

**2012 VCAA Mathematical Methods CAS
Written Examination 1
MAV Suggested Solutions**

Question 1

a. $y = (x^2 - 5x)^4$ and so $\frac{dy}{dx} = 4(2x - 5)(x^2 - 5x)^3$

b. $f(x) = \frac{x}{\sin(x)}$
 $f'(x) = \frac{\sin(x) \times 1 - x \times \cos(x)}{\sin^2(x)}$
 $f'\left(\frac{\pi}{2}\right) = \frac{\sin\left(\frac{\pi}{2}\right) \times 1 - \frac{\pi}{2} \times \cos\left(\frac{\pi}{2}\right)}{\sin^2\left(\frac{\pi}{2}\right)}$
 $= \frac{1 \times 1 - \frac{\pi}{2} \times 0}{1^2}$
 $= 1$

Question 2

$$\begin{aligned} \int \frac{1}{(2x-1)^3} dx &= \int (2x-1)^{-3} dx \\ &= \frac{1}{-2} \times \frac{1}{2} \times (2x-1)^{-2} \\ &= \frac{-1}{4(2x-1)^2} \end{aligned} \quad \text{(no } c \text{ required for *an* anti-derivative).}$$

Question 3

Let $y = 2x^3 + 1$

Interchanging x and y : $x = 2y^3 + 1$ and so $2y^3 = x - 1$

So $y^3 = \frac{x-1}{2}$

Answer: $h^{-1}(x) = \sqrt[3]{\frac{x-1}{2}}$ for $x \in \mathbb{R}$.

Question 4

x	0	1	2	3
$\Pr(X=x)$	0.2	0.2	0.5	0.1

a. Mean = $0 \times 0.2 + 1 \times 0.2 + 2 \times 0.5 + 3 \times 0.1$
 $= 0 + 0.2 + 1.0 + 0.3$
 $= 1.5$

b. $0.2 \times 0.2 \times 0.2 = 0.008$

c. Getting no phone calls is impossible. The probability table now is:

x	1	2	3
$\Pr(X=x)$	$\frac{0.2}{0.8} = \frac{1}{4}$	$\frac{0.5}{0.8} = \frac{5}{8}$	$\frac{0.1}{0.8} = \frac{1}{8}$

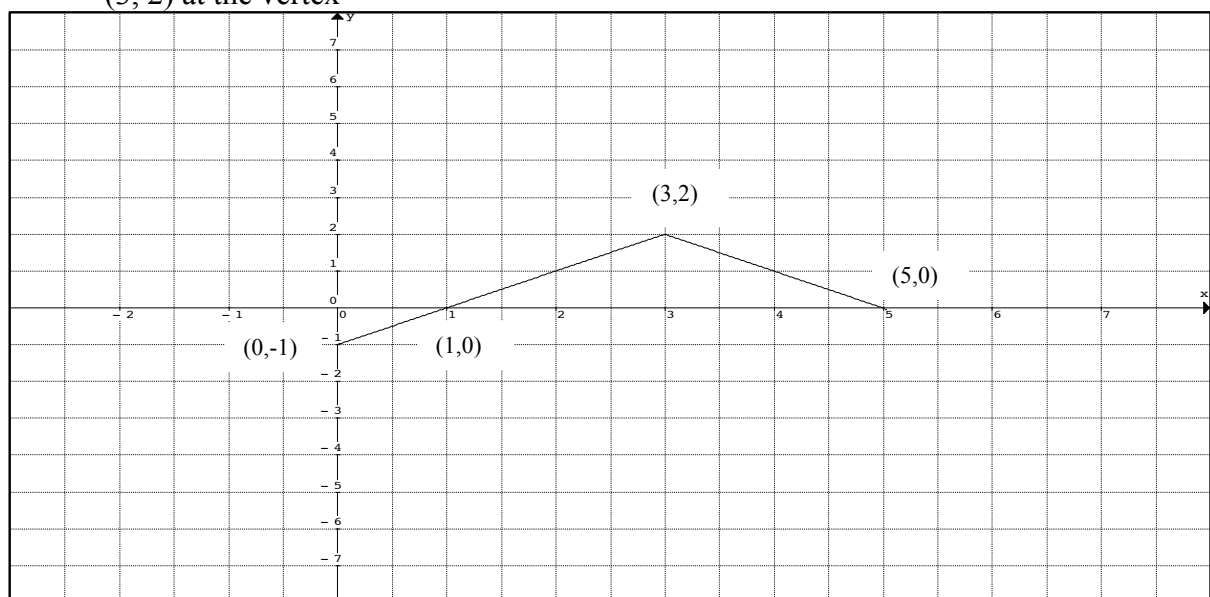
To get a total of four calls over two days means 1 & 3; 2 & 2 and 3 & 1.

The probability is: $\frac{1}{4} \times \frac{1}{8} + \frac{5}{8} \times \frac{5}{8} + \frac{1}{8} \times \frac{1}{4} = \frac{29}{64}$

Question 5

a. The following sets of coordinates **MUST** be put on the graph below:

- x – intercepts at (1, 0) and (5, 0)
- y – intercept at (0, -1)
- full dots at the end-points (5, 0) and (0, -1)
- (3, 2) at the vertex



b. Reflecting $(3, 2)$ in the x -axis gives the point $(3, -2)$. Translating this 5 units to the right gives $(8, -2)$.

c. Reflecting $f(x) = -|x-3| + 2$ in the x -axis gives $g(x) = -(-|x-3| + 2)$ which simplifies to $g(x) = |x-3| - 2$.

Translating this 5 units to the right gives: $h(x) = |(x-5)-3| - 2$.

Answer: $h(x) = |x-8| - 2$ for the domain $[5, 10]$.

Question 6

a. $a \sin\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$ so $a \tan\left(\frac{\pi}{3}\right) = 1$

Answer: $a = \frac{1}{\sqrt{3}}$

Alternatively: $a \sin\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$

$$a \times \frac{\sqrt{3}}{2} = \frac{1}{2}$$

$$a = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

b. $a \sin(x) = \cos(x)$
 $\frac{a \sin(x)}{\cos(x)} = \frac{\cos(x)}{\cos(x)}$ and so $a \tan(x) = 1$

$$\tan(x) = \frac{1}{a} = \sqrt{3}$$

Tangent is positive in 1st and 3rd quadrants. The 3rd quadrant angle is $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

Question 7

$$2\log_e(x+2) - \log_e(x) = \log_e(2x+1) \text{ for } x > 0.$$

$$\log_e(x+2)^2 - \log_e(x) = \log_e(2x+1)$$

$$\log_e\left(\frac{(x+2)^2}{x}\right) = \log_e(2x+1)$$

$$\text{So } \frac{(x+2)^2}{x} = 2x+1$$

$$(x+2)^2 = x(2x+1)$$

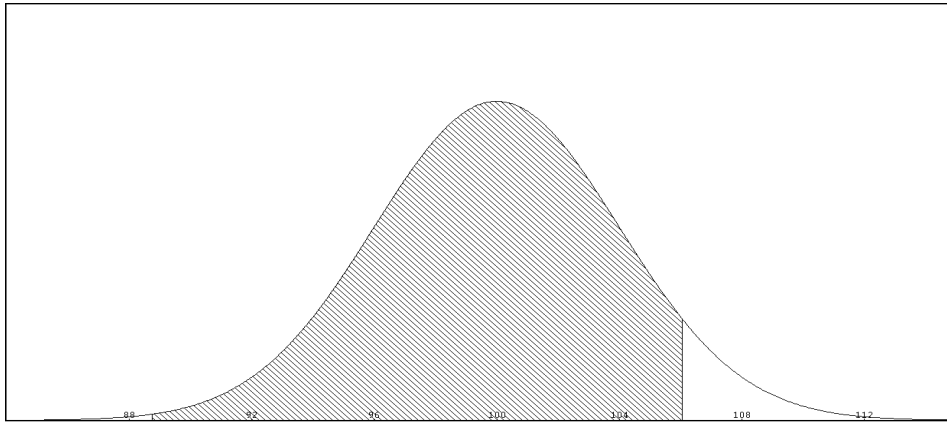
$$x^2 + 4x + 4 = 2x^2 + x$$

$$x^2 - 3x - 4 = 0 \text{ and so } (x-4)(x+1) = 0$$

Hence $x = 4$ is the only answer as $x > 0$.

Question 8

a. The area associated with $\Pr(X < 106) = q$ is shown on the normal curve below.



By symmetry, $\Pr(X < 94) = \Pr(X > 106) = 1 - q$

Hence $\Pr(94 < X < 106) = \frac{1}{2} - (1 - q) = q - \frac{1}{2}$

Answer: $q - \frac{1}{2}$

b. $\int_0^b \frac{bx+1}{12} dx = \frac{5}{8}$

$$\frac{1}{12} \int_0^b (x+1) dx = \frac{5}{8}$$

$$\left[\frac{1}{2}x^2 + x \right]_0^b = \frac{12 \times 5}{8}$$

$$\frac{1}{2}b^2 + b = \frac{15}{2} \text{ which simplifies to } b^2 + 2b - 15 = 0$$

$$(b+5)(b-3) = 0 \text{ and so the only answer is } b = 3 \text{ as } 0 \leq b \leq 4$$

Question 9

If $f(x) = x \sin(x)$ then $f'(x) = 1 \times \sin(x) + x \times \cos(x) = \sin(x) + x \cos(x)$

$$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} f'(x) dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin(x) dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \cos(x) dx$$

$$\left[f(x) \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \left[-\cos(x) \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \cos(x) dx$$

$$\left[x \sin(x) \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \left[-\cos(x) \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \cos(x) dx$$

$$\left(\frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) - \frac{\pi}{6} \sin\left(\frac{\pi}{6}\right) \right) = \left(-\cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{6}\right) \right) + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \cos(x) dx$$

$$\left(\frac{\pi}{2} \times 1 - \frac{\pi}{6} \times \frac{1}{2} \right) = \left(-0 + \frac{\sqrt{3}}{2} \right) + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \cos(x) dx$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \cos(x) dx = \frac{5\pi}{12} - \frac{\sqrt{3}}{2} \text{ which is in the form } a\pi + b \text{ where } a = \frac{5}{12} \text{ and } b = -\frac{\sqrt{3}}{2}.$$

Question 10

a. i. $f(x) = e^{-mx} + 3x$

$$f'(x) = -me^{-mx} + 3$$

= 0 for a stationary value.

$$me^{-mx} = 3 \text{ and so } e^{-mx} = \frac{3}{m}$$

$$-mx = \log_e\left(\frac{3}{m}\right)$$

$$x = -\frac{1}{m} \log_e\left(\frac{3}{m}\right) \text{ or } \frac{1}{m} \log_e\left(\frac{m}{3}\right)$$

ii. This value of x will be positive if $m > 3$.

b. The gradient of the curve is $f'(-6) = -me^{6m} + 3$

The tangent passes through the origin so its equation is $y = (-me^{6m} + 3)x$

When $x = -6$, $y = f(-6) = e^{6m} - 18$

So substituting $e^{6m} - 18 = ((-me^{6m} + 3) \times (-6))$

$$e^{6m} - 18 = 6me^{6m} - 18$$

$$e^{6m} = 6me^{6m}$$

$$\therefore e^{6m}(1 - 6m) = 0$$

$$1 = 6m \text{ as } e^{6m} \text{ is never zero}$$

$$m = \frac{1}{6}$$