

2012 VCAA Specialist Mathematics**Written Examination 1****MAV Suggested answers and solutions****Question 1**

$$\begin{aligned} & \int \frac{6+x}{x^2+4} dx \\ &= \int \left(\frac{6}{x^2+4} + \frac{x}{x^2+4} \right) dx \\ &= 3 \int \frac{2}{x^2+4} dx + \frac{1}{2} \int \frac{2x}{x^2+4} dx \\ &= 3 \tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{2} \log_e |x^2+4| + c \end{aligned}$$

Since $x^2+4 > 0$, an antiderivative is

$$3 \tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{2} \log_e (x^2+4)$$

Question 2

$$\begin{aligned} 2 \cos x &= \sqrt{3} \cot x \\ \therefore 2 \cos x &= \frac{\sqrt{3} \cos x}{\sin x} \\ \therefore 2 \cos x \sin x &= \sqrt{3} \cos x \\ \therefore 2 \cos x \sin x - \sqrt{3} \cos x &= 0 \\ \therefore \cos x (2 \sin x - \sqrt{3}) &= 0 \end{aligned}$$

Either

$$\begin{aligned} \cos x &= 0 \\ \therefore x &= \dots, \frac{\pi}{2}, \frac{3\pi}{2}, \dots \\ \therefore x &= \frac{(2n+1)\pi}{2}, n \in \mathbb{Z} \end{aligned}$$

$$\text{Alternatively, } x = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$$

OR

$$\begin{aligned} \sin x &= \frac{\sqrt{3}}{2} \\ \therefore x &= \dots, \frac{\pi}{3}, \frac{2\pi}{3}, \dots \\ \therefore x &= 2n\pi + \frac{\pi}{3}, 2n\pi + \frac{2\pi}{3}, n \in \mathbb{Z} \end{aligned}$$

Note : Do not cancel by $\cos(x)$, as you will lose a valid set of solutions.

There are other valid ways of generalizing your solutions for each alternative such as

$$\begin{aligned} \sin x &= \frac{\sqrt{3}}{2} \\ \therefore x &= n\pi + (-1)^n \frac{\pi}{3}, n \in \mathbb{Z} \end{aligned}$$

Question 3(a)

$$\begin{aligned} P(z) &= z^3 - z^2 - 2z - 12 \\ P(3) &= 27 - 9 - 6 - 12 \\ \therefore P(3) &= 0 \end{aligned}$$

$\therefore z = 3$ is a solution of the equation

Since the coefficients of the equation are real, by the Conjugate Complex Root Theorem, we know that both $z = 2\text{cis}\left(\frac{2\pi}{3}\right)$ and its conjugate will both be solutions

Now

$$\begin{aligned} z &= 2\text{cis}\left(\frac{2\pi}{3}\right) \\ \therefore z &= 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) \end{aligned}$$

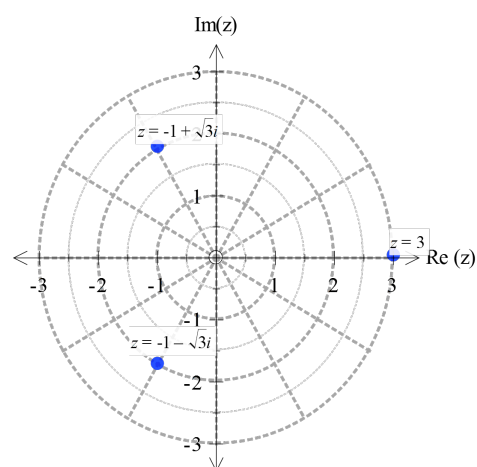
$$\therefore z = 2\left(\frac{-1}{2} + \frac{i\sqrt{3}}{2}\right)$$

$$\therefore z = -1 + i\sqrt{3}$$

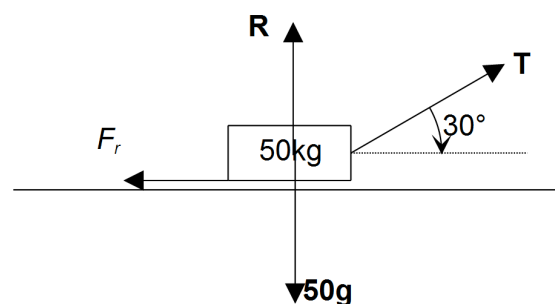
$\therefore$ the solutions are

$$z = 3 + 0i, -1 + i\sqrt{3}, -1 - i\sqrt{3}$$

Question 3 (b)



Question 4(a)



Question 4(b)

Resolving vertically

$$R + T \sin 30^\circ = 50g$$

$$\therefore R + \frac{T}{2} = 50g$$

T is maximum when $R = 0$

$$\therefore T = 100g$$

$$\text{or } T = 980N$$

Question 4 (c)

Resolving vertically

$$R + T \sin 30^\circ = 50g$$

$$\therefore R = 50g - \frac{T}{2}$$

Resolving horizontally

$$Fr = T \cos 30^\circ \dots\dots\dots(i)$$

Limiting Friction

$$Fr = 0.2R$$

$$\therefore Fr = 0.2\left(50g - \frac{T}{2}\right) \dots\dots\dots(ii)$$

Sub (i) into (ii)

$$\frac{T\sqrt{3}}{2} = 10g - \frac{T}{10}$$

$$\therefore \frac{5\sqrt{3}T}{10} = \frac{100g}{10} - \frac{T}{10}$$

$$\therefore T(1 + 5\sqrt{3}) = 100g$$

$$\therefore T = \frac{100g}{1 + 5\sqrt{3}}$$

Question 5

$$y = \arctan(2x)$$

Method 1

$$\therefore y = \tan^{-1}\left(\frac{x}{0.5}\right)$$

$$\therefore \frac{dy}{dx} = \frac{0.5}{0.25 + x^2}$$

Question 5 continued

$$\therefore \frac{dy}{dx} = \frac{2}{1+4x^2}$$

Method 2

$$\text{Use } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+(2x)^2} \times 2$$

$$\therefore \frac{dy}{dx} = \frac{2}{1+4x^2}$$

Now find second derivative

$$\frac{d^2y}{dx^2} = -2(1+4x^2)^{-2}(8x)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-16x}{(1+4x^2)^2}$$

$$\text{Using } \frac{d^2y}{dx^2} = ax\left(\frac{dy}{dx}\right)^2$$

$$\frac{-16x}{(1+4x^2)^2} = \frac{ax(4)}{(1+4x^2)^2}$$

$$\therefore -16x = 4ax$$

$$\therefore a = -4$$

Question 6

Differentiate with respect to x

$$xy^2 + y + (\log_e(x-2))^2 = 14$$

$$\frac{d(xy^2)}{dx} + \frac{dy}{dx} + \frac{d(\log_e(x-2))^2}{dx} = \frac{d(14)}{dx}$$

$$\therefore y^2 + x \frac{dy^2}{dy} \frac{dy}{dx} + \frac{dy}{dx} + \frac{2 \log_e(x-2)}{(x-2)} = 0$$

$$\therefore \frac{dy}{dx}(1+2xy) = -y^2 - \frac{2 \log_e(x-2)}{(x-2)}$$

When $x = 3$ and $y = 2$

$$\frac{dy}{dx} = \frac{-2 \log_e(1) - 4}{1+12}$$

$$\therefore \frac{dy}{dx} = \frac{-4}{13}$$

Question 7Note: $y \geq 0, 1 \leq x \leq 2$

$$\therefore \text{Area} = \int_1^2 (x-1)\sqrt{2-x} dx$$

Let $u = 2-x$ when $x = 1, u = 0$ $x = 2-u$ when $x = 2, u = 0$

$$\frac{du}{dx} = -1$$

$$\therefore \text{Area} = \int_1^0 (2-u-1)u^{\frac{1}{2}}(-du)$$

$$\therefore \text{Area} = \int_0^1 (u^{\frac{1}{2}} - u^{\frac{3}{2}}) du$$

$$\therefore \text{Area} = \left[\frac{2u^{\frac{3}{2}}}{3} - \frac{2u^{\frac{5}{2}}}{5} \right]_0^1$$

$$\therefore \text{Area} = \frac{2}{3} - \frac{2}{5}$$

$$\therefore \text{Area} = \frac{4}{15} \text{ units}^2$$

Question 8

$$v = \frac{2x}{\sqrt{1+x^2}}$$

Method 1

$$a = \frac{d\left(\frac{v^2}{2}\right)}{dx} = \frac{d\left(\frac{2x^2}{1+x^2}\right)}{dx}$$

Using quotient rule

$$\therefore a = \frac{(1+x^2) \times 4x - 2x^2 \times 2x}{(1+x^2)^2}$$

$$\therefore a = \frac{4x}{(1+x^2)^2}$$

Method 2

Using long division and chain rule

$$a = \frac{d(\frac{2x^2}{1+x^2})}{dx} = \frac{d(\frac{2(x^2+1)-2}{1+x^2})}{dx}$$

$$\therefore a = \frac{d(2 - \frac{2}{1+x^2})}{dx}$$

$$\therefore a = \frac{4x}{(1+x^2)^2}$$

Method 3

$$\text{Use } a = v \frac{dv}{dx}$$

$$\therefore a = \frac{2x}{\sqrt{1+x^2}} \frac{d(\frac{2x}{\sqrt{1+x^2}})}{dx} = \frac{2x}{\sqrt{1+x^2}} \left(\frac{2\sqrt{1+x^2} - \frac{2x^2}{\sqrt{1+x^2}}}{(1+x^2)} \right)$$

$$\therefore a = \frac{4x - \frac{4x^3}{1+x^2}}{1+x^2}$$

$$\therefore a = \frac{4x + 4x^3 - 4x^3}{(1+x^2)^2}$$

$$\therefore a = \frac{4x}{(1+x^2)^2}$$

Question 9 a

$$\vec{r}(t) = (2\sqrt{t^2+2} - t^2)\vec{i} + (2\sqrt{t^2+2} + 2t)\vec{j}$$

$$\text{Since } \frac{d(2(t^2+2)^{\frac{1}{2}})}{dt} = \frac{4t(t^2+2)^{-\frac{1}{2}}}{2} \text{ then}$$

Velocity is given by

$$\vec{v}(t) = \left(\frac{2t}{\sqrt{t^2+2}} - 2t\right)\vec{i} + \left(\frac{2t}{\sqrt{t^2+2}} + 2\right)\vec{j}$$

Question 9 b

$$\vec{v}(1) = \left(\frac{2}{\sqrt{3}} - 2\right)\vec{i} + \left(\frac{2}{\sqrt{3}} + 2\right)\vec{j}$$

$$\therefore \text{speed} = \sqrt{\left(\frac{2}{\sqrt{3}} - 2\right)^2 + \left(\frac{2}{\sqrt{3}} + 2\right)^2}$$

$$\therefore \text{speed} = \sqrt{\left(\frac{4}{3} - \frac{8}{\sqrt{3}} + 4\right) + \left(\frac{4}{3} + \frac{8}{\sqrt{3}} + 4\right)}$$

$$\therefore \text{speed} = \sqrt{\frac{8}{3} + 8}$$

$$\therefore \text{speed} = \sqrt{\frac{32}{3}}$$

$$\therefore \text{speed} = \frac{\sqrt{96}}{3}$$

$$\therefore \text{speed} = \frac{4\sqrt{6}}{3}$$

Question 9 c

$$\vec{v}(t) = \left(\frac{2t}{\sqrt{t^2+2}} - 2t\right)\vec{i} + \left(\frac{2t}{\sqrt{t^2+2}} + 2\right)\vec{j}$$

$$\therefore \vec{v}(1) = \left(\frac{2}{\sqrt{3}} - 2\right)\vec{i} + \left(\frac{2}{\sqrt{3}} + 2\right)\vec{j}$$

$$\therefore \frac{dx}{dt} = \left(\frac{2}{\sqrt{3}} - 2\right) \quad \& \quad \frac{dy}{dt} = \left(\frac{2}{\sqrt{3}} + 2\right)$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ \therefore \frac{dy}{dx} &= \frac{2+2\sqrt{3}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2-2\sqrt{3}} \\ \therefore \frac{dy}{dx} &= \frac{2(1+\sqrt{3})}{2(1-\sqrt{3})} \\ \therefore \frac{dy}{dx} &= \frac{1+\sqrt{3}}{1-\sqrt{3}}\end{aligned}$$

Question 9 d

Let $\underline{a} = -\sqrt{3}\underline{i} + \underline{j}$ where

$$\tan \theta_1 = \frac{-1}{\sqrt{3}}, \theta_1 \in \left(\frac{\pi}{2}, \pi\right)$$

$$\therefore \theta_1 = \pi - \frac{\pi}{6}$$

$$\therefore \theta_1 = \frac{5\pi}{6}$$

Let

$$\underline{b} = r(0)$$

$$\therefore \underline{b} = 2\sqrt{2}\underline{i} + 2\sqrt{2}\underline{j}$$

where

$$\tan \theta_2 = \frac{2\sqrt{2}}{2\sqrt{2}}, \theta_2 \in \left(0, \frac{\pi}{2}\right)$$

$$\therefore \tan \theta_2 = 1$$

$$\therefore \theta_2 = \frac{\pi}{4}$$

The required angle is the difference between θ_1 and θ_2

$$\therefore \text{size of angle} = \frac{5\pi}{6} - \frac{\pi}{4}$$

$$\therefore \text{size of angle} = \frac{7\pi}{12}$$

Question 10 a i

$$y = \arcsin(\theta)$$

$$\text{Domain} = [-1, 1]$$

$$f_1(x) = \arcsin\left(\frac{x}{2}\right)$$

$$\therefore -1 \leq \frac{x}{2} \leq 1$$

$$\therefore -2 \leq x \leq 2$$

$$\therefore \text{Dom } f_1(x) = [-2, 2]$$

Question 10 a ii

$$f_2(x) = \frac{3}{\sqrt{25x^2 - 1}}$$

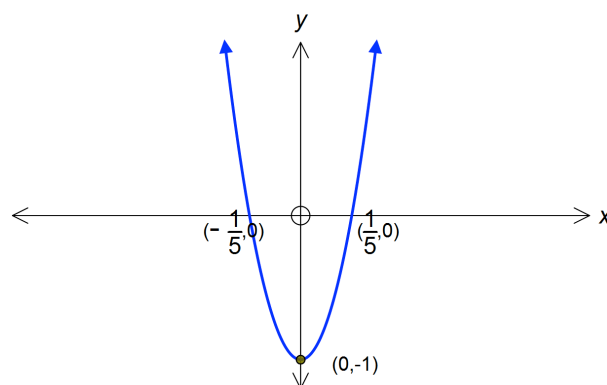
$$\text{We need } 25x^2 - 1 > 0$$

$$\text{Consider } y = 25x^2 - 1$$

$$\text{when } 25x^2 - 1 = 0$$

$$\therefore x = \pm \frac{1}{5}$$

Graphically



Question 10 a ii continued

Now $y > 0$ when $x < \frac{-1}{5}$ or $x > \frac{1}{5}$

$$\therefore \text{Domain } f_2 = (-\infty, \frac{-1}{5}) \cup (\frac{1}{5}, \infty)$$

Question 10 a iii

$$\text{Domain } f(x) = \text{Domain } f_1(x) \cap \text{Domain } f_2(x)$$

$$\therefore \text{Dom } f(x) = [-2, \frac{-1}{5}) \cup (\frac{1}{5}, 2]$$

Question 10 b

$$h(x) = f(x) + g(x)$$

$$\therefore h(\frac{1}{4}) = f(\frac{1}{4}) + g(\frac{1}{4}) = \theta$$

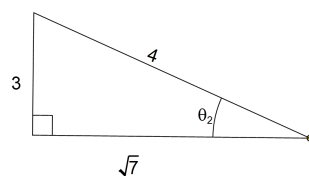
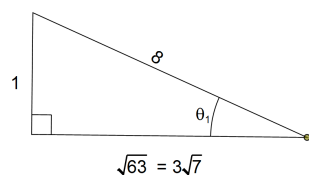
$$\therefore h(\frac{1}{4}) = \arcsin(\frac{1}{8}) + \frac{3}{\sqrt{\frac{25}{16} - 1}} + \arcsin(\frac{3}{4}) - \frac{3}{\sqrt{\frac{25}{16} - 1}}$$

$$\therefore h(\frac{1}{4}) = \arcsin(\frac{1}{8}) + \arcsin(\frac{3}{4})$$

$$\therefore \theta = \theta_1 + \theta_2$$

$$\text{where } \theta_1 = \arcsin(\frac{1}{8}) \text{ and } \theta_2 = \arcsin(\frac{3}{4})$$

as illustrated below



$$\text{Now } \sin \theta = \sin(\theta_1 + \theta_2)$$

$$\therefore \sin \theta = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2$$

$$\therefore \sin \theta = \frac{1}{8} \cos(\arcsin(\frac{3}{4})) + \frac{3}{4} \cos(\arcsin(\frac{1}{8}))$$

$$\therefore \sin \theta = \frac{1}{8} \cdot \frac{\sqrt{7}}{4} + \frac{3}{4} \cdot \frac{3\sqrt{7}}{8}$$

$$\therefore \sin \theta = \frac{10\sqrt{7}}{32} = \frac{5\sqrt{7}}{16}$$