ABC$ is isosceles)

$\triangle BDC \equiv \triangle BEC$ (AAS)

b $\triangle EDB \equiv \triangle DEC$ (SSS)

$\angle EDB = \angle DEC$ (matching angles)

$\angle AED = 180^\circ - 90^\circ - \angle DEC$

$\angle ADE = 180^\circ - 90^\circ - \angle EDB$

Therefore $\angle AED = \angle ADE$.

- 7 a $\angle ABE = \angle BED$ (alternate angles, $AB \parallel ED$)

$\angle CBE = \angle FEB$ (alternate angles, $BC \parallel FE$)

$\angle ABC = \angle ABE + \angle CBE = \angle BED + \angle BEF = \angle FED$

b $\triangle ABC \equiv \triangle DEF$ (SAS)

$AC = FD$. Let BE meet AC at X and FD at Y . Thus $\angle AXB = \angle DYE$ (angle sum of triangles) and $\angle DYE = \angle FYB$ (vertically opposite).

$ACDF$ is a parallelogram.

- 8 a $\angle BAE = \angle AED$ (alternate angles, $AB \parallel CD$)

$\angle ABE = \angle CEB$ (alternate angles, $AB \parallel CD$)

$\angle DAE = \angle BAE$ (given)

$\triangle AED$ is isosceles and $\triangle BEC$ is isosceles.

$AD = BC$ (opposite sides of parallelogram)

$AD = DE$ and $BC = EC$ ($\triangle AED$ is isosceles and $\triangle BEC$ is isosceles.)

Therefore $DE = EC$.

b $\triangle BEC$ is isosceles, so $EC = BC$

$\triangle AED$ is isosceles, so $AD = DE$

$AD = BC$

so $DE = EC$

Therefore $DC = 2CB$

c $2\angle BAE + 2\angle ABE = 180^\circ$, and therefore $\angle BAE + \angle ABE = 90^\circ$

$\angle AEB = 180^\circ - (\angle BAE + \angle ABE) = 90^\circ$

- 9 a $\triangle AXD$ is isosceles and $\triangle BYC$ is isosceles.

$\angle ADX = \angle CBY$ (alternate angles, $AD \parallel BC$)

$\triangle AXD \equiv \triangle CYB$ (SAS)

$AX = CY$ (matching sides)

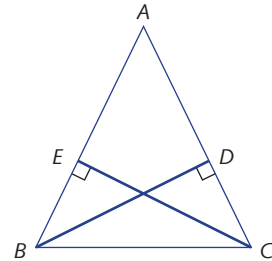
b $\angle XDC = \angle YBA$ (alternate angles, $AB \parallel CD$)

$DX = BY$ (part a)

$\triangle AYB \equiv \triangle CXD$ (SAS)

$AY = CX$ (matching sides of congruent triangles)

c Opposite sides are of equal length



Challenge exercise

- 1 a Let $AM = a$, $BM = b$, $CM = c$ and $DM = d$. By Pythagoras' theorem,
 $AB^2 = a^2 + b^2$, $BC^2 = b^2 + c^2$, $CD^2 = c^2 + d^2$ and $AD^2 = a^2 + d^2$.
 So $AB^2 + CD^2 = a^2 + b^2 + c^2 + d^2$, and $AD^2 + BC^2 = a^2 + d^2 + b^2 + c^2$,
 so $AB^2 + CD^2 = AD^2 + BC^2$.
- b Let $AP = a$, $BQ = b$, $CQ = c$ and $DP = d$. By Pythagoras' theorem,
 $AB^2 = a^2 + (b + PQ)^2$, $BC^2 = b^2 + c^2$, $CD^2 = c^2 + (d + PQ)^2$ and $AD^2 = a^2 + d^2$.
 Now $AB^2 + CD^2 = AD^2 + BC^2$,
 so $a^2 + (b + PQ)^2 + c^2 + (d + PQ)^2 = a^2 + b^2 + c^2 + d^2$,

$$\begin{aligned}\text{and} \quad 2PQ^2 + 2b \times PQ + 2d \times PQ &= 0 \\ 2PQ(PQ + b + d) &= 0 \\ PQ &= 0\end{aligned}$$

So P and Q coincide, and the diagonals of the quadrilateral are perpendicular.

c Parts **a** and **b** still hold.

2 To prove: $OA = OB = OC$

$AR = BR$ (given), $\angle ARO = \angle BRO = 90^\circ$ (given R), $OR = OR$ (common),
so $\triangle ARO \equiv \triangle BRO$ (SAS). So $OA = OB$ (matching sides of congruent triangles).

$AQ = CQ$ (given), $\angle AQO = \angle CQO = 90^\circ$ (given), $OQ = OQ$ (common),
so $\triangle AQO \equiv \triangle CQO$ (SAS). So $OA = OC$ (matching sides of congruent triangles).

So $OA = OB = OC$.

To prove: $OP \perp BC$

$BP = CP$ (given), $OB = OC$ (from above), $OP = OP$ (common),

so $\triangle BPO \equiv \triangle CPO$ (SSS). So $\angle BPO = \angle CPO$ (matching angles of congruent triangles) $= 90^\circ$
(straight angle at P). So $OP \perp BC$.

3 **a** **i** 1 positive **ii** 1 negative **iii** 0 **iv** 2 positive

b For a convex polygon must complete one revolution. Angle sum of exterior angles is 360° .

4 $\frac{3}{4}$ of a positive revolution $= 270^\circ$. Sum of exterior angles $= 270^\circ$.

5 **a** $\triangle ABD$ and $\triangle ACD$ have the same height h and share a common base AD .

Therefore, area of $\triangle ABD$ = area of $\triangle ACD$ and area of $\triangle ABX$ + area of $\triangle AXD$ = area of $\triangle DXC$ + area of $\triangle AXD$.

Hence area of $\triangle ABX$ = area of $\triangle DCX$.

b If area of $\triangle ABX$ = area of $\triangle DCX$ then area of $\triangle ABD$ = area of $\triangle ACD$.

These two triangles have a common base AD and therefore their heights BY and CZ are equal and BC is parallel to AD . So $ABCD$ is a trapezium.

6 **a** $DC = DA$ (sides of a square)

Let $\angle DAQ = \alpha$

$\angle AQD = 90^\circ - \alpha$ (angle sum of $\triangle ADQ$)

$\angle MDQ = 90^\circ - (90^\circ - \alpha) = \alpha$ (angle sum of $\triangle DMQ$)

$\triangle DAQ \equiv \triangle CDR$ (AAS)

$AQ = DR$ (matching sides of congruent triangles)

b Consider suitable translations.

7 **a** $\triangle AHE \equiv \triangle BEF \equiv \triangle CFG \equiv \triangle DGH$ (SAS)

By Pythagoras' theorem, $EF = FG = GH = HE = \sqrt{5}$,
and the four angles are 90° . Area of square $= 5$

b Draw diagonals FH and EG to intersect at O' .

Show $\triangle HAO' \equiv \triangle EBO' \equiv \triangle FCO' \equiv \triangle GDO'$ (SAS)

This shows that O' is an equal distance from each vertex of the original square and hence is the centre.

8 The line through the intersection cuts opposite sides and two trapezia are formed. The opposite sides of a parallelogram being of equal length enables you to show that each of these trapezia is of equal area.

9 Construct the line from the vertex to the midpoint of the base, and note the two triangles formed are congruent. Hence the line is the perpendicular bisector. Now use the fact that there is only one perpendicular bisector of the base.

10 **a** $HK = XK$ (given)

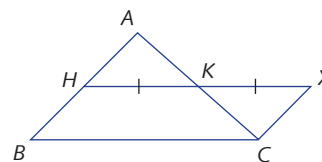
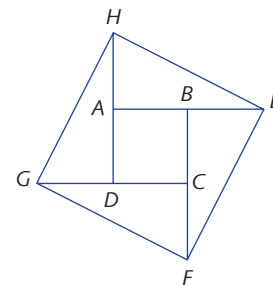
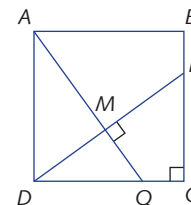
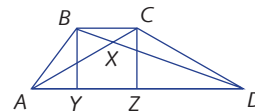
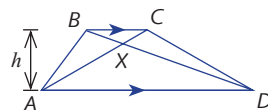
$\angle AKH = \angle CKX$ (vertically opposite)

$AK = KC$ (K is the midpoint of AC)

$\triangle AKM \equiv \triangle CKX$ (SAS)

$XC = AH$ (matching sides)

b $CXHB$ is a parallelogram and so $HX = BC$. $HK = \frac{1}{2} HX = \frac{1}{2} BC$ and $BC \parallel HK$.



Chapter 8 answers

Exercise 8A

1	a	b	c	d	e	f	
	base	6	7	8	10	5	6
	exponent	4	3	2	4	1	0
2	a 2 ³	b 3 ³	c 2 ⁶	d 3 ⁵	e 5 ³	f 3 ⁴	
3	a 81	b 128	c 3125	d 2401	e 1944	f 11 664	
4	a 2 × 3 ²	b 2 ³ × 3	c 2 ⁴ × 3 ²	d 2 × 3 ² × 5	e 2 ² × 5 ² × 7	f 2 ² × 3 × 7	
5	a 2 ⁵ g a ¹¹	b 2 ¹⁰ h b ¹⁹	c 3 ⁹ i 3a ⁵	d 3 ¹¹ j 12x ⁵	e 3 ⁷ k 6y ⁵	f 3 ¹² l 12b ⁶	
6	a a ² b ⁵	b a ⁵ b ⁴	c x ⁵ y ²	d 6x ³ y ³	e 4a ⁵ b ⁶	f 10a ⁵ b ⁴	
7	a 3 ⁵ g a ³	b 2 ⁴ h a ²	c 5 ³ i 2x	d 7 ² j 3x ³	e 10 ⁵ k 2y ⁹	f 10 ⁸ l 3p ³	
8	a ab ² e 3a ⁴ b	b x ² y f 5xy	c ab ² g $\frac{4a^2b}{3}$	d xy ⁵ h $\frac{3xy}{2}$			
9	a a ³ b ² g 4ab ²	b x ⁴ y ⁴ h 2x ⁴ y ²	c 4a ² b ²	d 4xy	e $\frac{3a^2b}{2}$	f $\frac{2x^3y}{3}$	
10	a a ⁶ g b ⁶ m 3ab ²	b b ⁹ h d ⁴ n a ⁵ b ³	c 3a ⁴ i 5d ⁵ o 3x ⁵ y ³	d 3d j 3d ⁵ p ℓ ⁴ m ²	e a ⁴ k a ³ b ⁵ q 3m ⁵ n ⁴	f x ⁴ l m ⁶ n ² r 6pq ⁵	
11	a 6x g 5	b 2x h 2x ²	c x	d 2x ³	e 3x ²	f 2x	
12	a 1	b 2	c x	d 7			
13	a 3 g 1	b 6 h 1	c 1 i 3	d 1 j −4	e 7 k 1	f 13 l 1	
14	a 2 ¹²	b 3 ⁶	c a ¹⁰	d y ³⁰			
15	a a ¹⁸ f 4b ²	b x ¹⁷ g $\frac{x^4}{2}$	c b ² h $\frac{3a^2b}{2}$	d y ⁴ i x ⁴	e 6a ² b ⁸		
16	a 4 g a ²	b 7 h m ⁵	c 2 i (m ¹) ²⁰ or (m ²) ¹⁰ or (m ⁴) ⁵ or (m ⁵) ⁴ or (m ¹⁰) ² or (m ²⁰) ¹	d 5	e p ⁶	f p ⁵	
17	a Yes	b Yes	c (a ^m) ⁿ = (a ⁿ) ^m				
18	a 9a ² g $\frac{a^5}{b^5}$	b 8x ³ h $\frac{x^6}{y^3}$	c x ² y ⁶	d a ⁸ b ⁴	e $\frac{a^2}{25}$	f $\frac{8}{x^3}$	
19	a 12a ⁵ b ⁵	b 27x ⁷ y ⁸	c m ⁷ n ⁹	d 125x ⁷ y ¹²	e 24a ⁹ b ³	f 27x ⁶ y ³	
20	a xy ⁴	b 2ab	c x ³ y ²	d 2x ¹⁴ y			
21	a 2x ⁴ y ⁵	b $\frac{3ab^4}{2}$	c 2xy ⁸	d 3a ² b			
22	a a ² b ³ g 2a ²	b m ⁵ n ⁴ h 3q ³	c 2 i 2m	d 3 j 7m ³	e 0 k 4ℓ ³ m	f 0 l 5m ⁵ n ³	

Exercise 8B

1	a $\frac{1}{2^1} = \frac{1}{2}$ f $\frac{1}{10^2} = \frac{1}{100}$	b $\frac{1}{5^1} = \frac{1}{5}$ g $\frac{1}{2^4} = \frac{1}{16}$	c $\frac{1}{3^2} = \frac{1}{9}$ h $\frac{1}{2^5} = \frac{1}{32}$	d $\frac{1}{6^2} = \frac{1}{36}$ i $\frac{1}{3^3} = \frac{1}{27}$	e $\frac{1}{9^2} = \frac{1}{81}$ j $\frac{1}{5^3} = \frac{1}{125}$
---	---	---	---	--	---

$$\mathbf{k} \quad \frac{1}{10^4} = \frac{1}{10000}$$

$$\mathbf{p} \quad \frac{1}{8^2} = \frac{1}{64}$$

$$\mathbf{2} \quad \begin{array}{l} \mathbf{a} \quad 2^{-3} \\ \mathbf{g} \quad 2^{-4} \\ \mathbf{m} \quad 5^{-4} \end{array}$$

$$\mathbf{3} \quad \begin{array}{l} \mathbf{a} \quad \frac{1}{a^3} \\ \mathbf{g} \quad \frac{4}{a^5} \end{array}$$

$$\mathbf{m} \quad \frac{1}{9a^2}$$

$$\mathbf{4} \quad \begin{array}{l} \mathbf{a} \quad 4 \\ \mathbf{e} \quad 27 \end{array}$$

$$\mathbf{5} \quad \begin{array}{l} \mathbf{a} \quad 2^{-3} = \frac{1}{8} \\ \mathbf{g} \quad 8^{-1} = \frac{1}{8} \end{array}$$

$$\mathbf{6} \quad \begin{array}{l} \mathbf{a} \quad 3x^{-1} \\ \mathbf{g} \quad 7x^{-11} \end{array}$$

$$\mathbf{7} \quad \begin{array}{l} \mathbf{a} \quad 2 \\ \mathbf{g} \quad \frac{27}{8} \end{array}$$

$$\mathbf{8} \quad \begin{array}{l} \mathbf{a} \quad \frac{y^2}{x^4} \\ \mathbf{g} \quad \frac{4}{a^{10}} \\ \mathbf{m} \quad 12x^{10}y \end{array}$$

$$\mathbf{9} \quad \begin{array}{l} \mathbf{a} \quad 6^{-2} \\ \mathbf{g} \quad d^{-22} \\ \mathbf{m} \quad 5m^{-3} \end{array}$$

$$\mathbf{r} \quad \left(\frac{m^2}{n^3}\right)^3 \text{ or } \left(\frac{n^3}{m^2}\right)^{-3}$$

$$\mathbf{10} \quad \begin{array}{l} \mathbf{a} \quad \frac{27}{4a^2b^6} \\ \mathbf{g} \quad \frac{2a^{15}}{b^8c^3} \end{array}$$

$$\mathbf{l} \quad \frac{1}{10^6} = \frac{1}{1000000}$$

$$\mathbf{q} \quad \frac{1}{10^5} = \frac{1}{100000}$$

$$\mathbf{b} \quad \begin{array}{l} 3^{-2} \\ 2^{-6} \\ 3^{-4} \end{array}$$

$$\mathbf{b} \quad \begin{array}{l} \frac{1}{x^7} \\ \frac{9}{m^7} \end{array}$$

$$\mathbf{n} \quad \frac{1}{16x^2}$$

$$\mathbf{b} \quad \begin{array}{l} \frac{25}{4} \\ 125 \end{array}$$

$$\mathbf{b} \quad \begin{array}{l} 4^{-2} = \frac{1}{16} \\ 20^{-2} = \frac{1}{400} \end{array}$$

$$\mathbf{b} \quad \begin{array}{l} 4x^{-1} \\ \frac{3}{2}x^{-4} \end{array}$$

$$\mathbf{b} \quad \begin{array}{l} \frac{3}{2} \\ \frac{25}{16} \end{array}$$

$$\mathbf{b} \quad \begin{array}{l} \frac{a^2}{b^8} \\ \frac{2}{a^9} \\ \frac{2a^3}{b} \end{array}$$

$$\mathbf{b} \quad \begin{array}{l} 9^{-1} \\ e^{-2} \\ \frac{b^3}{a^2} \end{array}$$

$$\mathbf{b} \quad \begin{array}{l} \frac{125}{x^9y^{21}} \\ mn^3p^9 \end{array}$$

$$\mathbf{m} \quad \frac{1}{9^1} = \frac{1}{9}$$

$$\mathbf{r} \quad \frac{1}{2^7} = \frac{1}{128}$$

$$\mathbf{c} \quad \begin{array}{l} 3^{-3} \\ 2^{-9} \\ 2^{-8} \end{array}$$

$$\mathbf{c} \quad \begin{array}{l} \frac{1}{m^5} \\ x^3 \end{array}$$

$$\mathbf{o} \quad \frac{1}{343a^5}$$

$$\mathbf{c} \quad \begin{array}{l} \frac{9}{100} \\ \frac{1}{49} \end{array}$$

$$\mathbf{c} \quad \begin{array}{l} 3^{-1} = \frac{1}{3} \\ 3^{-4} = \frac{1}{81} \end{array}$$

$$\mathbf{c} \quad \begin{array}{l} 9x^{-1} \\ \frac{2x^{-6}}{5} \end{array}$$

$$\mathbf{c} \quad \begin{array}{l} \frac{5}{4} \\ \frac{16}{81} \end{array}$$

$$\mathbf{c} \quad \begin{array}{l} \frac{15y^3}{x^9} \\ \frac{9}{2a^9} \\ \frac{b^4c^5}{3a^3} \end{array}$$

$$\mathbf{c} \quad \begin{array}{l} b^{-2} \\ -5 \\ \frac{p}{m^2n^3} \end{array}$$

$$\mathbf{c} \quad \begin{array}{l} \frac{2n^{12}}{25m^8} \\ \frac{a^8}{b^7} \end{array}$$

$$\mathbf{n} \quad \frac{1}{3^4} = \frac{1}{81}$$

$$\mathbf{p} \quad \frac{1}{2^7} = \frac{1}{128}$$

$$\mathbf{d} \quad \begin{array}{l} 7^{-2} \\ 7^{-3} \\ 3^{-5} \end{array}$$

$$\mathbf{d} \quad \begin{array}{l} \frac{1}{n^4} \\ x^5 \end{array}$$

$$\mathbf{p} \quad \frac{1}{25x^7}$$

$$\mathbf{d} \quad \begin{array}{l} \frac{27}{8} \\ \frac{1}{16} \end{array}$$

$$\mathbf{d} \quad \begin{array}{l} 6^{-3} = \frac{1}{216} \\ 2^{-6} = \frac{1}{64} \end{array}$$

$$\mathbf{d} \quad \begin{array}{l} 5x^{-2} \\ \frac{4}{3}x^{-7} \end{array}$$

$$\mathbf{d} \quad \begin{array}{l} \frac{8}{11} \\ \frac{125}{216} \end{array}$$

$$\mathbf{d} \quad \begin{array}{l} 14b^2 \\ \frac{3}{m} \\ \frac{3m^6}{7p^7} \end{array}$$

$$\mathbf{d} \quad \begin{array}{l} m^{-11} \\ -3 \\ p^{-2}q^3 \end{array}$$

$$\mathbf{d} \quad \begin{array}{l} \frac{b^6}{108a^9} \\ \frac{8a^{14}}{b^6} \end{array}$$

$$\mathbf{o} \quad \frac{1}{5^2} = \frac{1}{25}$$

$$\mathbf{e} \quad \begin{array}{l} 11^{-2} \\ 2^{-2} \\ 31^{-1} \end{array}$$

$$\mathbf{e} \quad \begin{array}{l} \frac{3}{a^4} \\ 3a^4 \end{array}$$

$$\mathbf{q} \quad \frac{n^6}{169}$$

$$\mathbf{e} \quad \begin{array}{l} 7^{-2} = \frac{1}{49} \\ 10^{-4} = \frac{1}{10000} \end{array}$$

$$\mathbf{e} \quad \begin{array}{l} 6x^{-3} \\ \frac{3x^{-4}}{4} \end{array}$$

$$\mathbf{e} \quad \begin{array}{l} 4 \\ \frac{4}{25} \end{array}$$

$$\mathbf{e} \quad \begin{array}{l} \frac{56}{a^2m^7} \\ \frac{7}{t^5} \end{array}$$

$$\mathbf{e} \quad \begin{array}{l} a^{-3} \\ \frac{2}{m^2} \\ (a^3b^{-2})^2 \text{ or } (b^2a^{-3})^{-2} \end{array}$$

$$\mathbf{e} \quad \begin{array}{l} \frac{x}{y^7} \\ \frac{16c^{10}}{27a^5} \end{array}$$

$$\mathbf{f} \quad \begin{array}{l} 5^{-3} \\ 13^{-2} \\ 2^{-5} \end{array}$$

$$\mathbf{f} \quad \begin{array}{l} \frac{5}{x^7} \\ 5x^5 \end{array}$$

$$\mathbf{r} \quad \frac{x^3}{y^4}$$

$$\mathbf{f} \quad \begin{array}{l} 5^{-3} = \frac{1}{125} \\ 12^{-2} = \frac{1}{144} \end{array}$$

$$\mathbf{f} \quad \begin{array}{l} 8x^{-4} \\ \frac{2x^{-5}}{3} \end{array}$$

$$\mathbf{f} \quad \begin{array}{l} 243 \\ \frac{12}{rs^2} \end{array}$$

$$\mathbf{f} \quad \begin{array}{l} \frac{12}{rs^2} \\ \frac{4}{h^5} \end{array}$$

$$\mathbf{f} \quad \begin{array}{l} b^{-8} \\ 3a^3 \end{array}$$

$$\mathbf{f} \quad \begin{array}{l} \frac{x^8}{3y^9} \\ \frac{1}{9m^3n^6p} \end{array}$$

There is more than one answer for parts **q** and **r**.

Exercise 8C

- | | | | | | |
|---------------------------------|---|------------------------------|-------------------------------|------------------------------|----------------------|
| 1 a 2 | b 2 | c 6 | d 3 | e 4 | f 4 |
| 2 a $14^{\frac{1}{2}}$; 3.7417 | b $64^{\frac{1}{4}} = 8^{\frac{1}{2}}$; 2.8284 | c $7^{\frac{1}{5}}$; 1.4758 | d $11^{\frac{1}{7}}$; 1.4085 | e $2^{\frac{7}{3}}$; 5.0397 | |
| 3 a 2 | b 3 | c 3 | d 3 | e 8 | f 5 |
| g 5 | h 4 | i 2 | j 5 | k 6 | l 7 |
| 4 a 32 | b 125 | c 25 | d 32 | e 4 | f 27 |
| g 36 | h 27 | i 8 | j 9 | k 16 | l 4 |
| 5 a a | b b^2 | c c^3 | d c^2 | e x^2 | f y |
| g $p^{\frac{23}{20}}$ | h $q^{\frac{13}{6}}$ | i x | j $y^{\frac{1}{3}}$ | k $p^{\frac{7}{20}}$ | l $q^{\frac{5}{6}}$ |
| m $2m^3$ | n $3n^4$ | o $8x^2$ | p $81y^2$ | | |
| 6 a $\frac{1}{2}$ | b $\frac{1}{5}$ | c $\frac{5}{2}$ | d $\frac{3}{4}$ | e $\frac{1}{4}$ | f 3 |
| g $\frac{1}{3}$ | h 5 | i $\frac{3}{2}$ | j $\frac{3}{2}$ | | |
| 7 a $\frac{1}{a}$ | b $\frac{1}{b^4}$ | c $\frac{1}{8x^2}$ | d $\frac{1}{81y^2}$ | e $\frac{1}{x}$ | f $\frac{1}{y^3}$ |
| g $p^{\frac{7}{20}}$ | h $q^{\frac{5}{6}}$ | i x^2 | j y | k $p^{\frac{23}{20}}$ | l $q^{\frac{13}{6}}$ |
| m $\frac{2}{m^3}$ | n $\frac{3}{n^4}$ | o $\frac{32}{x^2}$ | | | |

Exercise 8D

- | | | | | | |
|--|---------------------------|---------------------------|-------------------------|----------------------|---------------------|
| 1 a 10^1 | b 10^2 | c 10^3 | d 10^4 | e 10^6 | |
| f 10^9 | g 10^{100} | | | | |
| 2 a 10^{-1} | b 10^{-2} | c 10^{-3} | d 10^{-12} | e 10^{-5} | f 10^{-6} |
| 3 a 5.10×10^2 | b 7.9×10^2 | c 5.3×10^3 | d 2.6×10^4 | e 6.4×10^7 | |
| f 7.96×10^8 | g 5.76×10^{11} | h 4×10^{12} | i 8×10^{-3} | j 6×10^{-2} | |
| k 7.2×10^{-4} | l 4.1×10^{-5} | m 6×10^{-9} | n 2.06×10^{-7} | | |
| 4 a 32400 | b 7200 | c 860 | d 2700000 | e 5.1 | f 72 |
| g 0.056 | h 0.0017 | i 0.000872 | j 0.00201 | k 0.97 | l 0.00000026 |
| 5 6×10^{24} kg | 6 6.4×10^3 km | 7 2.99×10^5 km/s | | | |
| 8 10^{-4} m | 9 8.9×10^{-3} kg | 10 4×10^{-8} m | | | |
| 11 a 8×10^{11} | b 6.3×10^{13} | c 2×10^{-4} | d 2.4×10^4 | e 2.5×10^1 | f 2×10^6 |
| g $\approx 7.5 \times 10^0$ | h 3×10^{-2} | i 1.94481×10^9 | j 2.7×10^{-5} | k 5×10^6 | l 6.4×10^2 |
| m 8×10^4 | n 9.6×10^{12} | o 1×10^7 | p 5×10^0 | | |
| 12 $\approx 7.57 \times 10^{17}$ km | 13 1840 | | | | |
| 14 8.3 minutes ($\approx$ 8 minutes 18 seconds) | | | | | |
| 15 $\approx 4.35 \times 10^{22}$ | | | | | |
| 16 5656 days and 7 hours | | | | | |

Exercise 8E

- | | | | | | |
|--------------------------|-------------------------|------------------------|------------------------|--------------------------|-------------------------|
| 1 a 2.70×10^0 | b 6.35×10^2 | c 8.76×10^3 | d 2.56×10^5 | e 3.61×10^{-3} | f 2.42×10^{-2} |
| 2 a 3.7×10^2 | b 2.8×10^5 | c 4.3×10^{-3} | d 2.2×10^{-5} | | |
| 3 a 2.746×10^2 | b 2.75×10^2 | c 2.7×10^2 | d 3×10^2 | e 4.124×10^{-2} | f 4.12×10^{-2} |
| g 4.1×10^{-2} | h 4×10^{-2} | i 1.704×10^3 | j 1.70×10^3 | k 1.7×10^3 | l 2×10^3 |
| m 1.993×10^{27} | n 1.99×10^{27} | o 2.0×10^{27} | p 2×10^{27} | | |

- 4 a 2.17×10^{-1} b 2.40×10^2 c 1.11×10^8 d 8.70×10^{-3} e 9.61×10^3 f 6.48×10^{-4}
g 6.40×10^1 h 2.55×10^1 i 1.72×10^0 j 2.02×10^0 k -4.02×10^2
- 5 a 2.453×10^{-1} b 3.851×10^{11} c 6.207×10^0 d 6.141×10^{31} e 7.241×10^0 f 6.271×10^{43}
g 9.870×10^{-1} h 2.205×10^{-6} i 5.247×10^{-1} j 1.792×10^{13} k 3.956×10^{-2} l 2.646×10^4
m 5.577×10^{-1} n 4.547×10^2 o 1.380×10^3

Review exercise

- 1 a 64 b 64 c 64 d 1000000
- 2 a $2^6 \times 3^2 \times 5^2$ b $2^6 \times 3^6 \times 5^6$ c $3^8 \times 5^4 \times 7^4$ d $2^5 \times 5^5 \times 7^{10}$
- 3 a a^{13} b b^{13} c $15a^9$ d $10x^9$ e a^3 f m^{18}
g $2b^5$ h $2p^9$ i a^{12} j b^{30} k $8a^{21}$ l $81m^8$
m 1 n 3 o 5 p 1
- 4 a $20a^3b^7$ b $10m^{10}n^{10}$ c $4a^2b$ d $\frac{4}{3}m^3n^2$ e $81a^{12}b^4$ f $100a^8b^5$
g $15a^7b^7$ h $\frac{16}{3}m^7n^{12}$ i $\frac{x^9}{5y^5}$ j $\frac{16a^3}{5}$ k $\frac{2x^5y^5}{3}$ l $144a^6b^3$
- 5 a $\frac{1}{36}$ b $\frac{1}{512}$ c $\frac{1}{128}$ d $\frac{1}{64}$ e $\frac{25}{16}$ f $\frac{81}{16}$
g $\frac{1}{2\sqrt{2}}$ h $\frac{1}{8}$
- 6 a $\frac{m^4}{4p^9}$ b $\frac{1}{9x^2}$ c $\frac{1}{64y^3}$ d $\frac{1}{2^{10}y^{15}}$ e $\frac{5^{10}}{x^{15}}$ f $\frac{7^4}{a^2b^4}$
g $\frac{3^{12}b^4}{a^8}$ h $\frac{b^4}{a^6}$ i $\frac{p^{10}}{a^8}$ j $\frac{a^6b^9}{27}$ k $\frac{1}{25g^4h^6}$ l $\frac{m^6}{2^8n^8}$
- 7 a $\frac{20}{a}$ b $\frac{40n^3}{m^2}$ c $2a$ d $\frac{2}{m^2n^2}$ e $\frac{n^6}{64m^6}$ f $\frac{1}{250m^8n^5}$
g $\frac{b^{16}}{162a^{14}}$ h $\frac{4m^2n^8}{15}$ i $\frac{5a^8}{b^8}$
- 8 a 7 b 2 c 25 d 16 e 27 f $\frac{1}{64}$
g 4 h $\frac{9}{16}$
- 9 a $\frac{5}{a^6}$ b $12b^{\frac{5}{3}}$ c $p^{\frac{1}{6}}$ d $m^{\frac{3}{2}}$ e $256x^{\frac{4}{3}}$ f $\frac{2}{x^{\frac{3}{4}}}$
g $3a^{\frac{1}{6}}$ h $\frac{2^{\frac{3}{2}}q^{\frac{3}{2}}}{p}$
- 10 a 4.7×10^4 b 1.64×10^8 c 4.7×10^{-3} d 3.5×10^{-3} e 8.4×10^2 f 8.40×10^{-1}
- 11 a 68000 b 7500 c 0.0026 d 0.094 e 6.7 f 0.00032
- 12 a 8×10^8 b 6.2×10^2 c 3×10^6 d 2.6×10^{-10}
e 2.7×10^{13} f 2.5×10^{-7} g 8×10^{-1} h 3×10^{14}
- 13 a 1.9×10^1 b 1.86×10^1 c 2×10^1 d 4.3×10^{-3} e 6.0×10^3 f 4.740×10^{-1}

Challenge exercise

- 1 a 12006 b 35.6 c 77%
d 2.4, 125.0, 105.8, 4581.4, 12.6, 28.4
i 52.2 ii Australia iii Singapore
- e i 1.95×10^7 , 1.27×10^9 , 2.18×10^8 , 3.06×10^6 , 3.50×10^6 , 2.74×10^8
ii 2.09×10^7 , 1.34×10^9 , 2.36×10^8 , 3.24×10^6 , 3.61×10^6 , 2.88×10^8
iii 7.31×10^7 , 3.58×10^9 , 9.83×10^8 , 8.66×10^6 , 6.18×10^6 , 7.06×10^8
f 9.6 , 3.7×10^2 , 5.2×10^2 , 1.4×10^4 , 2.3×10^1 , 7.7×10^1

- 2 a 9 cm b i 1 cm ii 12 cm iii $1\frac{1}{3}$
 c i 48 ii $\frac{1}{3}$ cm iii 16 cm iv $a = \frac{4}{3}$
 d i $\left(\frac{4}{3}\right)^3 \times 9$ ii $\left(\frac{4}{3}\right)^4 \times 9$ iii $\left(\frac{4}{3}\right)^{10} \times 9$ iv $\left(\frac{4}{3}\right)^n \times 9$
 e i 9 ii 17 iii 33 f becomes very large
 3 a $100 \text{ mm} \times 136 \text{ mm}$ b $80 \text{ mm} \times 108.8 \text{ mm}$
 c i $(125 \times 0.8^3 \text{ mm}), (170 \times 0.8^3 \text{ mm})$ ii $(125 \times 0.8^4 \text{ mm}), (170 \times 0.8^4 \text{ mm})$
 iii $(125 \times 0.8^n \text{ mm}), (170 \times 0.8^n \text{ mm})$
 d $21\,250 \text{ mm}^2$
 e i $21\,250 \times 0.8^2 \text{ mm}^2$ ii $21\,250 \times 0.8^4 \text{ mm}^2$
 iii $21\,250 \times 0.8^6 \text{ mm}^2$ iv $21\,250 \times 0.8^{2n} \text{ mm}^2$
 f i $x \times 0.8^n \text{ cm}, y \times 0.8^n \text{ cm}$ ii $xy \times 0.8^{2n} \text{ cm}^2$
 4 2^{n+2} 5 a $\left(\frac{2}{3}\right)^n$ b 2^{-3n^2+4n+2} c $\frac{16}{3 \times 2^n}$ d $x^n(a + bx^2)$
 6 a $10p^2$ b $\frac{10}{p^3}$ 7 $(xyz)^{a+b+c}$
 8 a $x^2 - 2 + \frac{1}{x^2}$ b $x^4 + 2x^{\frac{5}{2}} + x$ c $x^{2n} - 6 + \frac{9}{x^{2n}}$
 9 a $-\frac{1}{2}$ b $-(xy)^{\frac{1}{2}}$ c $\frac{\frac{1}{x^2} - 1}{\frac{1}{x^2} - 2}$ d $x^{\frac{1}{2}} + y^{\frac{1}{2}}$

Chapter 9 Answers

Exercise 9A

- 3 b centre: lamp, enlargement factor: $\frac{5}{2}$ 4 b centre the light, enlargement factor: 11
 5 4 m
 6 Venus 1.87 m; Earth 2.58 m; Mars 3.94 m; Jupiter 13.44 m; Saturn 24.64 m; Uranus 49.57 m; Neptune 77.65 m
 7 Carbon 2.8 cm; gold 5.4 cm; radium 8.6 cm
 8 a 125 km b 500 km c 75 km d 12.5 km

Exercise 9B

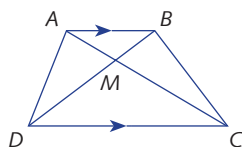
- 1 a i $\frac{PR}{AC} = \frac{PQ}{AB}$ ii $\frac{QR}{BC} = \frac{RP}{CA}$ b i $\frac{DE}{RS} = \frac{DF}{RT}$ ii $\frac{FE}{TS} = \frac{FD}{TR}$
 c i $\frac{AB}{SR} = \frac{AD}{ST}$ ii $\frac{CB}{UR} = \frac{CD}{UT}$ d i $\frac{JK}{AE} = \frac{MN}{DB}$ ii $\frac{JL}{AC} = \frac{MN}{DB}$
 2 a $\frac{x}{5} = \frac{2}{6}; x = \frac{5}{3}$ b $\frac{x}{2} = \frac{5}{3}; x = \frac{10}{3}$ c $\frac{x}{4} = \frac{6}{9}; x = \frac{8}{3}$
 d $\frac{x}{3} = \frac{3}{6}; x = \frac{3}{2}$ e $\frac{x}{2} = \frac{8}{3}; x = \frac{16}{3}$ f $\frac{x}{3} = \frac{9}{7}; x = \frac{27}{7}$
 3 a 7.5 b 1.2
 4 a 0.7 b 0.6 c 6 d 2.2 cm
 5 a $\angle ACB = \angle PRQ$ b $\angle CAB = \angle RPQ$ c $\frac{AB}{PQ} = \frac{AC}{PR} = \frac{BC}{QR}$
 6 a 29° b 32 cm c 104° d 16 cm

- 7 a 36° b 88° c 21.6 cm d 12.6 cm
 8 a 2 b $\frac{4}{3}$ c $\frac{8}{3}$ d 2

Exercise 9C

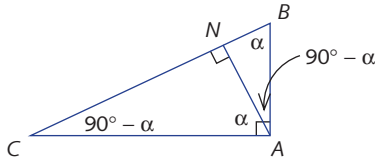
- 1 a $\triangle ABC$ is similar to $\triangle AMX$ (AAA); $\frac{AB}{AM} = \frac{BC}{MX}$
 b $\triangle ABC$ is similar to $\triangle APB$ (AAA); $\frac{AB}{AP} = \frac{BC}{PB}$
 c $\triangle ABC$ is similar to $\triangle PQC$ (AAA); $\frac{AB}{PQ} = \frac{BC}{QC}$
 d $\triangle ABC$ is similar to $\triangle ADB$ (AAA); $\frac{AB}{AD} = \frac{BC}{DB}$
- 2 a $\triangle ACQ$ is similar to $\triangle ABP$ (AAA); $\angle BAP = \angle CAQ$ (common);
 $\angle QCA = \angle PBA$ (corresponding angles, $BP \parallel CQ$); $\frac{AB}{AC} = \frac{AP}{AQ}$
 b $\triangle KFL$ is similar to $\triangle MGL$ (AAA); $\angle KLF = \angle GLM$ (vertically opposite);
 $\angle FKL = \angle GML$ (alternate angles, $KF \parallel GM$); $\frac{GL}{FL} = \frac{GM}{FK}$
- 3 a $\angle ABC = \angle DEC$ (given); $\angle ACB = \angle DCE$ (vertically opposite at C);
 $\triangle BCA$ is similar to $\triangle ECD$ (AAA); $\frac{x}{12} = \frac{6}{8}$; $x = 9$
 b $\angle PRQ = \angle TRS$ (given); $\angle PQR = \angle TSR = 90^\circ$;
 $\triangle PQR$ is similar to $\triangle TSR$ (AAA); $\frac{x}{9} = \frac{9}{6}$; $x = 13\frac{1}{2}$
 c $\angle CMD = \angle BMA$ (vertically opposite at M); $\angle DCM = \angle ABM$ (alternate angles, $AB \parallel CD$);
 $\triangle AMB$ is similar to $\triangle DMC$ (AAA); $\frac{x}{5} = \frac{7}{3}$; $x = 11\frac{2}{3}$
 d $\angle KMP = \angle LMQ$ (common); $\angle KPM = \angle LQM$ (corresponding angles, $KP \parallel LQ$);
 $\triangle KPM$ is similar to $\triangle LQM$ (AAA); $\frac{x}{4} = \frac{7}{8}$; $x = 3\frac{1}{2}$
 e $\angle QRS = 60^\circ$ (angle sum of triangle); $\angle PRQ = \angle QRS = 60^\circ$;
 $\angle RQS = \angle RPQ = 90^\circ$; $\triangle PRQ$ is similar to $\triangle QRS$ (AAA); $\frac{x}{1} = \frac{1}{2}$; $x = \frac{1}{2}$
 f $\angle DAB = \angle EAC$ (common); $\angle AEC = \angle ADB$ (corresponding angles, $DB \parallel EC$);
 $\triangle ACE$ is similar to $\triangle ABD$ (AAA); $\frac{x+4}{4} = \frac{10}{6}$; $x = 2\frac{2}{3}$
- 4 a $6\frac{2}{3}$ m b $7\frac{1}{2}$ m c 3000 m d 1.28 m
- 5 a $\alpha = 65^\circ$, $\beta = 25^\circ$, $\gamma = 65^\circ$
 b $\triangle ABC$ is similar to $\triangle AMB$ is similar to $\triangle BMC$
 c i $\frac{a}{b} = \frac{y}{h} = \frac{h}{x}$ ii $\frac{h}{a} = \frac{b}{x+y} = \frac{x}{b}$

6



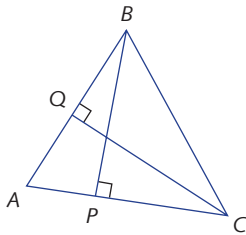
$\angle BMA = \angle DMC$ (vertically opposite at M);
 $\angle ABM = \angle CDM$ (alternate angles, $AB \parallel DC$);
 $\triangle ABM$ is similar to $\triangle CDM$ (AAA)

7



- a Let $\angle NBA = \alpha$.
 $\angle BAN = \angle ACN = 90^\circ - \alpha$ (angle sum of triangle)
 $\angle ANB = \angle CNA = 90^\circ$ (given)
 $\triangle ABN$ is similar to $\triangle CAN$ (AAA)
 b $\frac{AN}{CN} = \frac{BN}{AN}$, $AN^2 = BN \times CN$

8



- a $\angle BAP = \angle CAQ$ (common);
 $\angle APB = \angle AQC = 90^\circ$;
 $\triangle APB$ is similar to $\triangle AQC$ (AAA)
 b $\frac{BP}{CQ} = \frac{AB}{AC}$ (matching sides of similar triangles)
 c $\frac{1}{2} \times AC \times BP = \frac{1}{2} \times AB \times CQ$ (both $\frac{1}{2} \times \text{base} \times \text{height} = \text{area of same triangle } ABC$)
 d So $\frac{BP}{CQ} = \frac{AB}{AC}$

Exercise 9D

- 1 a $QP = PR = 7\text{ cm}$ ($\triangle PQR$ is isosceles), $\angle BAC = \angle QPR$ and $\frac{AB}{PQ} = \frac{AC}{PR} = \frac{5}{7}$. Thus $\triangle PQR$ is similar to $\triangle ABC$ (SAS).

- b $\angle BAC = \angle EDF = 50^\circ$ and $\frac{BA}{ED} = \frac{AC}{DF} = 2$. Thus $\triangle BAC$ is similar to $\triangle EDF$ (SAS).

- c $\angle ABC = \angle PQR = 120^\circ$ and $\frac{CB}{RQ} = \frac{AB}{PQ} = 2$. Thus $\triangle BAC$ is similar to $\triangle QPR$ (SAS).

- d $\angle BAC = \angle QPR = 40^\circ$ and $\frac{AB}{PQ} = \frac{AC}{PR} = 5$. Thus $\triangle BAC$ is similar to $\triangle QPR$ (SAS).

- 2 a $AB = 8$ and $AC = 12.8$. $\angle BAC = \angle PAQ$. $\frac{AB}{AP} = \frac{AC}{AQ} = \frac{8}{5}$. Thus $\triangle BAC$ is similar to $\triangle PAQ$ (SAS).

Hence $\angle APQ = \angle ABC$ (matching angles in similar triangles). Thus $PQ \parallel BC$ (corresponding angles equal)

- b $AB = 10$ and $AC = 7.5$. $\angle BAC = \angle PAQ$. $\frac{AB}{AP} = \frac{AC}{AQ} = \frac{5}{4}$. Thus $\triangle BAC$ is similar to $\triangle PAQ$ (SAS).

Hence $\angle APQ = \angle ABC$ (matching angles in similar triangles). Thus $PQ \parallel BC$ (corresponding angles equal)

- 3 $\angle BAC = \angle DAE$, $\angle ADE = \angle ABC$ (corresponding angles, $DE \parallel BC$), $\angle AED = \angle ACB$ (corresponding angles, $DE \parallel BC$). Thus $\triangle BAC$ is similar to $\triangle DAE$ (AAA).

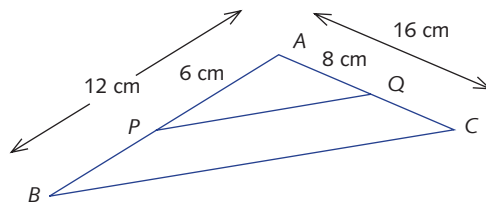
$$\frac{AB}{AD} = \frac{AC}{AE} = \frac{4}{3}$$

- 4 $\angle HAI$ is common to all triangles. Then, for example $\frac{AD}{AB} = \frac{AE}{AC} = 2$. Thus $\triangle ABC$ is similar to $\triangle ADE$ (SAS).

The other two triangles are similar to these two triangles.

$$z = 8, y = 16 \text{ and } x = 24.$$

5



$$\angle BAC = \angle PAQ \quad (\text{common})$$

$$\frac{AB}{AP} = \frac{AC}{AQ} = 2$$

$\triangle BAC$ is similar to $\triangle PAQ$ (SAS)

Therefore, $\angle ACB = \angle AQP$ (matching angles in similar triangles)

Thus $PQ \parallel BC$ (corresponding angles equal)

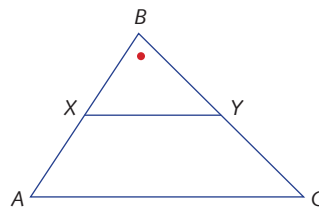
- 6 $\angle P = \angle Z$ and $\frac{PQ}{XZ} = \frac{PR}{YZ}$. Therefore $\triangle RPQ$ is similar to $\triangle YZX$ (SAS).

$$\angle Q = \angle X, \angle R = \angle Y$$

- 7 $\triangle ABD$ is similar to $\triangle CDB$ (SAS). Therefore $\angle CBD = \angle ADB$ and $\angle CDB = \angle ABD$ (matching angles of similar triangles). Hence $BC \parallel AD$ and $CD \parallel AB$ and $ABCD$ is a parallelogram.

Exercise 9E

- 1 a $\triangle ABC$ is similar to $\triangle GHI$ (AAA) b $\triangle KML$ is similar to $\triangle RTS$ (SAS)
c $\triangle DEF$ is similar to $\triangle GHI$ (SSS) d $\triangle KLM$ is similar to $\triangle NPQ$ (AAA)
e $\triangle LKM$ is similar to $\triangle UST$ (SAS) f $\triangle DEF$ is similar to $\triangle LMK$ (SSS)
- 2 a $\triangle ABC$ is similar to $\triangle APB$ (SAS); $\frac{AB}{AP} = \frac{BC}{PB}$
b $\triangle ABC$ is similar to $\triangle APB$ (SSS); $\frac{AB}{AP} = \frac{BC}{PB}$
c $\triangle ABC$ is similar to $\triangle BMC$ (RHS); $\frac{AB}{BM} = \frac{BC}{MC}$
d $\triangle ABC$ is similar to $\triangle ACD$ (AAA); $\frac{AB}{AC} = \frac{BC}{CD}$
- 3 a $a = 6$ b $b = 9$ c $x = \frac{6}{5}$
- 4 a $x = \frac{32}{3}$ b $x = 6$ c $y = \frac{5}{2}$ d $y = 2.25$
e $x = \frac{10}{3}$ f $a = \frac{9}{2}, b = 6$ g $x = \frac{15}{2}$ h $x = \frac{8}{5}$
- 5 a i 30° ii 30° b AAA c $6\sqrt{3}$ cm e $3\sqrt{3}$ cm
- 6 a $\frac{8}{3}$ b $\frac{8}{5}$ c $\frac{32}{15}$
- 7 a i $\frac{AB}{AL} = \frac{AC}{AM}$ and $\angle BAC = \angle LAM$ (common); $\triangle BAC$ is similar to $\triangle LAM$ (SAS)
ii $\angle ABC = \angle ALM$ (matching angles of similar triangles). Therefore $LM \parallel BC$ (corresponding angles equal)
iii $\frac{x}{3} = \frac{4}{1}, x = 12$
b i $\frac{CM}{MA} = \frac{CD}{AB}$ and $\angle CMD = \angle AMB = 90^\circ$; $\triangle AMB$ is similar to $\triangle CMD$ (RHS)
ii $\angle ABM = \angle CDM$ (matching angles of similar triangles).
iii No (alternate angles not equal)
c i $\angle QOP = \angle GOF$ (common); $\frac{OG}{OQ} = \frac{OF}{OP} = \frac{3}{2}$; $\triangle OPQ$ is similar to $\triangle OFG$ (SAS)
ii $\angle OPQ = \angle OFG$ (matching angles of similar triangles). Therefore $PQ \parallel FG$ (corresponding angles equal)
iii $x = 21$
d i BDC ; $\frac{BD}{BA} = \frac{DC}{BC} = \frac{BC}{AC}$; $\triangle BDC$ is similar to $\triangle ABC$ (SSS) ii $\beta = \theta$
e i $\angle BAC = \angle FCB$ (given) and $\angle FBC = \angle ABC$ (common); $\triangle BAC$ is similar to $\triangle FCB$ (AAA)
ii $x = 3, y = 9$ ($\triangle ABC$ isosceles)
f i CBD ; $\frac{BA}{BC} = \frac{BC}{BD} = a$. $\angle ACB = \angle BDC = 90^\circ$. $\triangle ABC$ is similar to $\triangle CBD$ (RHS) ii α
- 8 a Let X and Y be the midpoints of AB and BC respectively.
 $BA = 2BX$ and $BC = 2BY$
Therefore $\frac{BA}{BX} = 2$ and $\frac{BC}{BY} = 2$
 $\angle ABC = \angle XBY$ (common angle)
 $\triangle ABC$ is similar to $\triangle XBY$
Therefore $XY = \frac{1}{2} AC$ (matching sides of similar triangles) and $\angle BXY = \angle BAC$
(matching angles of similar triangles)
Thus $AC \parallel XY$ (corresponding angles equal)

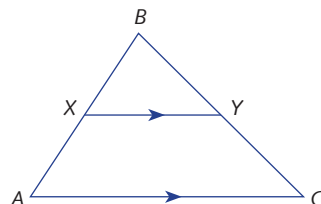


- b** Let X be the midpoint of AB and Y a point on BC such that $XY \parallel AC$.

$\angle BXY = \angle BAC$ (corresponding angles, $XY \parallel AC$). Therefore $\triangle ABC$ is similar to

$$\triangle XBY \quad \frac{BY}{BC} = \frac{BX}{BA} = \frac{1}{2}. \text{ (matching sides of similar triangles)}$$

Thus Y is the midpoint of BC .



- c** Draw diagonals AC and BD . $\triangle ABC$ and $\triangle ADC$, $PQ \parallel AC \parallel SR$ and similarly $PS \parallel BD \parallel QR$. Opposite sides of a quadrilateral are parallel implies that $PQRS$ is a parallelogram.
- 9** We know the interval joining midpoints of two sides of a triangle is parallel to the third side.
Therefore $PR \parallel AC$ and $PQ \parallel AB$ and $RQ \parallel BC$ and $RP = \frac{1}{2} AC$, $PQ = \frac{1}{2} AB$ and $RQ = \frac{1}{2} BC$.
 $\triangle ARQ \equiv \triangle RBP \equiv \triangle QPC$ (SSS) (Each side lengths is equal to half the length of a side of $\triangle ABC$)
 $\triangle ABC$ is similar to $\triangle PQR$ (AAA)
- 10 a** $\triangle APQ$ is similar to $\triangle ABC$ (SAS) with $\frac{BA}{PA} = \frac{CA}{AQ} = \frac{7}{2}$ and $\angle BAC = \angle PAQ$ (common)
b $\frac{BC}{PQ} = \frac{7}{2}$ (matching sides of similar triangles).
c $\angle APQ = \angle ABC$ (matching sides of similar triangles). Hence $PQ \parallel BC$.
- 11 a** True, AAA (all angles are 60°)
b False. For example, an isosceles triangle with base angles of 30° is not similar to an isosceles triangle with base angles 60° .
c True, AAA. If the apex angle is a° , each of the two base angles $= 90^\circ - \frac{a}{2}$
d False. Draw two isosceles triangles with common base but different base angles.
e False. The right-angled triangles with sides 3, 4, 5 is not similar to the right-angled triangle with sides 5, 12, 13.
f False. Draw a circle with the fixed hypotenuse as diameter. Draw the triangles with the third vertex on the circle.
g True (SAS) **h** True (AAA)
i False. The angle between the two sides can vary.
j False. The ratio could involve the hypotenuse or not involve the hypotenuse.
- 12 a** True **b** False **c** True **d** True **e** False **f** False **g** True
- 13 a** $AD = 15$, $DC = 20$, $BC = 16$ **b** $AM = 12$, $BM = 16$, $DM = 9$

Review exercise

- 1 a** $\angle BAC = \angle CDE$ (alternate angles) $AB \parallel ED$;
 $\angle BCA = \angle ECD$ (vertically opposite angles at C);
 $\triangle BCA$ is similar to $\triangle ECD$ (AAA); $e = 7\frac{1}{2}$ and $c = 9$
- b** $\angle NJK = \angle MJL$ (common), $\angle NKJ = \angle MLK = 90^\circ$;
 $\triangle JNK$ is similar to $\triangle JML$ (AAA); $x = 4$ and $y = 2\frac{2}{3}$
- 2 a** $\triangle BAD$ is similar to $\triangle CAB$ is similar to $\triangle CBD$; $a = 31\frac{1}{5}$ and $b = 28\frac{4}{5}$ **b** $\triangle MJL$ is similar to $\triangle NJK$; $x = 3\frac{1}{5}$
- 3 a** $\triangle ABD$ is similar to $\triangle ACE$, $b = 2.5$, $a = 2$ **b** $x = 2$ and $y = 4.8$, $\triangle SPR$ is similar to $\triangle TPQ$
- 4** $16\frac{2}{3}$ cm
- 5 a** $\angle BAC = \angle PAQ$ (common angle);
 $\angle APQ = \angle ABC$ (corresponding angles, $PQ \parallel BC$), $\triangle PAQ$ is similar to $\triangle BAC$ (AAA).
b $x = 3.75$
- 6 a** $CB = 9$ cm **b** $CB = 1\frac{1}{4}$ cm
- 7** $\angle ACB = \angle DFE = 20^\circ$
 $\angle CAB = \angle FDE = 120^\circ$
 $\triangle ABC$ is similar to $\triangle DEF$ (AAA); $2\frac{2}{5}$ cm

- 8 a $\angle BAD = \angle ABE$ (alternate angles, $AD \parallel BC$); $\angle AEB = \angle ABE$ (isosceles);
 $\angle ADF = \angle BAD$ (alternate angles, $AB \parallel CD$); $\angle AFD = \angle ADF$ (isosceles);
 $\triangle AEB$ is similar to $\triangle ADF$ (AAA)
- b In $\triangle ABF$ and $\triangle AED$, $AB = AE$ (given); $AD = AF$ (given);
 $\angle EAD = \angle BAD + \angle EAB$ (adjacent angles) and
 $\angle FAB = \angle BAD + \angle DAF$ (adjacent angles);
 $\angle DAF = \angle EAB$ (matching angles of similar triangles);
 $\triangle ABF \equiv \triangle AED$ (SAS) and $DE = BF$ (matching sides of congruent triangles)
- 9 a $\frac{BA}{AM} = \frac{CA}{AN}$, so $\triangle AMN$ is similar to $\triangle ABC$ (SAS). Therefore $BC \parallel MN$
- b $\angle PBC = \angle PNM$ (alternate angles, $BC \parallel MN$); $\angle MPN = \angle BPC$ (vertically opposite).
 So $\triangle MPN$ is similar to $\triangle CPB$; $\frac{BP}{PN} = \frac{BC}{MN} = 3$

10 20 m

11 50 m

Challenge exercise

Only outlines are given for some proofs in this section.

1 *Hint:* Draw the diagonals of the quadrilateral.

2 By similar triangle, $\frac{x}{ka} = \frac{c}{c + kc}$

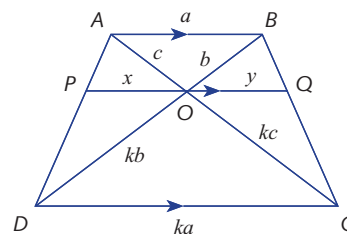
$$x = \frac{ka}{1 + k}$$

$$\frac{y}{ka} = \frac{b}{kb + b}$$

$$y = \frac{ak}{1 + k}$$

Therefore,

$$x = y$$



3 a $\triangle AKM \equiv \triangle CLM$ b 1 : 2 c 1 : 12

5 a $\triangle ABC$ is similar to $\triangle ADE$ is similar to $\triangle AFG$ (AAA)

$$\frac{x}{y} = \frac{x + b}{y + d} \quad \text{and} \quad \frac{x}{y} = \frac{x + a + b}{y + c + d}$$

$$xy + xd = yx + yb \quad \text{and} \quad xy + xc + xd = yx + ya + yb$$

$$xd = yb$$

$$\frac{x}{y} = \frac{b}{d}$$

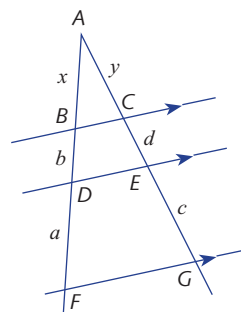
and

$$x(c + d) = y(a + b)$$

$$\frac{x}{y} = \frac{a + b}{c + d} = \frac{b}{d}$$

$$ad + bd = bc + bd$$

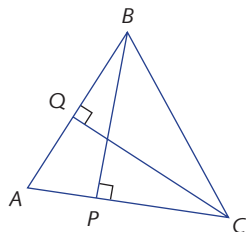
$$\text{Therefore } \frac{a}{b} = \frac{c}{d}$$



b Yes

c $\frac{8}{3}$

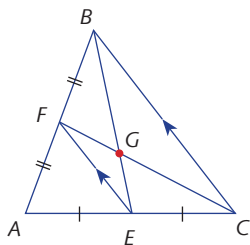
6



$$\triangle APB \text{ is similar to } \triangle AQC. \quad \frac{BA}{CA} = \frac{PB}{CQ}$$

If $PB = CQ$ then $BA = CA$. The triangle is isosceles.

7 a



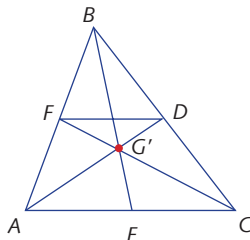
Draw FE .

$FE \parallel BC$ (E and F are midpoints of AB and AC respectively)

$BC = 2FE$ (E and F are midpoints of AB and AC respectively)

$\triangle FGE$ is similar to $\triangle CGB$. $CG = 2GF$ and $BG = 2GE$

b



Let D be the midpoint of BC . AD is the median from A .

$FD \parallel AC$ (F and D are midpoints)

$AC = 2FD$ (D and F are midpoints)

$\triangle AG'C$ is similar to $\triangle DG'F$. $G'C = 2G'F$.

Hence $G' = G$

c If $BE = CF$, then $BG = CG$ ($BG = \frac{2}{3}BE$ and $CG = \frac{2}{3}CF$), $\triangle BGC$ is isosceles and therefore $\angle BCG = \angle CBG$,

$\triangle CFB \equiv \triangle BEC$ (SAS), $BF = CE$ (matching sides), $AB = AC$ and triangle ABC is isosceles.

8 First, we prove the result when $k = 2$.

Let F be the midpoint of QR . Then $QF = FR = a$.

Construct G on RP and H on QP so that $FG \parallel QP$ and $FH \parallel RP$. Omitting the details of the angle-chasing, $\triangle RFG \equiv \triangle FQH$ (AAS congruence test).

Using opposite sides of parallelograms and matching sides of congruent triangles.

$HP = FG = QH$ and $RG = FH = GP$, as required.

Secondly, we prove the result when $k = 3$.

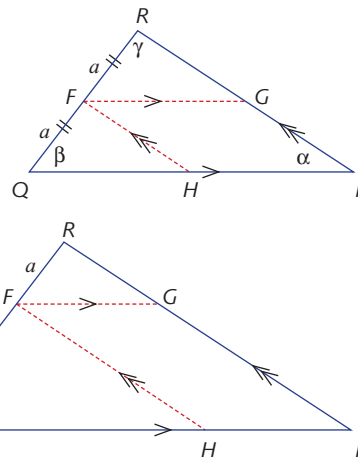
Let F be the divide RQ in the ratio $1 : 2$. Then $FR = a$ and $FQ = 2a$

Construct G on RP and H on QP so that $FG \parallel QP$ and $FH \parallel RP$.

Omitting the details of the angle-chasing, $\triangle RFG$ is similar to $\triangle FQH$ (AAA similarity test with ratio $1 : 2$).

Using opposite sides of parallelograms and matching sides of congruent triangles.

$HP = FG = \frac{1}{2}QH$ and $RG = \frac{1}{2}FH = \frac{1}{2}GP$, as required.



Chapter 10 answers

Chapter 10A review

Chapter 1: Algebra

- | | | | | | |
|--------------------------|----------------------|-----------------------|-----------------------|---------------------|-------------------|
| 1 a 24 | b 23 | c 87 | d $\frac{111}{50}$ | | |
| 2 a $2a + 6b$ | b $4x^2y + 4xy$ | c $14n^2 - 2m$ | d $p^2 - 7p + 15$ | | |
| 3 a $14a^2b$ | b $6xy$ | c $4y$ | d $15a$ | | |
| 4 a $\frac{-7a}{15}$ | b $\frac{-x}{40}$ | c $\frac{2a^2}{15}$ | d $\frac{a^2}{7}$ | e $\frac{7}{8}$ | f $\frac{ab}{18}$ |
| 5 a $3a + 12$ | b $2b + 12$ | c $4b - 20$ | d $6x - 6$ | e $6b + 4$ | f $20d - 5$ |
| g $-9d + 6$ | h $-10\ell + 8$ | i $-6x^2 - 2x$ | j $8x^2 + 12x$ | | |
| 6 a $7a + 26$ | b $5b + 14$ | c $17x + 2$ | d $23d - 38$ | e $27e + 29$ | f 3 |
| g $2x^2 + 11x - 6$ | h $26x^2 - 18x$ | i $-5x^2 + 14x$ | j $-8x^2 + 50x$ | | |
| 7 a $\frac{7x + 15}{12}$ | b $\frac{5x - 8}{6}$ | c $\frac{5x + 1}{12}$ | d $\frac{5x - 1}{12}$ | | |
| 8 a $x^2 + 8x + 15$ | b $x^2 + 11x + 28$ | c $x^2 - 4x - 12$ | d $x^2 + 4x - 21$ | e $x^2 + 4x - 45$ | f $x^2 + 5x - 24$ |
| g $6x^2 - x - 2$ | h $12x^2 + 29x + 15$ | i $10x^2 + 11x - 6$ | j $6x^2 - 29x + 28$ | k $x^2 - 25$ | l $x^2 - 49$ |
| m $4x^2 - 9$ | n $9x^2 - 25$ | o $x^2 + 14x + 49$ | p $x^2 + 6x + 9$ | q $4x^2 - 20x + 25$ | |
| r $9x^2 - 24x + 16$ | s $12x - 12$ | t $24x$ | u $8x^2 + 6x + 1$ | v $-4x^2 + 9x - 6$ | |

- 9 a 7, 21 b 3, 6 c $x, 5$ d $x, 6$ e 6, $11x$ f $2x, 7, 25x$

Chapter 2: Pythagoras' theorem and surds

- 1 a 14.4 b 13.2 c 13.1 d 37.9 2 no
 3 a 5 cm b 13 cm c 8.5 cm d 0.5 cm e 2.6 cm f 25.5 cm
 4 5.25 m 5 436 m
 6 a $2\sqrt{5}$ b $5\sqrt{3}$ c $6\sqrt{2}$ d $20\sqrt{2}$ e $30\sqrt{3}$ f $27\sqrt{3}$
 7 a $\sqrt{12}$ b $\sqrt{18}$ c $\sqrt{500}$ d $\sqrt{112}$
 8 a $11\sqrt{2}$ b $3\sqrt{3}$ c $20\sqrt{6}$ d $12\sqrt{35}$ e $7\sqrt{2}$ f $\sqrt{3}$
 g $23\sqrt{3}$ h $26\sqrt{2}$
 9 a $\sqrt{6} + 2\sqrt{5}$ b $12 - 5\sqrt{3}$ c $6\sqrt{10} - 60$ d $6\sqrt{6} + 16$ e $16 - \sqrt{3}$
 f $19 - 13\sqrt{2}$ g $19 - 6\sqrt{2}$ h $6 + 2\sqrt{5}$ i -78 j -13
 10 a $\sqrt{3}$ b 2 c $\frac{\sqrt{3}}{6}$ d $\frac{5\sqrt{6}}{6}$
 11 a $3\sqrt{10} - 6\sqrt{2}$ b $\frac{6 + \sqrt{3}}{11}$ c $3\sqrt{10} + \sqrt{5} - 6\sqrt{2} - 2$ d $\sqrt{6} + \sqrt{3} - 2 - \sqrt{2}$
 12 a 17 cm b $5\sqrt{13}$ cm c 10 cm d $\frac{\sqrt{481}}{2}$ cm e 12.5 cm f $\frac{3\sqrt{41}}{2}$ cm
 13 12.8 cm

Chapter 3: Consumer arithmetic

- 1 a $\frac{9}{50}$ b $\frac{16}{25}$ c $\frac{13}{500}$ d $\frac{17}{200}$ e $\frac{3}{8}$ f $\frac{1}{15}$
 2 a 0.08 b 0.27 c 0.096 d 0.458 e 0.1225 f 0.385
 3 a 40% b 62.5% c 61% d 23% e 2% f 62%
 g $57\frac{1}{7}\%$ h $55\frac{5}{9}\%$
 4 a $\frac{1}{4}, 0.25$ b 30%, 0.3 c 26%, $\frac{13}{50}$ d $66\frac{2}{3}\%$, 0.6 e $\frac{2}{25}, 0.08$ f 7.5%, $\frac{3}{40}$
 5 a 9.6 b 8.64 c \$340 d \$570
 6 351 7 \$9.60 8 1 350 000 9 32% 10 9.5%
 11 a 112 b 159 c 211.2 d 153.6
 12 \$520 13 \$665.60
 14 a \$300 b \$150 c \$750 d \$144.35
 15 8.82% 16 11.9%
 17 a 18.72% increase b 1% decrease c 22.72% decrease d 0.32% increase
 18 a \$3009.55 b 15.752% increase 19 11.11%
 20 a \$7440 b \$8160 c \$6749.18 d \$7899.20
 21 a \$18 762.40 b \$17 636.66 c \$15 583.75 d \$14 648.72 e \$10 750.76 f \$41.02

Chapter 4: Factorisation

- 1 a $5(a + 2)$ b $6(b + 3)$ c $2(3c - 4)$ d $3(3d - 8)$ e $3e(e + 3)$ f $2f(3f + 5)$
 g $-2g(2g + 7)$ h $-3h(h + 5)$ i $2ab(2a + 3b)$ j $3mn(3n + 4)$
 2 a $(x + 3)(x + 4)$ b $(x + 6)(x + 3)$ c $(x - 6)(x + 1)$ d $(x + 7)(x - 4)$ e $(x - 5)(x - 6)$
 f $(x - 2)(x - 12)$ g $(x + 10)(x - 7)$ h $(x - 11)(x + 5)$ i $3(x^2 + 2x + 3)$ j $4(x^2 - 2x + 3)$
 k $(x - 10)(x + 10)$ l $(x - 7)(x + 7)$ m $(3x - 4y)(3x + 4y)$ n $(1 - 4a)(1 + 4a)$
 3 a $\frac{x + 1}{x - 1}$ b $\frac{1}{(x + 1)(x + 4)}$ c $\frac{1}{4}$ d $\frac{1}{2(p + 2)}$
 e $2a$ f $\frac{3}{a + 2}$ g $\frac{x + 2}{2x - 1}$ h $x - 2$ i $\frac{3(x - 2)(x + 4)}{(x - 1)}$ j $2x^2y$

Chapter 5: Linear equations and inequalities

- | | | | | | |
|------------------------------|-------------------------|--|------------------------------|--------------------------|-----------------------|
| 1 a $x + 2$ | b $b - 4$ | c $\frac{c}{2}$ | d $3d$ | e $2e + 1$ | f $\frac{f}{3} - 2$ |
| g $2(s + 3)$ | h $\frac{1}{2}(h - 1)$ | | | | |
| 2 a $a = 9$ | b $b = 8$ | c $x = 9$ | d $x = 9$ | e $y = 11$ | f $m = 2$ |
| g $x = 12$ | h $x = -5$ | i $x = \frac{7}{2}$ | j $x = \frac{9}{8}$ | k $x = -5$ | l $x = -4$ |
| 3 a $x = 2$ | b $x = -16$ | c $x = -24$ | d $x = 2$ | e $x = 3$ | f $x = \frac{24}{5}$ |
| g $x = \frac{9}{5}$ | h $x = -\frac{33}{2}$ | i $a = -\frac{7}{5}$ | j $a = 1$ | k $a = -\frac{4}{7}$ | l $a = -\frac{19}{6}$ |
| 4 $\frac{13}{8}$ | 5 13 | 6 82 cents | 7 36 | 8 35 km/h | |
| 9 a $x = \frac{3-a}{7}$ | b $x = \frac{b+c}{a}$ | c $x = a(c-b)$ | d $x = a-b$ | e $x = ab - bc = b(a-c)$ | f $x = \frac{c}{a+b}$ |
| g $x = \frac{ab}{c-a}$ | h $x = \frac{-ab}{c}$ | i $x = \frac{ab}{c} + ad = \frac{ab + cad}{c}$ | j $x = \frac{ad(e-b)}{d-ac}$ | | |
| 10 a $x = \frac{c-b}{a}$ | | | | | |
| b i 2 | ii 4 | iii -2 | iv 4 | v -3 | vi $12\frac{1}{2}$ |
| 11 a $x = \frac{ac}{d} - ab$ | | | | | |
| b i $-\frac{36}{7}$ | ii 7 | iii 2 | iv $-\frac{1}{4}$ | v 2 | vi $\frac{10}{9}$ |
| 12 a $\frac{3b-a}{2}$ | b $\frac{ab+a}{2b}$ | c $\frac{a^2+b^2}{2b}$ | d $\frac{-2}{a+b}$ | | |
| 13 a $x < 7$ | b $x \geq \frac{11}{2}$ | c $x \leq -4$ | d $m > -\frac{14}{3}$ | e $d \geq -1$ | |
| f $m > -\frac{14}{3}$ | g $m \geq 8$ | h $q \leq \frac{8}{3}$ | i $\ell \leq -\frac{29}{4}$ | j $m \leq -22$ | |

Chapter 6: Formulas

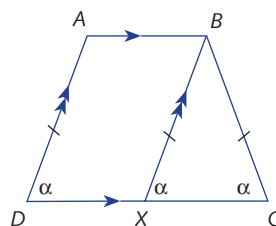
- | | | | | | |
|--------------------------------|--------------------------------------|--------------------------|--------------------------------|-------------------------|-------------------------------|
| 1 a 257.6 | b 6.936 | c 96.8 | d 13.3225 | e 1360.5 | f $\approx 3.163 \times 10^7$ |
| 2 a $a = \frac{A}{\pi b}$ | b $\ell = \frac{A - \pi r^2}{\pi r}$ | c $a = \frac{2S + n}{n}$ | d $n = \frac{\ell + d - a}{d}$ | e $c = 2s - a - b$ | |
| f $t = \frac{v}{a+b}$ | g $t = \frac{2s}{u+v}$ | h $p = \frac{b^2}{a+b}$ | i $b = \frac{ac}{c-a}$ | j $r = R - \frac{h}{a}$ | |
| k $\ell = \frac{gT^2}{4\pi^2}$ | l $y = \sqrt{\frac{3x}{m^2a}}$ | | | | |
| 3 a 2 | b 5 | c 20 | d $\frac{10}{61}$ | | |
| 4 a i 1249°C | ii 98.4°F | b i -1.077 | ii -6 | | |
| 5 a -3 | b $-\frac{11}{4}$ | c 4 | d $-\frac{3\sqrt{2}}{4}$ | e 0 | f $-\frac{5\pi}{14}$ |

Chapter 7: Congruence and special quadrilaterals

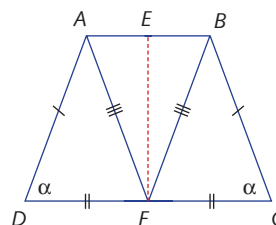
- | | | | | | |
|--|---|-----------------------|---|-----------------------|----------------------|
| 1 a acute | b right | c obtuse | d straight | e acute | f right |
| 2 a $\angle ABC$ and $\angle DEF$ | b $\angle ABC$ and $\angle PQR$ | | | | |
| 3 a $\alpha = 118^\circ, \beta = 62^\circ$ | b $\alpha = 69^\circ, \beta = 21^\circ, \gamma = 159^\circ$ | c $\alpha = 42^\circ$ | d $\beta = 100^\circ$ | | |
| 4 a $\alpha = 71^\circ$ | b $\beta = 77^\circ$ | c $\gamma = 60^\circ$ | d $\beta = 71^\circ, \theta = 38^\circ$ | e $\alpha = 45^\circ$ | f $\beta = 87^\circ$ |

- 5 a $\alpha = 100^\circ, \beta = 100^\circ$ b $\alpha = 106^\circ, \beta = 74^\circ, \gamma = 106^\circ$ c $\beta = 55^\circ, \phi = 125^\circ, \alpha = 45^\circ, \gamma = 80^\circ, \theta = 100^\circ, \psi = 100^\circ$
d $\alpha = 68^\circ, \beta = 60^\circ, \theta = 52^\circ, \gamma = 60^\circ, \phi = 68^\circ$
- 6 a $\angle AFE$ b $\angle ACD$ c $\angle BCF$ d $\angle ACD$ e $\angle DCF$
- 7 a $\alpha = 50^\circ$ b $\beta = 75^\circ$ c $\gamma = 70^\circ$
- 8 a $\alpha = 40^\circ$ b $\beta = 36^\circ, \alpha = 26^\circ, \theta = 26^\circ, \gamma = 118^\circ$ c $\alpha = 61^\circ, \beta = 29^\circ$ d $\alpha = 18^\circ$
- 9 a kite b rhombus c rectangle d square e trapezium
f isosceles trapezium
- 10 a $a = 4, b = 5$ b $\alpha = 90^\circ, \beta = 45^\circ$ c $\alpha = 96^\circ, \beta = 96^\circ$
d $\theta = 49^\circ, \beta = 131^\circ$ e $\alpha = 42^\circ, \beta = 96^\circ, \theta = 96^\circ$ f $\alpha = \beta = \theta = 25^\circ$
- 11 a parallelogram, rhombus, rectangle, square, kite (one pair only)
b isosceles trapezium, rectangle, square
c rhombus, square, kite d rhombus, square
e parallelogram, rhombus, rectangle, square, trapezium and isosceles trapezium (one pair only)
f parallelogram, rhombus, rectangle, square, trapezium (two pairs only)
- 12 a $\triangle ABC \equiv \triangle DEF$ (SAS) b $\triangle MNP \equiv \triangle STR$ (ASA)
c $\triangle XYZ \equiv \triangle MNL$ (SAS) d $\triangle TUV \equiv \triangle PRQ$ (RHS)
e $\triangle STU \equiv \triangle DFE$ (ASA)
- 13 a $\triangle ABC \equiv \triangle HGI$ (ASA) b $\triangle ABC \equiv \triangle RQP$ (SSS)
c $\triangle ABC \equiv \triangle MLN$ (RHS) d $\triangle ABC \equiv \triangle YXZ$ (SAS)
- 14 a $\triangle ABO \equiv \triangle DCO$ (RHS), $a = 8, \alpha = 44^\circ$
b $\triangle KLM \equiv \triangle MNK$ (AAS), $\beta = 110^\circ, \gamma = 23^\circ, \theta = 47^\circ$
c $\triangle ACD \equiv \triangle ACB$ (RHS), $\alpha = 48^\circ, \beta = 42^\circ, c = 2$
d $\triangle ALT \equiv \triangle NLM$ (SSS), $\alpha = 52^\circ, \beta = 67^\circ, \gamma = 61^\circ$
e $\triangle POS \equiv \triangle ROQ$ (SAS), $\gamma = 106^\circ, \beta = 37^\circ, \theta = 37^\circ$
- 15 $ABCD$ is a parallelogram. Therefore $AD = BC$.
Hence $AF = GC$, $\angle FAE = \angle GCE$ (alternate angles, $AB \parallel DC$),
 $\angle AEF = \angle CEG$ (vertically opposite). Hence $\triangle AEF \equiv \triangle CEG$ (AAS),
 $AE \equiv EC$ (matching sides of congruent triangles)

- 16 Let $BX \parallel AD$.
Therefore $ABXD$ is a parallelogram.
Therefore $BX = AD = BC$.
Therefore BXC is isosceles.
 $\angle ADC = \angle BXC$ (corresponding angles),
 $= \angle BCD$ (isosceles triangle).



Hence $\triangle ADF \equiv \triangle BCF$ (SAS),
 $\therefore AF \equiv BF$ (matching sides),
 $\therefore EF \perp DC$ (property of isosceles triangle)



- 17 AE and FC are equal in length and parallel.
Hence $AECF$ is a parallelogram.
- 18 $\triangle ABM \equiv \triangle ACM$ (SSS), $\angle ABC = \angle ACB$ (matching angles)
- 19 $\triangle AOB \equiv \triangle COD$ (SSS), $\angle AOB = \angle COD$ (matching angles)
- 20 a 7 b 5.6 m

Chapter 8: Index laws

- 1 a 3^4 b 5^3 c a^6 d m^6
2 a 216 b 625 c 128 d 1024
3 a 17.576 b 2.89 c 6274.2241 d 0.002 209

- 4 a 5^{13} b 3^{12} c $10x^6$ d $15m^{10}$ e 5^5
f 7 g $\frac{3a^4}{2}$ h $\frac{3n^3}{2}$ i 5^6 j 5^{10}
k $8n^{18}$ l $16b^8$ m 2 n 1 o $27a^3b^6$ p 1
5 a m^5n^2 b p^3q^3 c $54m^{10}n^4$ d $2m^7n^2$ e $\frac{2pq^6r^3}{3}$ f $\frac{2x^3y^4}{3}$
6 a $\frac{1}{4^2} = \frac{1}{16}$ b $\frac{1}{5^2} = \frac{1}{25}$ c $\frac{1}{6^3} = \frac{1}{216}$ d $\frac{1}{4^4} = \frac{1}{256}$
e $\frac{1}{10^4} = \frac{1}{10000}$ f $\frac{1}{8}$ g $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ h $\left(\frac{4}{3}\right)^3 = \frac{64}{27}$
7 a $\frac{1}{a^2}$ b $\frac{1}{b^6}$ c $\frac{1}{a^2b^4}$ d $\frac{1}{mn^3}$ e x^4
f $2y^3$ g $\frac{3}{m^2}$ h $\frac{5}{m^6}$ i $\frac{n^6}{m^4}$ j $\frac{3q^2}{p}$
8 a $\frac{1}{x^2y^8}$ b $\frac{1}{p^4q^5}$ c $\frac{12a^4}{n^2b^5}$ d $\frac{10}{\ell^2m^3}$ e $\frac{7}{t^6}$ f $\frac{5\ell^4}{3}$
g $\frac{4y}{3x^4}$ h $\frac{4u^4v^8}{3}$ i $\frac{n^2}{4m^2}$ j $\frac{m^4}{81n^8}$ k $\frac{3a^{18}b}{8}$ l $\frac{a^{10}}{b^{17}}$
m $\frac{b^{13}}{324a^5}$ n $\frac{1}{a^{10}b^4}$ o $\frac{a^7}{b^5}$ p $\frac{12b^4}{a^8}$ q $\frac{12y^7}{x^5}$ r $\frac{24n^6}{m^4}$
9 a 3 b 2 c 2 d 3 e 8 f 9
g 343 h 16
10 a $2a^{\frac{2}{3}}$ b $2ab^{\frac{1}{5}}$ c $2a^{\frac{1}{2}}$ d $5x^{\frac{8}{3}}$ e $6x^4$ f $4a^4b^2$
g $6^{\frac{2}{3}}x^2y^3$ h $12a^2b^2$
11 a 2.1×10^4 b 4.1×10^2 c 6.1×10^{10} d 2.4×10^3 e 6.2×10^{-3}
f 4.71×10^{-2} g 7×10^{-7} h 3.8×10^{-4} i 4.6×10^1 j 2.9×10^0
12 a 72000 b 380 c 0.97 d 0.0206 e 152 f 4070
g 0.00016 h 0.087
13 a 2×10^9 b 6.3×10^8 c 3×10^{-1} d 2×10^6 e 1.764×10^7
f 1.331×10^{-6} g $\approx 5.56 \times 10^7$ h 5×10^9
14 a 2.4×10^5 b 6×10^2 c 6.49×10^{-4} d 8.7×10^{-3} e 2.765×10^7 f 2.68×10^{-1}
15 a 1×10^3 b 2×10^3 c 5×10^{-2} d 1×10^{-5}
16 a 6.1 b 91.4 c 552.6 d 0.00168 e 100 f 0.094
g 2.533 h 1.641 i 0.0052

Chapter 9: Enlargements and similarity

- 1 a $\triangle ABC$ is similar to $\triangle DEF$ (AAA), $g = \frac{9}{2}$, $h = 6$
b $\triangle MNP$ is similar to $\triangle QRS$ (AAA), $a = 6$, $b = 4.5$
c $\triangle GHI$ is similar to $\triangle LJK$ (AAA), $m = 10$, $n = 2.4$
d $\triangle ABC$ is similar to $\triangle DEC$, $p = \frac{35}{2}$, $q = 3$
2 a $\triangle LOT$ is similar to $\triangle YOB$ (AAA), $a = 6$, $b = \frac{16}{3}$
b $\triangle RST$ is similar to $\triangle RUV$ (AAA), $\alpha = 63^\circ$, $\beta = 33^\circ$, $g = \frac{21}{5}$

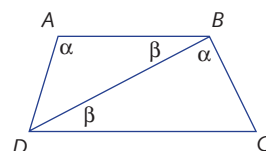
c $\triangle GHF$ is similar to $\triangle GJE$ (AAA), $\alpha = 42^\circ$, $\beta = 100^\circ$, $\ell = 10$

d $\triangle ADB$ is similar to $\triangle ABC$ (AAA), $\alpha = 51^\circ$, $x = \frac{27}{5}$, $y = \frac{36}{5}$

e $\triangle KLO$ is similar to $\triangle MNO$ (AAA), $x = \frac{15}{4}$, $y = \frac{32}{15}$

- 3 9.75 m 4 10.15 m 5 a $\frac{20}{3}$ cm b 4 cm c $\frac{16}{3}$ cm
- 6 a i $\frac{2}{3}$ m ii $\frac{4}{3}$ m b $2\sqrt{10}$ m c i 33 m ii \$151.80
- 7 a 4 m b 2.5 m c 1.8 m
- 8 a i 2136 mm ii 375 mm b i 350 mm ii 1143 mm iii 2270 mm
- 9 b i 2.5 cm ii 5 cm
- 10 b 2 cm c $\triangle DAE$ is similar to $\triangle CFE$, $\triangle DAE$ is similar to $\triangle BFA$

- 11 $\angle ABD = \angle BDC$ (alternate angles, $AB \parallel DC$),
 $\triangle ABD$ is similar to $\triangle BDC$ (AAA) $\frac{AB}{BD} = \frac{AD}{BC}$ (matching sides of similar triangles)
 $AB \times BC = DB \times AD$



- 12 a $\triangle BEA$ is similar to $\triangle CEF$ (AAA).

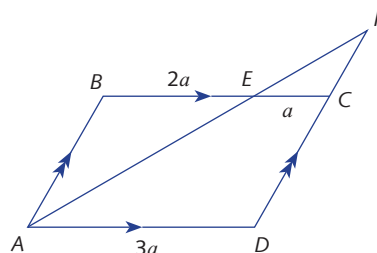
$$\frac{AB}{CF} = \frac{BE}{CE} = 2$$

$$\therefore AB = 2CF$$

- b $\triangle AFD$ is similar to $\triangle EFC$.

$$\frac{AF}{EF} = \frac{AD}{EC} = 3$$

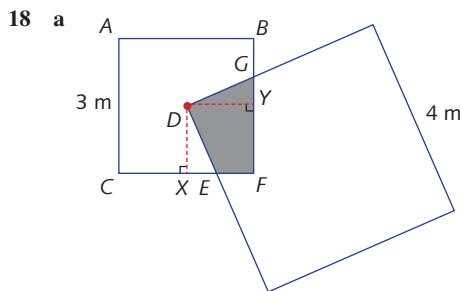
$$\therefore AF = 3EF$$



10B Problem-solving

- 1 a i \$15 750 ii \$60 750 iii \$1012.50
 b i \$61 040 ii 7.13% c the bank
- 2 a i \$30 600 ii \$31 824 iii \$126 072
 b i \$31 500 ii \$126 000 c The union's claim
- 3 a 35% b $(25 + 20x)\%$ c 0.8 litres
- 4 a i 38 km ii $48 - \frac{x}{5}$ km b 240 km
- 5 a 32 minutes b 255 km c 3 hours 4 minutes
- 6 a \$35 456 b \$3456 c 2.16% d i \$16 887 ii \$659.48
- 7 a \$1.25/litre b 40 litres c $C = VR$
- 8 a $\frac{4x}{9}$ km b i distance = $\frac{5xt}{3}$ ii 16 minutes
- 9 a 22.58 litres b $y = 0.86x$
- 10 a i 9 m ii 10.8 m, 6 m iii 26 m iv 33.6 m v 7.6 m
 b i $(x + 5)$ m ii $1.2(x + 5)$ m, $1.5x$ m iii $(4x + 10)$ m
 iv $(5.4x + 12)$ m v $(1.4x + 2)$ m c 15
- 11 a i $2x$ m ii $(x + 4)$ m iii $(2x + 3)$ m iv $(3x + 7)$ m b $C = 22x + 48$
 c i $2x^2 + 11x + 12$ iii 56 m^2 iv 4.5 m, 9 m
- 12 a 12 cm^2
 b i 17.28 cm^2 ii 5.28 cm^2 iii increase by 44%
 c i 8.67 cm^2 ii 3.33 cm^2 iii decrease by 27.75%
- d i $3\left(1 + \frac{x}{100}\right) \text{ cm}$ ii $4\left(1 + \frac{x}{100}\right) \text{ cm}$ iii $12\left(1 + \frac{x}{100}\right)^2 \text{ cm}^2$ iv 11.8

- 13 a 140 km b x km c $\frac{x}{35}$ hours, $\frac{140}{x}$ hours d 70 km/h
- 14 a $\frac{40}{x}$ items b $\frac{40}{x+0.4}$ items c $\frac{40}{x} = \frac{40}{x+0.4} + 5$ d \$1.60
- 15 a $y = 21 - x$ b $(2x^2 - 42x + 441) \text{ cm}^2$
 c i 9 or 12 ii 9 cm, 12 cm, 15 cm d $x = 10.5$
- 16 a i AAA ii $5\frac{1}{7}$ iii 49.0% b i $x = \frac{3y}{2}$ ii 44.4%
- 17 d $\frac{9}{4}$ of the area of $\triangle PQR$



- b ASA
- c Area $DEFG = \text{Area } DEFY + \text{Area } \triangle DYG$
 $= \text{Area } DEFY + \text{Area } \triangle DXE$
 $= \text{Area } DXFY$
- d $\frac{9}{4} \text{ m}^2$

- 19 b $BY = AX$ (matching sides of congruent triangles)
 $= PQ$ (opposite sides of rectangle $AXPQ$) = 5
 c 6 d 40 e 58
- 20 a i $\frac{1}{2}ab$ ii $\frac{1}{2}c^2$ iii $ab + \frac{1}{2}c^2$
 b $\frac{1}{2}(a+b)^2$
 c Equate the 2 expressions for area $ABCD$

How does a sextant work?

- a i $a = c = e = 70^\circ$, $b = d = 40^\circ$ ii Yes iii Yes iv They are parallel
 b i 50° ii 78° iii The angle of elevation is twice the angle measured by the sextant
 iv $(2x)^\circ$ v Yes

How long is a piece of string?

- 1 a i 4 ii 6 iii 8
 b $2(n-1)$ c $\sqrt{d^2 + g^2}$ d $2(n-1)\sqrt{d^2 + g^2} + g + 2\ell$
- 2 a i 2 ii 3 iii 4 b $n-1$
 c i 2 ii 3 iii 4 d $n-1$
 e g f $\sqrt{d^2 + g^2}$ g i $2d$ ii $3d$ iii $4d$
 h $(n-1)d$ i $\sqrt{g^2 + (n-1)^2 d^2}$ j $(n-1)(g + \sqrt{d^2 + g^2}) + \sqrt{g^2 + (n-1)^2 d^2} + 2\ell$
- 3 Standard pattern uses less for $n > 2$.

Packaging

- 1 a $\frac{1}{2}\left(25 - \frac{25\pi}{4}\right) \text{ cm}^2$ b i 5 cm ii $\frac{5\sqrt{3}}{2} \text{ cm}$ iii $\left(\frac{25\sqrt{3}}{4} - \frac{25\pi}{8}\right) \text{ cm}^2$
- 2 a 2892.699 cm^3 b 2892.699 cm^3 c 2758.724 cm^3 d Packaging c is the best 3



