

2012 Specialist Mathematics

Written Examination 2

MAV Suggested answers and solutions

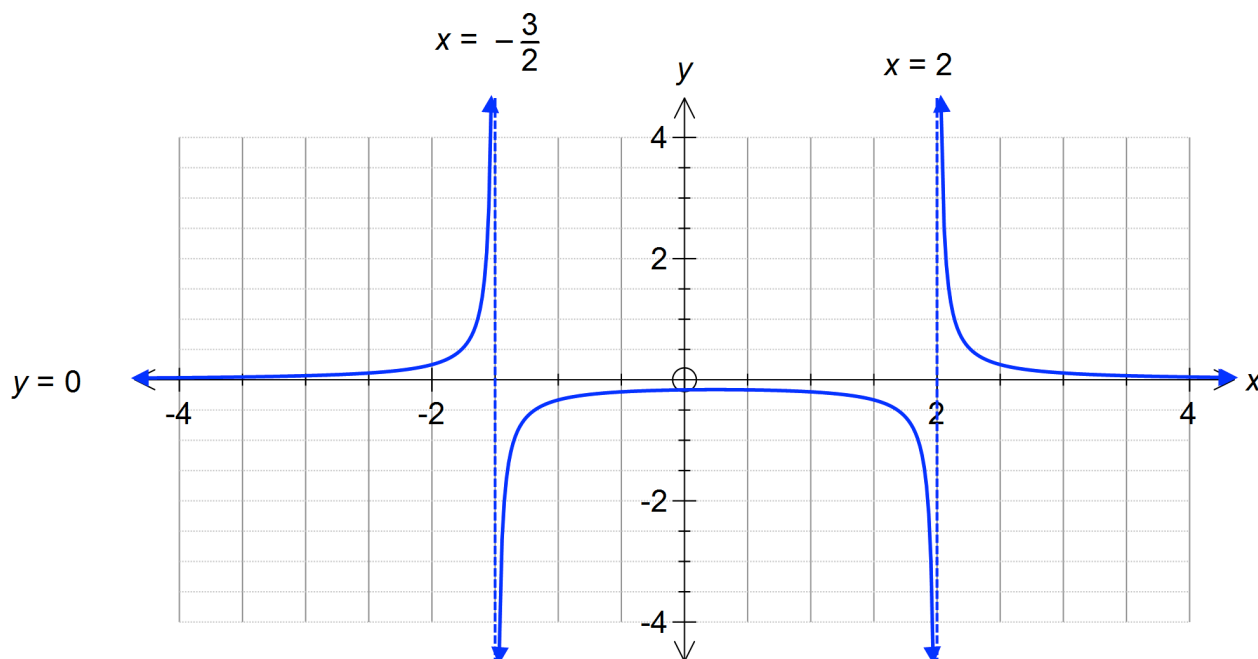
Section 1: Multiple Choice

Question 1 Answer D

The equation $y = \frac{1}{2x^2 - x - 6}$ can be expressed as $y = \frac{1}{(2x+3)(x-2)}$

Sketch a graph.

Vertical asymptotes will be at $x = -\frac{3}{2}$, $x = 2$ and horizontal asymptote at $y = 0$



Answer D

Question 2 Answer C

Horizontal spread from $x = -2$ to $x = 4$

Vertical spread from $y = -1$ to $y = 7$

Using symmetry the centre is therefore at the point $(1, 3)$

Horizontal spread ± 3 from centre

Vertical spread ± 4 from centre

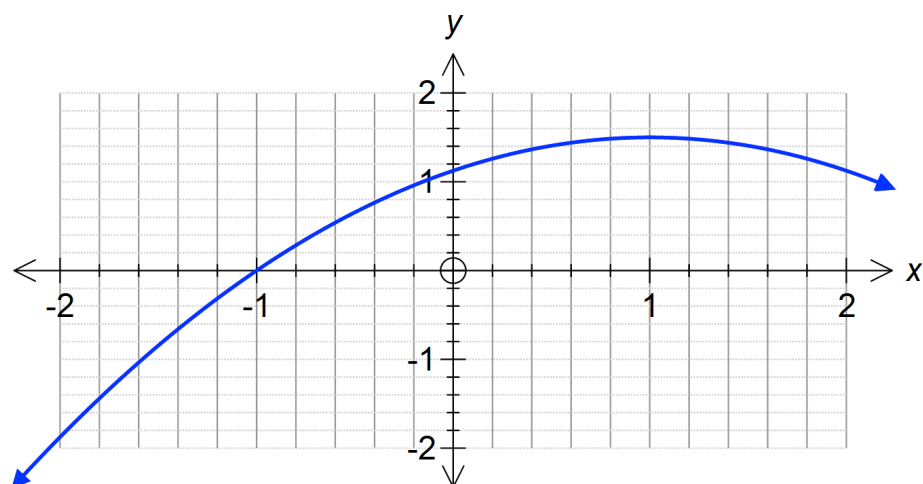
Ellipse equation $\frac{(x-1)^2}{3^2} + \frac{(y-3)^2}{4^2} = 1$

Answer C

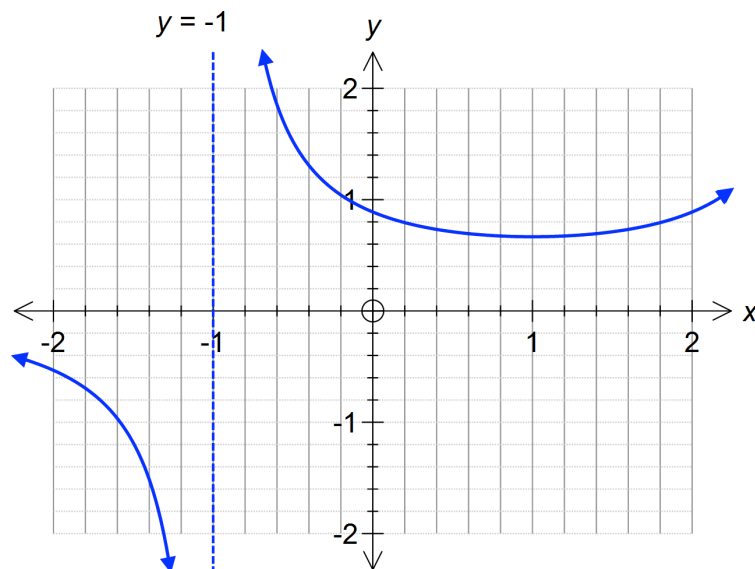
Question 3 Answer E

Method 1

Sketch a graph similar to the one shown.



and sketch its reciprocal.



Method 2

Consider crucial points and limiting behaviour

Original graph

x-intercept at $x = -1$

max height at $x = 1$

y value at $y = -1$

y value at $y = 1$

reciprocal graph

vertical asymptote at $x = -1$

local min point at $x = 1$

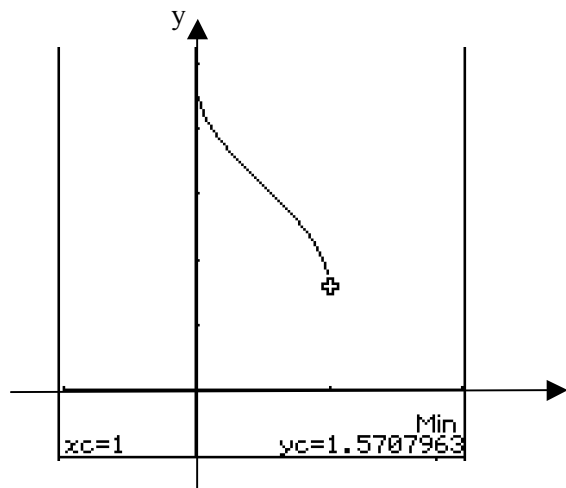
cross over y value at $y = -1$

cross over y value at $y = 1$

Answer E

Question 4 Answer D

$$f(x) = \cos^{-1}(2x-1) + \frac{\pi}{2}$$

Method 1: Sketch the graph

$$\therefore \text{Dom} = [0, 1], \text{Range} = \left[\frac{\pi}{2}, \frac{3\pi}{2} \right]$$

Method 2: Using properties of inverse trig graphs

Graph of $y = \cos^{-1}(x)$	Graph of $f(x) = \cos^{-1}(2x-1) + \frac{\pi}{2}$
Domain $-1 \leq x \leq 1$	Domain $-1 \leq 2x-1 \leq 1$ gives $0 \leq x \leq 1$
Range $[0, \pi]$	Range $\left[0 + \frac{\pi}{2}, \pi + \frac{\pi}{2} \right]$ gives $\left[\frac{\pi}{2}, \frac{3\pi}{2} \right]$

Gives **Answer D****Question 5 Answer C**

$$z = \sqrt{2} \text{cis} \left(-\frac{4\pi}{5} \right)$$

Using de Moivre's theorem

$$z^9 = \left(\sqrt{2} \right)^9 \text{cis} \left(-\frac{9 \times 4\pi}{5} \right)$$

$$\therefore z^9 = 16\sqrt{2} \text{cis} \left(-\frac{36\pi}{5} \right)$$

Convert using the principal domain for Arg where $\text{Arg } z \in (-\pi, \pi]$

$$z^9 = 16\sqrt{2} \text{cis} \left(\frac{4\pi}{5} \right)$$

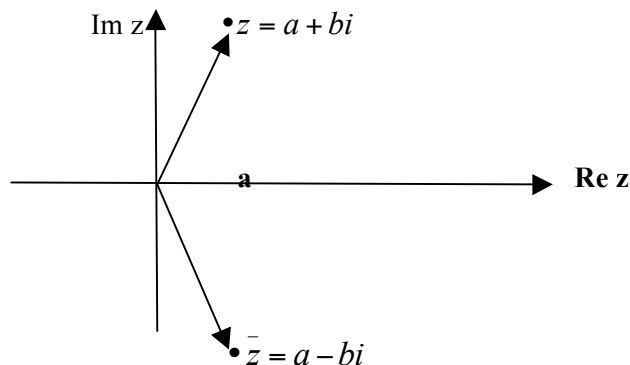
Giving **Answer C**

Question 6 Answer C

$$u = i^3 \bar{z}$$

Consider $z = a + bi$, where a, b are real numbers.

This means that $\bar{z} = a - bi$, the reflection of z over the Real axis.



Now consider that multiplying by i rotates the complex number anticlockwise by $\frac{\pi}{2}$.

So multiplying by i^3 rotates the complex number anticlockwise by $\frac{3\pi}{2}$.

This takes us to the same point as **Answer C**

Question 7 Answer B

$$|z + 2i| = |z|$$

Method 1

Replace z with $z = x + yi$

$$\text{gives } |x + yi + 2i| = |x + yi|$$

$$\therefore |x + (y + 2)i| = |x + yi|$$

$$\therefore \sqrt{x^2 + (y + 2)^2} = \sqrt{x^2 + y^2}$$

$$\therefore x^2 + y^2 + 4y + 4 = x^2 + y^2$$

$$\therefore 4y + 4 = 0$$

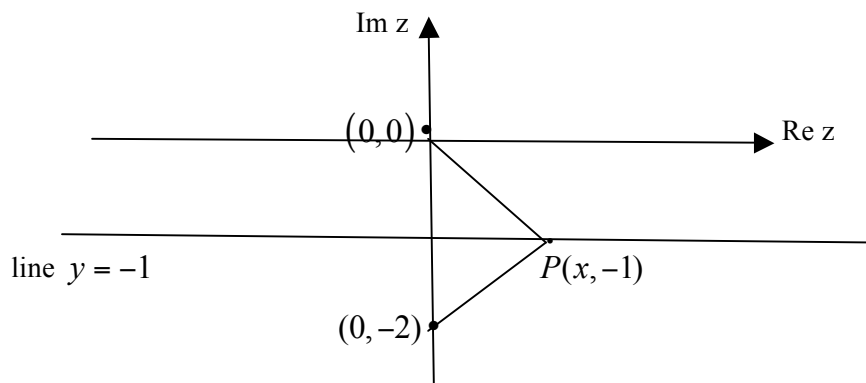
Giving the equation $y = -1$

This matches the line $\text{Im } z = -1$

Answer B**Method 2**

The answer to $|z + 2i| = |z|$ is the line that is equidistant from the line joining the 'zeros' of the brackets $(z + 2i)$ and (z)

The ‘zeros’ are therefore $-2i$ and 0 . So, we are interested in the perpendicular bisector of the line segment joining $(0, 0)$ and $(0, -2)$. The diagram represents this as $P(x, -1)$



This matches the line $\text{Im } z = -1$

Answer B

Question 8 Answer E

$$z = a + bi$$

Test each case

Option A

$$z + \bar{z} = a + bi + a - bi = 2a$$

Real, because a is real.

Option B

$$|z| = \sqrt{a^2 + b^2}$$

Real, because a and b are real.

Option C

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2$$

Real, because a and b are real.

Option D

$$z^2 - abi = (a + bi)^2 - abi = a^2 + abi + b^2 - abi = a^2 + b^2$$

Real, because a and b are real.

Option E

$$(z - \bar{z})(z + \bar{z}) = (a + bi - a + bi)(a + bi + a - bi) = 2bi \times 2a$$

Non-Real.

Answer E

Question 9 Answer B

$$\frac{dy}{dx} = \cos(x)$$

Use Euler's linear approximation $f(x+h) \approx f(x) + hf'(x)$ with initial point $(0,1)$ and $h = 0.1$.

Step 1

$$y_1 \approx f(0) + 0.1f'(0), x_1 = 0 + 0.1$$

$$\therefore y_1 \approx 1 + 0.1 \times 1 = 1.1, x_1 = 0.1$$

Step 2

$$y_2 \approx f(0.1) + 0.1f'(0.1), x_2 = 0.1 + 0.1$$

$$\therefore y_2 \approx 1.1 + 0.1 \times \cos(0.1) = 1.1995, x_2 = 0.2$$

This is an overestimation because of concavity of graph. The graph is concave down and hence the tangent approximations are above the graph.

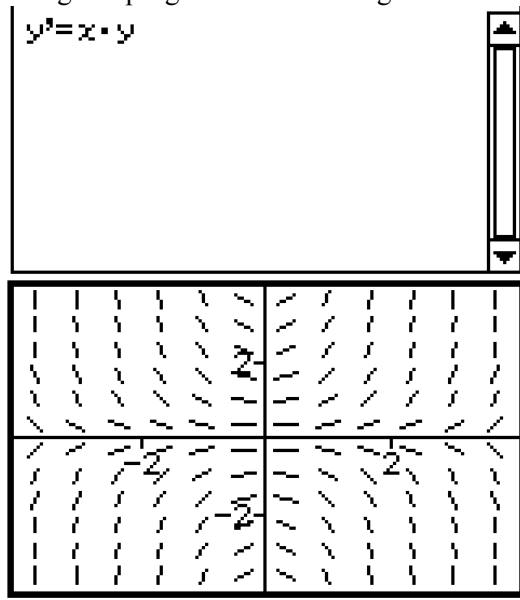
Answer B

Question 10 Answer A

$$\frac{dy}{dx} = xy$$

Method 1

Using DE program on the CAS gives



Method 2:

plot a table of values and check the slopes of each alternative

x	-1	-1	-1	0	0	0	1	1	1
y	-1	0	1	-1	0	1	-1	0	1
$\frac{dy}{dx}$	1	0	-1	0	0	0	-1	0	1

Answer A

Question 11 Answer E

$$\frac{d^2y}{dx^2} = x^2 - x$$

$$\therefore \frac{dy}{dx} = \int (x^2 - x) dx = \frac{x^3}{3} - \frac{x^2}{2} + c$$

$$\frac{dy}{dx} = 0 \text{ at } x = 0 \text{ gives } c = 0$$

$$\text{This gives stationary points at } \frac{x^3}{3} - \frac{x^2}{2} = 0$$

$$\text{Factorising gives } x^2 \left(\frac{x}{3} - \frac{1}{2} \right) = 0$$

$$\text{gives stationary points at } x = 0, x = \frac{3}{2}$$

The graph of $y = f(x)$ will be a concave upwards quartic function with stationary points at

$$x = 0, x = \frac{3}{2}.$$

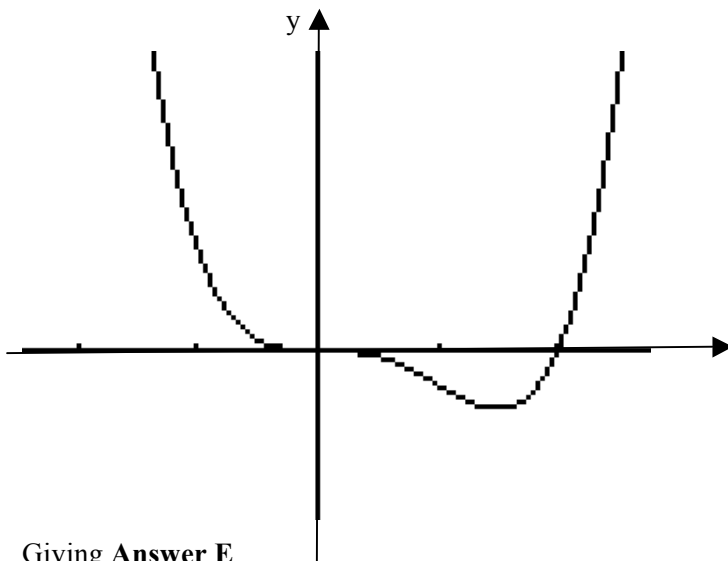
Given $\frac{d^2y}{dx^2} = x^2 - x$ means that points of inflexion can be found where $\frac{d^2y}{dx^2} = 0$.

Factorising $\frac{d^2y}{dx^2} = x^2 - x = x(x - 1)$ and solve for zero gives points of inflexion at

$$x = 1, x = 0.$$

This means that there will be a stationary point of inflexion at $x = 0$ and a non-stationary point of inflexion at $x = 1$

From CAS, the graph will have the approximate shape of



Giving **Answer E**

Question 12 Answer A

Method 1

Using volume of shape

$$y = \sqrt{9 - (x-1)^2}$$

Semicircle with radius of 3

Revolution of a semicircle about the x -axis gives a sphere of radius 3

$$\text{Use volume of sphere} = V = \frac{4\pi r^3}{3}, r = 3$$

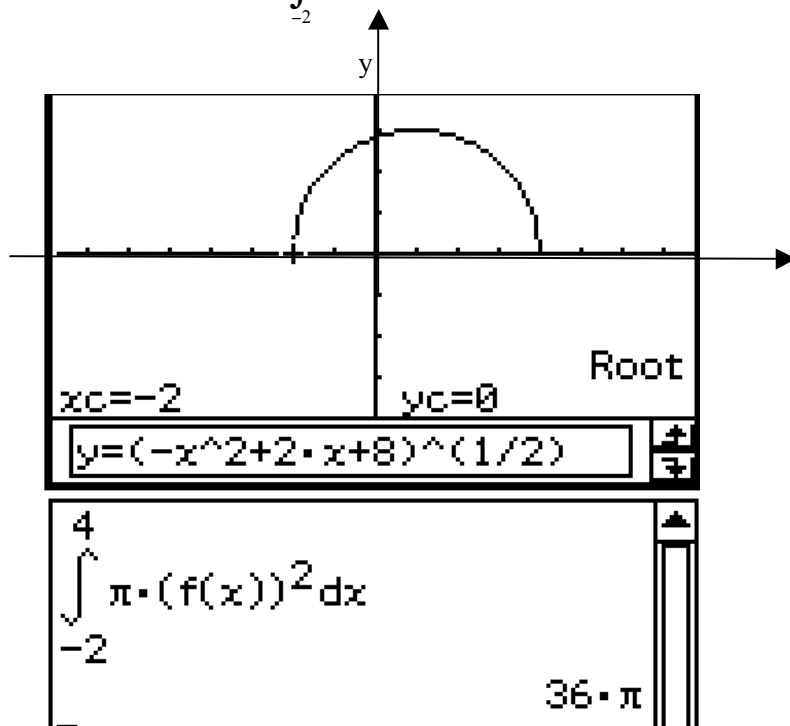
$$\therefore V = 4\pi(3)^2$$

gives **Answer A**

Method 2

Using integration and/or CAS

$$\text{Volume of solid} = \pi \int_{-2}^4 y^2 dx \text{ where } y^2 = 9 - (x-1)^2$$



gives **Answer A**

Question 13 Answer B

$$\int_0^{\frac{\pi}{3}} \sin^3(x) \cos^4(x) dx$$

Integrand

$$\begin{aligned}
 &= \int_0^{\frac{\pi}{3}} \sin^2(x) \sin(x) \cos^4(x) dx \\
 &= \int_0^{\frac{\pi}{3}} (1 - \cos^2(x)) \sin(x) \cos^4(x) dx
 \end{aligned}$$

Letting

$$u = \cos(x)$$

$$\therefore \frac{du}{dx} = -\sin(x)$$

$$x = 0, u = 1$$

$$\text{When } x = \frac{\pi}{3}, u = \frac{1}{2}$$

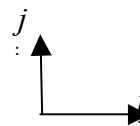
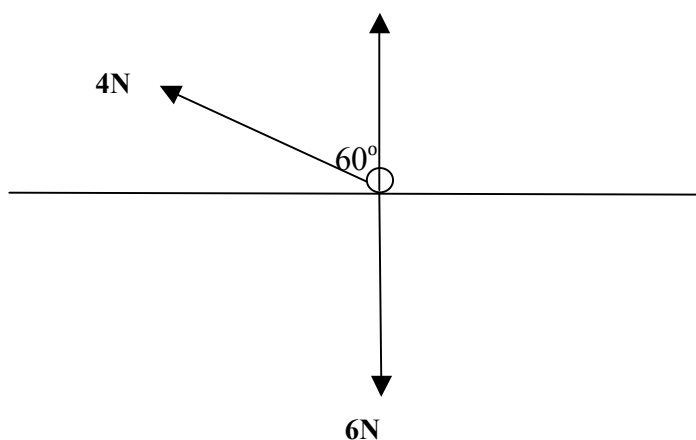
Integrand becomes

$$\begin{aligned}
 &\int_1^{\frac{1}{2}} (1 - u^2) \times -1 \times u^4 du \\
 &= -\int_1^{\frac{1}{2}} (u^4 - u^6) du \\
 &= \int_1^{\frac{1}{2}} (u^6 - u^4) du
 \end{aligned}$$

Answer B

Question 14 Answer B

Draw a diagram representing and labelling the forces.



$$\text{Resultant force} = R = -4 \sin 60^\circ i + (4 \cos 60^\circ - 6) j$$

$$\therefore |R| = \sqrt{16 \sin^2 60^\circ + (4 \cos 60^\circ - 6)^2}$$

$$\therefore |R| = 2\sqrt{7}$$

Answer B

Question 15 Answer D

$$a = 2i + mj - 3k \text{ and } b = m^2i - j + k$$

Perpendicular when $a \cdot b = 0$

$$a \cdot b = 2m^2 - m - 3$$

Solve for zero gives

$$(2m - 3)(m + 1) = 0$$

$$\therefore m = \frac{3}{2}, m = -1$$

Answer D

Question 16 Answer A

Considering the points P and Q, define position vectors $\vec{OP} = -2i + 4j + 3k$,

$$\vec{OQ} = i - 2j + k \text{ and } \vec{PQ} = 3i - 6j + 2k$$

Using distance formula $|\vec{PQ}| = \sqrt{3^2 + 6^2 + 2^2} = 7$

Answer A

Question 17 Answer D

$$\text{vectors } u = 2i - 2j + k \text{ and } v = 3i - 6j + 2k$$

Vector resolute is found by the formula $(v \cdot \hat{u})\hat{u}$

$$\text{First find } \hat{u} = \frac{1}{3}(2i - 2j + k)$$

$$\text{Then find } v \cdot \hat{u} = \frac{1}{3}(6 + 12 + 2) = \frac{20}{3}$$

$$(v \cdot \hat{u})\hat{u} = \frac{20}{3} \times \frac{1}{3}(2i - 2j + k)$$

$$\text{Giving } (v \cdot \hat{u})\hat{u} = \frac{20}{9}(2i - 2j + k)$$

Answer D

Question 18 Answer C

Distance found by area under the curve, taking into account the area that is below the x-axis.

$$\text{Distance found by } \left| \int_0^1 v dt \right| + \int_1^2 v dt.$$

$$\text{Simplify the absolute value to make } \left| \int_0^1 v dt \right| = - \int_0^1 v dt$$

$$\text{Answer} = - \int_0^1 v dt + \int_1^2 v dt$$

giving **Answer C**

Question 19 Answer E

Start with $v = x$ and select a kinematics technique that involves v and x and t .

$$v = x$$

$$\therefore \frac{dx}{dt} = x$$

$$\therefore \frac{dt}{dx} = \frac{1}{x}$$

This becomes a DE to solve.

$$t = \int \frac{1}{x} dx$$

$$\therefore t = \log_e |x| + c$$

Using $t = 3, x = -1$ gives

$$x < 0$$

$$\therefore t = \log_e(-x) + c$$

$$c = 3 - \log_e 1$$

$$\therefore c = 3$$

$$t = \log_e(-x) + 3$$

Rearrange to get $x = -e^{t-3}$

Answer E

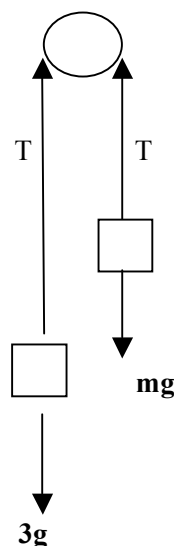
Question 20 Answer C

Connected masses.

Consider separately the 3kg mass and the mkg mass.

$$\text{Body } 3: T - 3g = 3a \dots\dots\dots(1)$$

$$\text{Body } m: mg - T = ma \dots\dots\dots(2)$$



Add the 2 equations to get

$$mg - 3g = 3a + ma$$

$$\text{Substitute } a = \frac{g}{2}$$

And solve to get $m = 9$

Answer C

Question 21 Answer E

Start with $v = x^2$ and select a kinematics technique that involves v and x .

$$\text{Select the formula } a = v \frac{dv}{dx}$$

$$v = x^2$$

$$\therefore \frac{dv}{dx} = 2x$$

$$\therefore v \frac{dv}{dx} = x^2 \times 2x = 2x^3$$

$$\text{giving } a = 2x^3$$

Now use $R = ma$ where $m = 3$

$$\text{giving } R = 3 \times 2x^3 = 6x^3$$

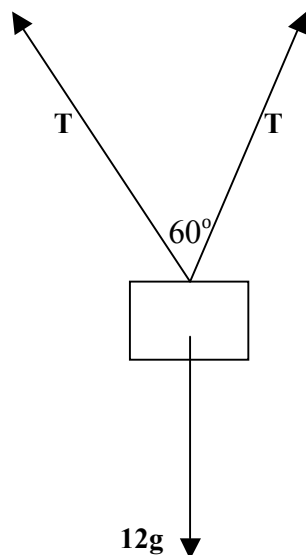
$$\text{at } x = 2, R = 48$$

Answer E

Question 22 Answer D

Method 1

Using Lami's theorem.



$$\frac{12g}{\sin 60^\circ} = \frac{T}{\sin 150^\circ}$$

$$\therefore T = \frac{12g \sin 150^\circ}{\sin 60^\circ}$$

$$\therefore T = 4\sqrt{3}g$$

Answer D

Method 2

Resolving horizontally and vertically

Horizontally

$$2T \cdot \frac{\sqrt{3}}{2} = 12g$$

:

$$\therefore T = \frac{12g}{\sqrt{3}} = 4\sqrt{3}g$$

Section 2: Extended Response**Question 1**

Curve defined by parametric equations

$$x = \operatorname{cosec}(\theta) + 1$$

$$y = 2 \cot(\theta)$$

- a.** Show that curve can be expressed as $(x-1)^2 - \frac{y^2}{4} = 1$

Rearrange given equations.

$$x = \operatorname{cosec}(\theta) + 1$$

$$\Rightarrow x - 1 = \operatorname{cosec}(\theta)$$

And

$$y = 2 \cot(\theta)$$

$$\Rightarrow \frac{y}{2} = \cot(\theta)$$

Use the trig identity $1 + \cot^2(\theta) = \operatorname{cosec}^2(\theta)$.

Substitute giving

$$1 + \left(\frac{y}{2}\right)^2 = (x-1)^2$$

Giving the result

$$(x-1)^2 - \frac{y^2}{4} = 1 \quad \text{QED}$$

- b.** Sketch the curve of the non-rectangular hyperbola $(x-1)^2 - \frac{y^2}{4} = 1$.

Equations of asymptotes.

Using the formula $y - k = \pm \frac{b}{a}(x - h)$

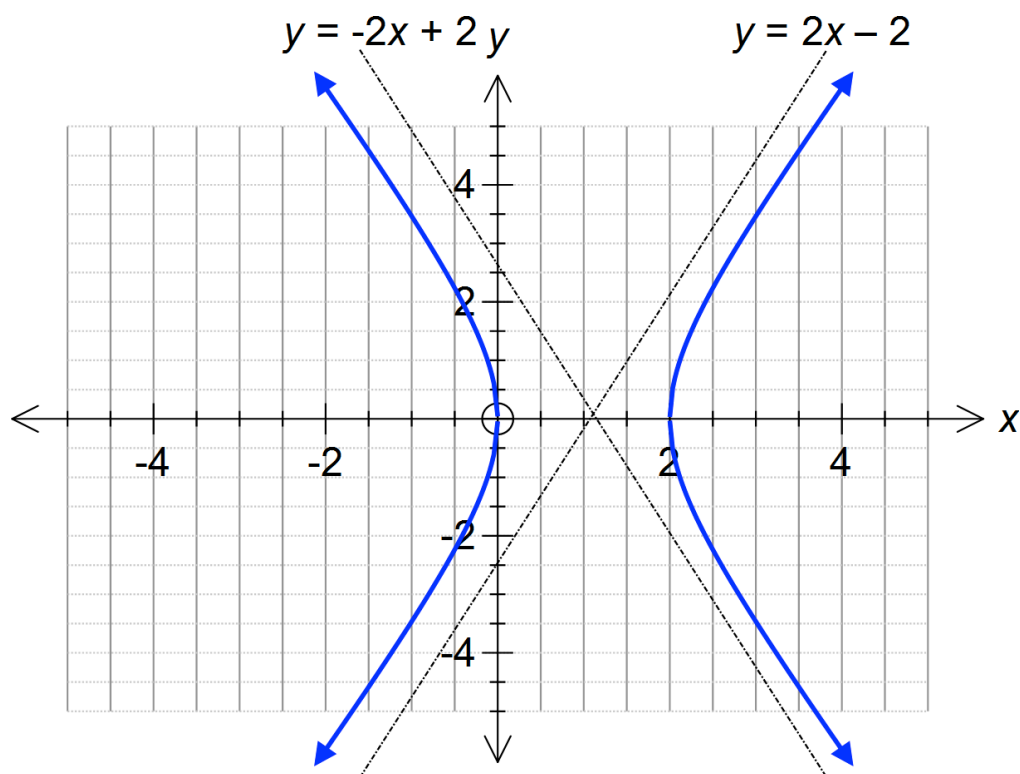
gives asymptotes

$$y - 0 = \pm \frac{2}{1}(x - 1)$$

$$\therefore y = \pm 2(x - 1)$$

$$\therefore y = 2x - 2, y = -2x + 2$$

Label asymptotes on the diagram.



- c. Volume of solid, rotating about x-axis.

Using the formula $\text{Volume} = \int_a^b \pi y^2 dx$

Rearrange $(x-1)^2 - \frac{y^2}{4} = 1$ to get $y^2 = 4(x-1)^2 - 4$

Volume

$$= \pi \int_2^3 (4(x-1)^2 - 4) dx$$

$$= \pi \left[\frac{4(x-1)^3}{3} - 4x \right]_2^3$$

Answer = $\frac{16\pi}{3}$ cubic units

- d. Gradient of curve where $\theta = \frac{7\pi}{6}$

Method 1

Using implicit differentiation on $(x-1)^2 - \frac{y^2}{4} = 1$

gives $2(x-1) - \frac{2y}{4} \times \frac{dy}{dx} = 0$

$$\therefore \frac{y}{2} \times \frac{dy}{dx} = 2(x-1)$$

$$\therefore \frac{dy}{dx} = \frac{4x-4}{y}$$

Now at $\theta = \frac{7\pi}{6}$ using $\begin{matrix} x = \operatorname{cosec}(\theta) + 1 \\ y = 2 \cot(\theta) \end{matrix}$

$$x = \operatorname{cosec} \frac{7\pi}{6} + 1$$

$$y = 2 \cot \frac{7\pi}{6}$$

giving

$$x = -1$$

$$y = 2\sqrt{3}$$

Substitute into $\frac{dy}{dx} = \frac{4x-4}{y}$

$$\frac{dy}{dx} = \frac{4(-1)-4}{2\sqrt{3}}$$

Answer: $\frac{dy}{dx} = \frac{-4}{\sqrt{3}}$ or $\frac{-4\sqrt{3}}{3}$

Method 2

Using chain rule involving parametric form

$$y = 2 \cot \theta$$

$$x = \operatorname{cosec} \theta = 1$$

When $\therefore \frac{dy}{d\theta} = -2 \operatorname{cosec}^2 \theta$ and $\therefore \frac{dx}{d\theta} = -\operatorname{cosec} \theta \cot \theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

Using $\therefore \frac{dy}{dx} = \frac{-2 \operatorname{cosec}^2 \theta}{-\operatorname{cosec} \theta \cot \theta}$

$$\therefore \frac{dy}{dx} = 2 \operatorname{cosec} \theta \tan \theta$$

When $\theta = \frac{7\pi}{6}$,

$$\frac{dy}{dx} = 2 \operatorname{cosec} \frac{7\pi}{6} \tan \frac{7\pi}{6}$$

$$\therefore \frac{dy}{dx} = \frac{-4\sqrt{3}}{3}$$

Question 2

a. Given that $\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{\sqrt{3}+2}}{2}$, show that $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2-\sqrt{3}}}{2}$

Method 1

Use the trig identity $\sin^2(\theta) + \cos^2(\theta) = 1$

$$\text{giving } \sin^2 \frac{\pi}{12} + \cos^2 \frac{\pi}{12} = 1$$

$$\text{Substitute: } \sin^2 \frac{\pi}{12} + \left(\frac{\sqrt{\sqrt{3}+2}}{2}\right)^2 = 1$$

gives

$$\sin^2 \frac{\pi}{12} = 1 - \left(\frac{\sqrt{\sqrt{3}+2}}{2}\right)^2$$

$$\therefore \sin^2 \frac{\pi}{12} = 1 - \left(\frac{\sqrt{3}+2}{4}\right)$$

$$\therefore \sin^2 \frac{\pi}{12} = 1 - \left(\frac{\sqrt{3}+2}{4}\right)$$

$$\therefore \sin^2 \frac{\pi}{12} = \frac{4 - \sqrt{3} - 2}{4}$$

$$= \frac{2 - \sqrt{3}}{4}$$

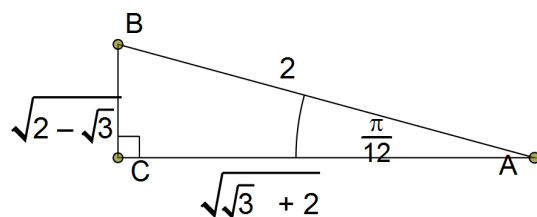
$$\text{Giving the answer } \sin \frac{\pi}{12} = \pm \frac{\sqrt{2 - \sqrt{3}}}{2}$$

But allowing that $\frac{\pi}{12}$ is in the 1st quadrant

$$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2 - \sqrt{3}}}{2} \quad \text{QED}$$

Method 2

Using Pythagoras' Theorem



Using

$$c^2 = a^2 + b^2$$

$$\therefore 4 = a^2 + 2 + \sqrt{3}$$

$$\therefore a^2 = 2 - \sqrt{3}$$

$$\text{Hence } \sin \frac{\pi}{12} = \frac{\sqrt{2 - \sqrt{3}}}{2}$$

b. i). Recognise that $\operatorname{cis}\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{12}\right) + i \sin\left(\frac{\pi}{12}\right)$

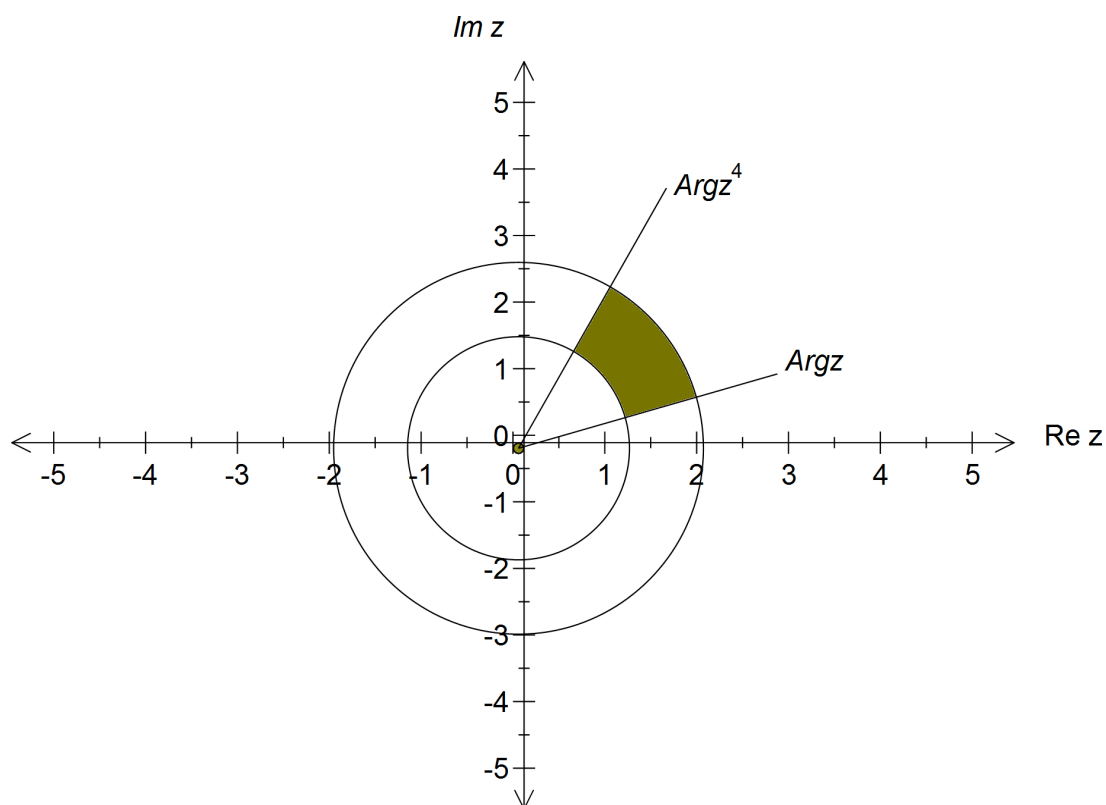
$$\text{gives } \operatorname{cis}\left(\frac{\pi}{12}\right) = \frac{\sqrt{\sqrt{3} + 2}}{2} + i \frac{\sqrt{2 - \sqrt{3}}}{2}$$

$$\therefore z_1 = \operatorname{cis}\left(\frac{\pi}{12}\right)$$

ii) $z_1^4 = 1^4 \operatorname{cis}\left(\frac{4\pi}{12}\right)$ using de Moivre's theorem

$$\text{gives } z_1^4 = \operatorname{cis}\left(\frac{\pi}{3}\right)$$

c. label $\operatorname{Arg} z_1^4 = \left(\frac{\pi}{3}\right)$ $\operatorname{Arg} z_1 = \left(\frac{\pi}{12}\right)$



- d. Using area of the circles, considering we are dealing with one-eighth of the annulus defined by circles of radius 1 and 2 units.

$$\text{Area} = \frac{1}{8}(\pi 2^2 - \pi 1^2)$$

$$\text{Giving area} = \frac{3\pi}{8}$$

e. i) **Method 1**

$$\text{Remember that } z_1 = \text{cis}\left(\frac{\pi}{12}\right)$$

$$\text{gives } z_1^n = 1^n \text{cis}\left(\frac{n\pi}{12}\right) = \text{cis}\left(\frac{n\pi}{12}\right)$$

$$\text{so } \text{Re}(z_1^n) = \cos\left(\frac{n\pi}{12}\right)$$

We are given

$$\text{Re}(z_1^n) = 0$$

$$\therefore \cos\left(\frac{n\pi}{12}\right) = 0$$

$$\text{Solving the equation } \cos\left(\frac{n\pi}{12}\right) = 0 \text{ gives } \frac{n\pi}{12} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

$$\therefore n = \frac{12}{2}, \frac{36}{2}, \frac{60}{2}, \dots$$

$$\therefore n = 6, 18, 30, \dots$$

$$\text{General solution answer } n = 6 \pm 12k, k \in \mathbb{Z}$$

i) **Method 2**

$$\text{Solving the equation } \cos\left(\frac{n\pi}{12}\right) = 0$$

$$\frac{n\pi}{12} = \dots, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

$$\text{gives } \therefore \frac{n\pi}{12} = \frac{(2k+1)\pi}{2}, k \in \mathbb{Z}$$

$$\therefore n = 6(2k+1), k \in \mathbb{Z}$$

$$\text{General solution answer is } n = 6(2k+1), k \in \mathbb{Z}$$

ii) **Method 1**

$$\text{allow } z_1^n = \text{cis}\left(\frac{n\pi}{12}\right)$$

$$\text{using } n = 6 \pm 12k, z_1^n = \text{cis}\left((6 \pm 12k)\frac{\pi}{12}\right), k \in \mathbb{Z}$$

$$z_1^n = \text{cis}\left(\frac{\pi}{2} \pm \pi k\right), k \in \mathbb{Z}$$

$$\text{gives } z_1^n = \pm i \text{ or } z_1^n = \text{cis}\left(\pm \frac{\pi}{2}\right)$$

ii) **Method 2**

$$\text{allow } z_1^n = \text{cis}\left(\frac{n\pi}{12}\right)$$

$$\text{using } n = 6 + 12k,$$

$$z_1^n = \text{cis}\left(\frac{6(1+2k)\pi}{12}\right), k \in \mathbb{Z}$$

$$\therefore z_1^n = \cos \frac{\pi(2k+1)}{2} + i \sin \frac{\pi(2k+1)}{2}, k \in \mathbb{Z}$$

$$\text{gives } z_1^n = i \text{ if } k \text{ is an even number}$$

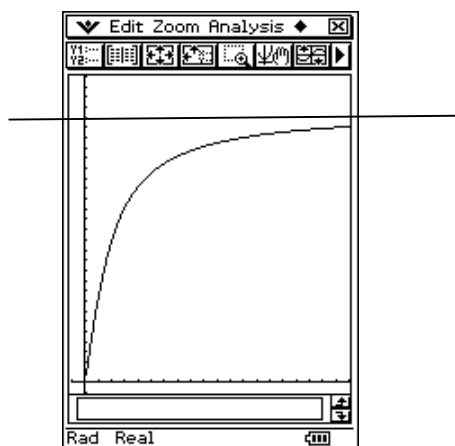
$$\text{or } z_1^n = -i \text{ if } k \text{ is an odd number}$$

$$\therefore z_1^n = \pm i \text{ or } z_1^n = \text{cis}\left(\pm \frac{\pi}{2}\right)$$

Question 3

$$\text{A car's speed after } T \text{ seconds is } V = 17 \tan^{-1}\left(\frac{\pi T}{6}\right), T \geq 0.$$

$$\text{It is helpful to sketch a graph of } V = 17 \tan^{-1}\left(\frac{\pi T}{6}\right), \text{ with horizontal asymptote at } V = \frac{17\pi}{2}$$



- a. Consider the horizontal asymptote.

As $T \rightarrow \infty$

$$V \rightarrow \frac{17\pi}{2}$$

$\therefore V = \frac{17\pi}{2}$ is the asymptote

- b. Using $V = 17 \tan^{-1}\left(\frac{\pi T}{6}\right)$ find acceleration = $\frac{dV}{dT}$

Find $\frac{dV}{dT}$ at $T=10$

```
define f(x)=17tan-1( $\frac{\pi x}{6}$ )
done
diff(f(x),x,1,10)
 $\frac{102 \cdot \pi}{100 \cdot \pi^2 + 36}$ 
diff(f(x),x,1,10)
0.3132500907
```

gives acceleration to nearest ms^{-2}

answer = 0.3ms^{-2}

'By-hand' algebra

Using

$$y = 17 \tan^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \frac{dy}{dx} = \frac{17a}{a^2 + x^2}$$

$$\therefore \frac{dV}{dT} = \frac{17 \times \frac{6}{\pi}}{\frac{36}{\pi^2} + T^2}$$

$$\therefore \frac{dV}{dT} = \frac{102\pi}{36 + \pi^2 T^2}$$

When $T = 10$

$$\frac{dV}{dT}; 0.3$$

c. Rearrange $V = 17 \tan^{-1}\left(\frac{\pi T}{6}\right)$ giving $T = \frac{6}{\pi} \tan \frac{V}{17}$

$$T(0) = 0, T(25) = 18.995$$

Answer 19 seconds

d. Using $\frac{dv}{dt} = -\frac{1}{100}(145 - 2t)$

i) $v = -\frac{1}{100} \int (145 - 2t) dt$

$$\therefore v = -\frac{1}{100} (145t - t^2) + c$$

using $t = 0, v = 25$ gives $c = 25$

$$\therefore v = -\frac{1}{100} (145t - t^2) + 25$$

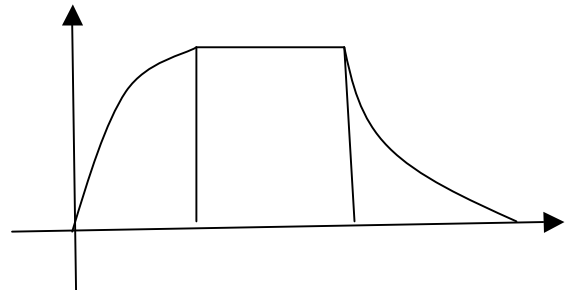
ii) Allow $v = 0 \therefore -\frac{1}{100} (145t - t^2) + 25 = 0$

gives $t = 20, t = 125$

at rest the first time

answer $t = 20$ seconds

e. It helps to sketch a quick graph of V against T.



i) distance = $\int_0^{18.995} 17 \tan^{-1} \frac{\pi T}{6} dT + (120 \times 25) + \int_0^{20} (25 - \frac{1}{100} (145t - t^2)) dt$

ii) answer = 3637 metres

Question 4

Position of International space station given by

$$\vec{r}(t) = 6800 \sin(\pi(1.3t - 0.1)) \vec{i} + (6800 \cos(\pi(1.3t - 0.1)) - 6400) \vec{j}$$

for $t \in [0, 0.154]$

a. allowing $\vec{i}$ component = 0

gives $6800 \sin(\pi(1.3t - 0.1)) = 0$

gives $t = \frac{1}{13}$

Substitute $t = \frac{1}{13}$ in the j component gives $6800 - 6400 = 400$ kms

Answer = 400km

b. given $\vec{r}(t) = 6800 \sin(\pi(1.3t - 0.1))\vec{i} + (6800 \cos(\pi(1.3t - 0.1)) - 6400)\vec{j}$

Hence velocity is given by

$$\vec{r}'(t) = 1.3\pi \times 6800 \cos(\pi(1.3t - 0.1))\vec{i} - 1.3\pi \times (6800 \sin(\pi(1.3t - 0.1)))\vec{j}$$

and

acceleration =

$$\vec{r}''(t) = -1.3^2 \pi^2 \times 6800 \sin(\pi(1.3t - 0.1))\vec{i} - 1.3^2 \pi^2 \times (6800 \cos(\pi(1.3t - 0.1)))\vec{j}$$

Find dot product = $\vec{r}'(t) \cdot \vec{r}''(t)$ and show $\vec{r}'(t) \cdot \vec{r}''(t) = 0$ for perpendicularity.

It is not necessary to write all terms exactly but need to show the pattern: here using the general case of θ

$$\vec{r}'(t) \cdot \vec{r}''(t) = -1.3^3 \pi^3 \times 6800^2 \cos \theta \sin \theta + 1.3^3 \pi^3 \times 6800^2 \cos \theta \sin \theta$$

where $\therefore \theta = \pi(1.3t - 0.1)$

$$\therefore \vec{r}'(t) \cdot \vec{r}''(t) = 0 \text{ as required}$$

Therefore acceleration and velocity are perpendicular

c. speed = $\left| \vec{r}'(t) \right|$

$$\left| \vec{r}'(t) \right| = \sqrt{1.3^2 \pi^2 6800^2 (\cos^2 \theta + \sin^2 \theta)} \text{ where } \theta = \pi(1.3t - 0.1)$$

Giving $\left| \vec{r}'(t) \right| = 1.3\pi \times 6800 = 8840\pi$

Answer = 27772 km/hr to nearest integer

d. equation of path.

Allow

$$x = 6800 \sin(\pi(1.3t - 0.1))$$

$$\Rightarrow \sin(\pi(1.3t - 0.1)) = \frac{x}{6800}$$

and

$$y = 6800 \cos(\pi(1.3t - 0.1)) - 6400$$

$$\Rightarrow \cos(\pi(1.3t - 0.1)) = \frac{y + 6400}{6800}$$

Using the trig identity $\sin^2(\theta) + \cos^2(\theta) = 1$

$$\text{gives } \left(\frac{x}{6800}\right)^2 + \left(\frac{y + 6400}{6800}\right)^2 = 1$$

gives the equation of the path:

$$x^2 + (y + 6400)^2 = 6800^2$$

$$\text{e. distance} = \sqrt{x^2 + y^2}$$

where $x = 6800 \sin(\pi(1.3t - 0.1))$ and $y = 6800 \cos(\pi(1.3t - 0.1)) - 6400$

allow distance = 1000

gives

$$\sqrt{6800^2(\cos^2 + \sin^2) - 2(6800)(6400)\cos(\pi(1.3t - 0.1)) + 6400^2} = 1000$$

Solve to get $t = 0.11097\dots, t = 0.0429\dots$

Answer: $t = 0.11, t = 0.04$

Question 5

$$\text{a. using kinematics formula } s = \left(\frac{u + v}{2}\right)t \text{ where } u = 0, v = 6, s = 10$$

$$\text{answer } t = \frac{10}{3} \text{ seconds}$$

$$\text{b. using } s = ut + \frac{1}{2}at^2 \text{ where } s = -6, u = 10, a = -9.8$$

$$\text{answer } t = 2.5 \text{ seconds}$$

$$\text{c. to travel down the slide we need } \frac{10}{3} \text{ seconds} + 4 \text{ seconds} = \frac{22}{3} \text{ seconds}$$

$$\text{using } s = ut + \frac{1}{2}at^2 \text{ where } s = -6, t = \frac{22}{3}, a = -9.8 \text{ find } u \text{ using CAS}$$

$$\text{answer } u = 35.1 \text{ ms}^{-1}$$

d. acceleration of boat given by $a = -\frac{1}{10}\sqrt{196-v^2}$

i) Gives the DE: $\frac{dv}{dt} = -\frac{1}{10}\sqrt{196-v^2}$

‘Flip’ to solve

$$\frac{dt}{dv} = -\frac{10}{\sqrt{196-v^2}}$$

$$t = -10 \int \frac{1}{\sqrt{196-v^2}} dv$$

$$\therefore t = -10 \sin^{-1} \frac{v}{14} + c$$

$$\text{where } t = 0, v = 7 \text{ gives } c = \frac{5\pi}{3}$$

$$\therefore t = -10 \sin^{-1} \frac{v}{14} + \frac{5\pi}{3} \quad \text{Simplify and rearrange giving}$$

$$\text{Answer : } v = 14 \sin \left(\frac{\pi}{6} - \frac{t}{10} \right)$$

$$\text{Alternative expressions exist such as } v = 14 \cos \left(\frac{t}{10} + \frac{\pi}{3} \right)$$

ii) Using $v=0$, solve $14 \sin \left(\frac{\pi}{6} - \frac{t}{10} \right) = 0$

$$\text{gives } t = \frac{5\pi}{3}$$

$$\text{iii) Distance} = 14 \int_0^{\frac{5\pi}{3}} \sin \left(\frac{\pi}{6} - \frac{t}{10} \right) dt$$

Answer = 18.8 metres, to 1 decimal place.