

The Mathematical Association of Victoria

Mathematical Methods

2005 Written Examinations

Solutions

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2005 Mathematical Methods Written Examination 1

Suggested Answers & Solutions

Part 1 (Multiple-choice) Answers

1. D 2. B 3. D 4. E 5. D
6. D 7. E 8. D 9. E 10. A
11. D 12. A 13. A 14. C 15. E
16. A 17. B 18. C 19. C 20. C
21. D 22. C 23. D 24. B 25. E
26. B 27. A

Part 1 (Multiple-choice) Solutions

Question 1 [D]

At least two hours means 2 or 3 or 4 hours. The sum of the proportions is

$$\frac{7}{30} + \frac{10}{30} + \frac{4}{30} = \frac{21}{30}$$

Question 2 [B]

Mean = $\sum (x \times \text{proportion})$

$$\begin{aligned} &= 0 \times \frac{3}{30} + 1 \times \frac{6}{30} + \dots + 4 \times \frac{4}{30} \\ &= \frac{0 + 6 + 14 + 30 + 16}{30} \\ &= \frac{66}{30} \end{aligned}$$

Question 3 [D]

InvNorm (0.95, 0, 1) = 1.645 from calculator.

Question 4 [E]

Hypergeometric distribution.

The probability of at least one box with a prize is the same as taking the probability of no prizes from 1.

$$\begin{aligned} &= 1 - \frac{{}^4C_2 \times {}^2C_0}{{}^6C_2} \\ &= 9/15 \end{aligned}$$

Question 5 [D]

Binomial: Bi(n, 0.3)

Pr(John wins no games) = 0.7^n

Therefore $0.7^n = 0.0576$

$n \log(0.7) = \log(0.0576)$

$$n = 8.0023$$

Answer: 8 games

Question 6 [D]

Using Factor 7 (or 9) the factorised form is $x(x-2)((x+2)(x+3))$.

So $(x-3)$ is not a factor.

Answer: $(x-3)$

Question 7 [E]

The turning point of this quadratic is at (3, 1). The function needs to be one-to-one for the inverse to exist. Since the domain extended to $+\infty$, it starts from the x value of the turning point. The domain is $[3, \infty)$ or $a \geq 3$.

Question 8 [D]

$$3e^{2x} = 4$$

$$e^{2x} = 4/3$$

$$\ln(e^{2x}) = \ln(4/3)$$

$$2x = \ln(4/3) \text{ and so } x = \frac{1}{2} \ln(4/3)$$

$$x = 0.144 \text{ to 3 decimal places.}$$

Question 9 [E]

$$2 \log_{10}(x) + 3 = 5 \log_{10}(x)$$

$$\text{Therefore } 3 = 5 \log_{10}(x) - 2 \log_{10}(x)$$

$$3 = 3 \log_{10}(x)$$

$$1 = \log_{10}(x) \text{ and so } x = 10.$$

Question 10 [A]

The amplitude is 2, eliminating alternatives C and D.

$$\text{Period: } \frac{2\pi}{n} = 1 \div \frac{5}{2} = \frac{2}{5} \text{ and so } n = 5\pi.$$

This eliminates E.

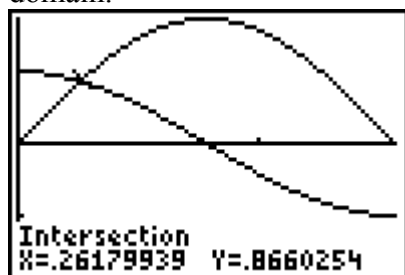
When $t = 0$ the value of $y = 3$. Out of the two alternatives remaining, A fulfils this requirement.

Question 11 [D]

The period of the graph $y = \tan(kx)$ is $\frac{\pi}{k}$ and so the period here is $\pi \div \frac{1}{4}$ or 4π .

Question 12 [A]

Using the Graphics Calculator with $y_1 = \cos 2x$ and $y_2 = \sqrt{3} \sin 2x$, only one point of intersection is found within the domain.



Dividing this x value shown on the screen by π gives 0.0833333, (the decimal equivalent for $\frac{1}{12}$) and so the

exact answer is $\frac{\pi}{12}$.

Or, algebraically,
 $\cos(2x) = \sqrt{3} \sin(2x)$

$$\tan(2x) = \frac{1}{\sqrt{3}}$$

$$2x = \frac{\pi}{6}$$

$$x = \frac{\pi}{12}$$

Question 13 [A]

If $a \sin(x) + c < 0$ then $c < -a$ as the maximum value of $\sin(x)$ is 1.

Question 14 [C]

The x and y values interchange for the inverse function. Because the original function had a horizontal asymptote with equation $y = 1$, this will become a vertical asymptote with equation $x = 1$ for the inverse. Only alternatives C and E display this feature. Also, a function and its inverse are mirror images in the line $y = x$. Only C displays this.

Question 15 [E]

A translation of -2 parallel to the x axis of $y = x^2$ results in a graph with equation $y = (x + 2)^2$.

This new graph is then dilated by a factor of $\frac{1}{2}$ from the y axis. This means that the x intercept of $(-2, 0)$ is now positioned at $(-1, 0)$, i.e. half the distance from the y axis. E is the only alternative to do this.

Answer: $y = (2x + 2)^2$.

Question 16 [A]

A factor of $(x - 1)^2$ is required because the graph touches the x axis at $x = 1$.

This eliminates B and C. It also requires factors of $(x - 3)$ and $(x + 2)$, or the negative of both of these. This eliminates D. If $x = 0$ the value of y is negative. A has this property whereas E does not.

Question 17 [B]

The vertical asymptote has equation $x - 1 = 0$ and so $x - 1 = x + b$. The value of b is -1 . Only alternatives A and B are possible answers. The graph's horizontal asymptote has equation $y = 2$. But $y = c$ is the horizontal asymptote in the given equation. Hence $c = 2$. B satisfies these two requirements.

Question 18 [C]

$$y = \frac{x - 2}{x + 3} = 1 - \frac{5}{x + 3}$$

So as $x \rightarrow \pm\infty$, $\frac{5}{x + 3} \rightarrow 0$, so $y \rightarrow 1$

So asymptotes occur at $y = 1$ and $x = -3$.

Question 19 [C]

The gradient of $f(x)$ should have the following features: It

- is always positive
- approaches zero as $x \rightarrow \infty$
- is large positive as $x \rightarrow 0$.

Only C has these properties.

Question 20 [C]

$$\text{Average rate of change} = \frac{f(1) - f(0)}{1 - 0}.$$

$$f(1) = 1^2 + e^1 \text{ and } f(0) = 0^2 + e^0 = 1$$

$$\text{Average rate} = \frac{1 + e - 1}{1} = e.$$

Question 21 [D]

Using Product Rule: $\frac{d}{dp}(10p(1-p)^9)$

$$= (1-p)^9 \frac{d}{dp}(10p) + 10p \frac{d}{dp}(1-p)^9$$

$$= 10(1-p)^9 + 10p(9)(-1)(1-p)^8$$

$$= 10(1-p)^8 [1-p-9p]$$

$$= 10(1-p)^8 (1-10p)$$

Question 22 [C]

$$\frac{d}{dx} f(e^{2x}) = \frac{d}{du}(f(u)) \times \frac{du}{dx}$$

where $u = e^{2x}$

$$= 2e^{2x} f'(u)$$

$$= 2e^{2x} f'(e^{2x})$$

Question 23 [D]

$$y = 2x^4 - 4x^3 \text{ and so } \frac{dy}{dx} = 8x^3 - 12x^2.$$

At $x = 2$, the values of y and $\frac{dy}{dx}$ are 0

and 16 respectively. The equation of the tangent is

$$y - 0 = 16(x - 2).$$

This is alternative D.

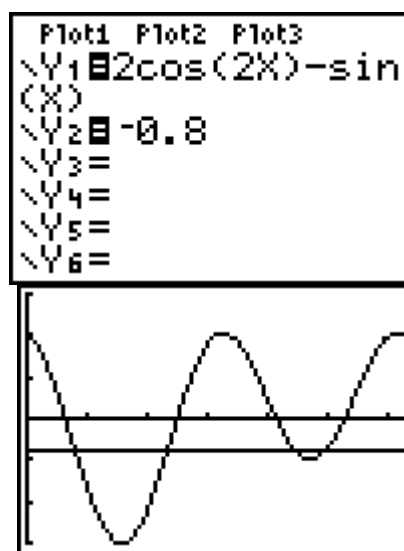
Question 24 [B]

The gradient of this curve is negative for $x > 2$ and positive for all other values of x except at $x = 0$ and $x = 2$ where it is zero.

Alternative B shows these features.

Question 25 [E]

If $f(x) = \sin 2x + \cos x$ then the derivative is $2 \cos 2x - \sin x$.



The domain for this window is $[0, 2\pi]$.
By observation there are 4 points of intersection.

Question 26 [B]

$$\int (ax + b)^n dx = \frac{1}{a} \times \frac{1}{n+1} (ax + b)^{n+1} + c.$$

In this case $a = 2$, $b = -1$, $n = -3/2$ and so the integral is:

$$3 \times \frac{1}{2} \times \frac{1}{-1.5+1} (2x-1)^{-1/2} + c$$

$$= -3(2x-1)^{-1/2} + c$$

Question 27 [A]

If any of the sets for x included zero then the definite integral could include

$$\int_0^0 f(t) dt \text{ which is zero and so } G(x)$$

would not be positive. This eliminates C and D. Alternative E is readily discarded because if $x = a/2$ the area is clearly positive.

Alternative B has certainly got a positive value for $G(x)$ but the word *only* prevents it from being a correct answer.

Alternative A is correct. If $x < 0$ then

$$\int_0^x f(t) dt \text{ will be the negative of a}$$

negative number, i.e. positive. If $x > 0$ then the integral is clearly positive.

Alternative A is correct.

PART II**Question 1**

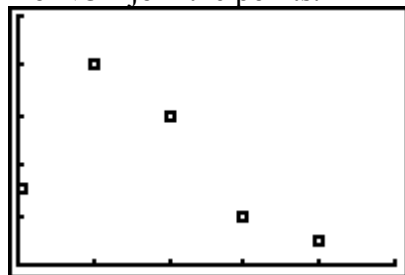
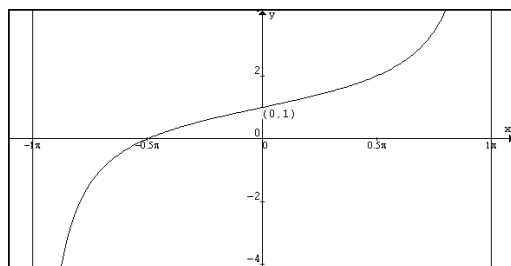
Normalcdf ($-\infty$, 46, 41, 3) = 0.9522.

Answer: 0.952 to three decimal places.

Question 2

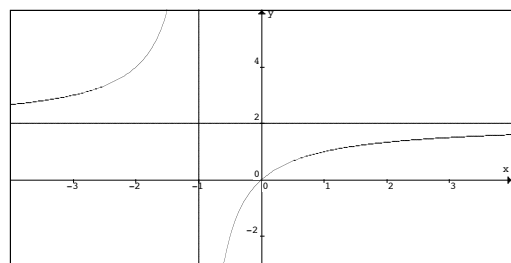
The display from a graphics calculator is shown. The x-values of 0,1,2,3,4 need to be placed horizontally and y values of 0.1, 0.2, 0.3, 0.4, 0.5 vertically.

Do NOT join the points.

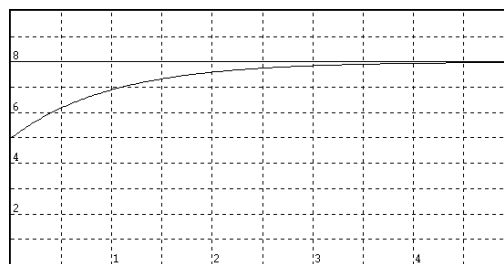
**Question 3**

The vertical asymptotes occur at $x = \pi$ and $x = -\pi$.

The intercepts are $(-\pi/2, 0)$ and $(0, 1)$.

Question 4**Question 5**

a.



Select a point other than (0,5) to indicate scale along the x axis. (1, 6.90) would suffice.

Horizontal asymptote $y = 8$.

Do NOT sketch anything for $x < 0$.

b.

Interchanging x and y :

$$\begin{aligned} x &= 8 - 3e^{-y} \\ 3e^{-y} &= 8 - x \\ e^{-y} &= \frac{8-x}{3} \\ -y &= \log_e\left(\frac{8-x}{3}\right) \end{aligned}$$

$$f^{-1}(x) = -\log_e\left(\frac{8-x}{3}\right) \text{ for } 5 \leq x < 8$$

Question 6

a. $y = (x+2)(x^2-4x+3)$

b. $y = (x+2)(x-1)(x-3)$ and so the x intercepts are 1, 3 and -2 .

c. $\int_{-2}^1 (x^3 - 2x^2 - 5x + 6) dx$

$$\int_1^3 (x^3 - 2x^2 - 5x + 6) dx$$

$$\begin{aligned} &= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x \right]_{-2}^1 \\ &- \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x \right]_1^3 \end{aligned}$$

$$\begin{aligned} &= \left(\frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6 \right) - \left(\frac{16}{4} + \frac{16}{3} - \frac{20}{2} - 12 \right) \\ &- \left(\frac{81}{4} - \frac{54}{3} - \frac{45}{2} + 18 \right) + \left(\frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6 \right) \\ &= 21.08 \end{aligned}$$

Question 7**a.**

$$3^x = e^{kx}$$

Taking logs to base e of both sides:

$$\log_e(3^x) = \log_e(e^{kx})$$

$$x \log_e 3 = kx \text{ so } k = \log_e(3) \text{ for all } x.$$

Answer: $k = \log_e 3$ **b.**

$$\begin{aligned} \frac{d}{dx}(3^x) &= \frac{d}{dx}(e^{x \log 3}) \\ &= \log_e(3) \times e^{x \log 3} \\ &\text{or} \\ &\log_e(3) \times 3^x \end{aligned}$$

VCAA 2005 Mathematical Methods Written Examination 2

Suggested Answers & Solutions

Question 1

a. i (0, 2]

ii Interchange t for y and y for t : $t = 2e^{-y}$ and so $\frac{t}{2} = e^{-y}$

Taking logarithm of both sides (to base e): $\log_e\left(\frac{t}{2}\right) = \log_e(e^{-y}) = -y$

Hence $f^{-1}(t) = -\log_e\left(\frac{t}{2}\right)$ and the domain is (0, 2]

b. i Using the Product Rule $g'(t) = 2(t-1)e^{-t} - (t-1)^2 e^{-t}$
 $= [2t - 2 - (t^2 - 2t + 1)] e^{-t}$
 $= (-t^2 + 4t - 3) e^{-t}$

Answers: $b = 4$, $c = -3$

ii $g'(t) = -(t-1)(t-3)e^{-t}$
 $= 0$ for stationary values.

This occurs at $t = 1$ and $t = 3$. It should be noted that e^{-t} is never zero.

When $t = 1$, $g(1) = 0$ and so one stationary value is (1, 0).

When $t = 3$, $g(3) = (3-1)^2 e^{-3} = 4e^{-3}$ and so the other stationary value is (3, $4e^{-3}$).

Answers: $p = 0$, $m = 3$ and $n = 4e^{-3}$

iii $q(t) = 2(t-1)^2 e^{-t} - 5$

Stationary points occur when $q'(t) = 0$.

ie. $2g'(t) = 0$

$g'(t) = 0$, so at $t = 1$ and 3

This has stationary points at (1, -5) and (3, $8e^{-3} - 5$)

c. i If this has one stationary value only then the discriminant of the quadratic factor is zero.

$$\Delta = (2-a)^2 + 4(a-10)$$

$$\text{Therefore } 4 - 4a + a^2 + 4a - 40 = 0$$

$$a^2 - 36 = 0 \text{ and so } a = \pm 6$$

ii If $h'(t) < 0$ for all t then the same discriminant must be negative as $e^{-t} > 0$ for all t .

Therefore $a^2 - 36 < 0$ and so $-6 < a < 6$.

Question 2

a. Greater than the A Standard:

$$\text{Probability} = \text{normalcdf}(81.8, 10^{10}, 80.8, 4.5) = 0.412$$

Greater than A but less than Olympic:

$$\text{Probability} = \text{normalcdf}(81.8, 90.17, 80.8, 4.5) = 0.393$$

Greater than Olympic:

$$\text{Probability} = \text{normalcdf}(90.17, 10^{10}, 80.8, 4.5) = 0.019$$

b. $\text{Invnorm}(0.1, 80.8, 4.5) = 75.03$ and so $M = 75.03$

$$\begin{aligned} \text{c. } \Pr(X > A \mid X < \text{Olympic}) &= \frac{0.4121 - 0.01866}{1 - 0.01866} \\ &= 0.401 \end{aligned}$$

d. The following table assists in finding this reward:

Length of throw	Amount paid \$	Probability
Under personal best (PB)	0	0.5
Between PB and A	1000	0.088
Between A and Olympic	2000	0.393
Over Olympic	10000	0.019

$$\begin{aligned} \text{Expected reward} &= 0 \times 0.5 + 1000 \times 0.088 + 2000 \times 0.393 + 10000 \times 0.019 \\ &= 0 + 88 + 786 + 190 \\ &= 1064 \end{aligned}$$

Answer: \$1060 (to nearest \$10)

e. i $5 \times 1064 = 5320$

Answer: \$5320

ii Binomial distribution: $n=5$ and $p = 0.393+0.019=0.412$

i.e. $\text{Bi}(5, 0.412)$

Probability of at least three throws = $\Pr(X=3) + \Pr(X=4) + \Pr(X=5)$

$$\begin{aligned} \Pr(3) + \Pr(4) + \Pr(5) &= {}^5C_3 (.412)^3 (.588)^2 + {}^5C_4 (.412)^4 (.588)^1 + (.412)^5 \\ &= 0.2418 + 0.0847 + 0.01187 \\ &= 0.3384 \end{aligned}$$

Answer: 0.338

iii Expected number = $np = 5 \times 0.412$

Answer: 2.06

iv $\Pr(\text{Throws at least one Olympic Record}) + \Pr(\text{Throws all five greater than Standard but less than Olympic})$

$$\begin{aligned} &= (1 - \Pr(\text{throwing no Olympic})) + 0.393^5 \\ &= (1 - 0.981^5) + 0.393^5 \\ &= (1 - 0.9085) + 0.0094 \\ &= 0.101 \end{aligned}$$

Question 3

a. 150 m

b. 50 m

c. i 800 m

ii $1200 - 800 = 400$ m

d. Solve $y_1 = 100 \cos\left(\frac{\pi(x-400)}{600}\right) + 50$ and $y_2 = 20$ to find intersection points on the graphics calculator. The two values of x found are 41.81 and 758.19.
The length of the tunnel = $758.19 - 41.81$
 $= 716$ m (to the nearest metre).

e.
$$\int_{1200}^{800} (100 \cos\left(\frac{\pi(x-400)}{600}\right) + 50) dx$$

$$= \left[\frac{100 \times 600}{\pi} \sin\left(\frac{\pi(x-400)}{600}\right) + 50x \right]_{1200}^{800}$$

$$= \left(\frac{60000}{\pi} \sin\left(\frac{400\pi}{600}\right) + 40000 \right) - \left(\frac{60000}{\pi} \sin\left(\frac{800}{600}\right) + 60000 \right)$$

$$= 13080 \text{ m}^2 \text{ (to the nearest square metre).}$$

f. i $800 - 2k$

ii $400 + 2k$

iii $C = (800 - 2k)^2 + (400 + 2k)^2$

iv $\frac{dC}{dk} = 2 \times -2(800 - 2k) + 2 \times 2(400 + 2k)$

$$= -3200 + 8k + 1600 + 8k$$

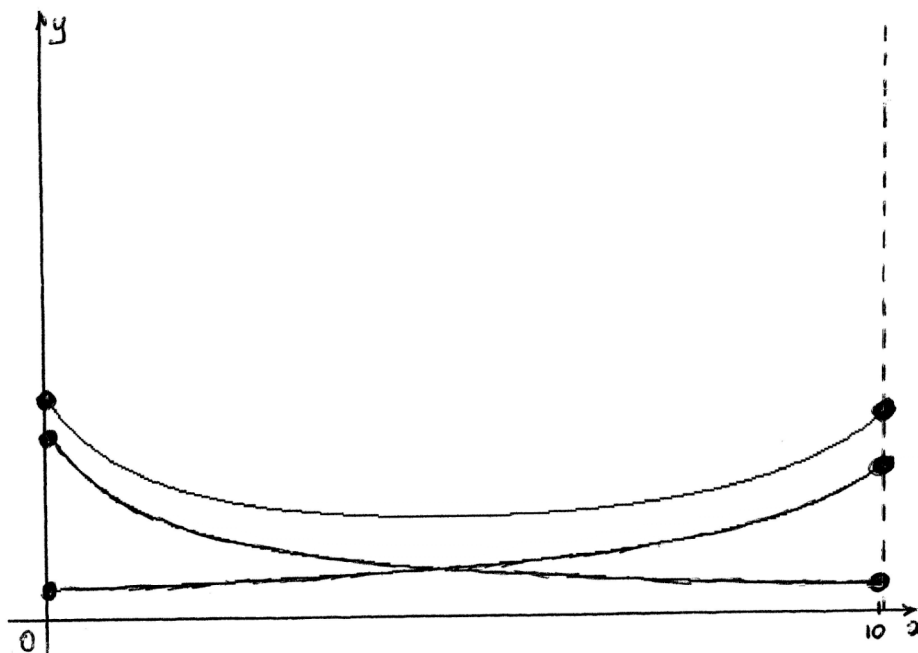
$$= 0 \text{ for a minimum}$$

$$0 = 16k - 1600 \text{ and so } k = 100.$$

Question 4

a. $y = \frac{p}{4} + \frac{q}{8}$

b.



Two features need to be shown here:

- the end-points need to be closed and filled in
- the addition graph is to have a minimum *above* the intersection of the two original graphs.

c. $y = \frac{9}{x+1} + \frac{4}{11-x}$

$$\frac{dy}{dx} = \frac{-9}{(x+1)^2} + \frac{4}{(11-x)^2}$$

$$\frac{-9}{(x+1)^2} + \frac{4}{(11-x)^2} = 0 \text{ for a minimum value.}$$

d. i Now adding these two fractions together gives:

$$\frac{-9(11-x)^2 + 4(x+1)^2}{(x+1)^2(11-x)^2} = 0$$

If this fraction is zero then the numerator is zero:

$$4(x+1)^2 - 9(11-x)^2 = 0$$

This can be factorised using the difference of two squares:

$$[2(x+1) - 3(11-x)][2(x+1) + 3(11-x)] = 0$$

$$[2x+2-33+3x][2x+2+33-3x] = 0$$

$$(5x-31)(-x+35) = 0$$

Hence $x = 6.2$ is the only answer within the domain.

When $x = 6.2$ then the minimum is 2.083 (correct to three decimal places).

ii Solve $\frac{9}{x+1} + \frac{4}{11-x} = 5$ using the calculator to give $x = 0.956$.

The pollution level will be less than 5 for a journey from $x = 0.956$ to $x = 10$, which is a distance of 9.044 km.

e. $\int_0^{10} \left(\frac{9}{x+1} + \frac{4}{11-x} \right) dx$

$$= [9 \log_e(x+1) - 4 \log_e(11-x)]_0^{10}$$

$$= (9 \log_e(11) - 4 \log_e(1)) - (9 \log_e(1) - 4 \log_e(11))$$

$$= 13 \log_e(11)$$

Answer: 31.17 correct to two decimal places.