

MATHEMATICAL METHODS (CAS)

Units 3 & 4 – Written examination 2



2007 Trial Examination

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: D

Explanation:

End points: when $x = -1$ $f(x) = -4$
when $x = 5$ $f(x) = -2$
 $\therefore$ Range = $(-2, 4]$

Question 2

Answer: B

Explanation:

When you solve $2\sin\left(\frac{x}{2}\right) = -\sqrt{3}$ without a specified domain, you get the two general solutions:

$$x = \frac{2(6n-1)\pi}{3} \text{ and } x = \frac{4(3n-1)\pi}{3}, \text{ where } n \in \mathbb{Z}$$

The first seven positive solutions are: $x = \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{20\pi}{3}, \frac{22\pi}{3}, \frac{32\pi}{3}, \frac{34\pi}{3}, \frac{44\pi}{3}$.

The seventh positive solution is $x = \frac{44\pi}{3}$

Question 3*Answer:* C*Explanation:*

$$f(x) = 3e^{x-1} + 2$$

$$x = 3e^{y-1} + 2$$

$$3e^{y-1} = x - 2 \quad \text{Domain} = [5, \infty)$$

$$e^{y-1} = \frac{x-2}{3}$$

$$y = \log_e\left(\frac{x-2}{3}\right) + 1$$

Question 4*Answer:* A*Explanation:*

Solve this system by using matrices:

$$\begin{bmatrix} 3 & 2 & -0.5 & -2 \\ -1 & 1 & 3 & 0.5 \\ 2 & -3 & 1.5 & -3 \\ 2 & 0 & -1 & 5 \end{bmatrix} \times \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4.3 \\ 10.2 \\ 15.3 \\ -17.4 \end{bmatrix}.$$

Which gives you the solution matrix:

$$\begin{bmatrix} 0.3 \\ -0.1 \\ 4.0 \\ -2.8 \end{bmatrix}$$

Question 5*Answer:* B*Explanation:*

$$n = b \quad \therefore \text{period} = \frac{2\pi}{b}$$

$$\text{amplitude} = a$$

$$\text{range} = [-a + c, a + c]$$

Question 6

Answer: C

Explanation:

Use transition matrices, in order: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \left(\begin{bmatrix} 3 \\ -5 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$, so the new image point is $(-5, 2)$.

Question 7

Answer: C

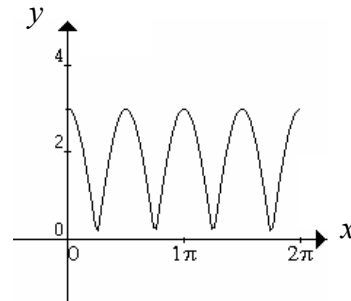
Explanation:

$$y = |3 \cos 2x|$$

Since the range of $3 \cos 2x$ is $[-3, 3]$

then the range of $y = |3 \cos 2x|$ is $[0, 3]$

From the graph of $y = |3 \cos 2x|$ below the range is $[0, 3]$



Question 8

Answer: D

Explanation:

$$2 \log_e 3x - 1 = \log_e a$$

$$\log_e 9x^2 - \log_e e = \log_e a$$

$$\log_e \frac{9x^2}{3} = \log_e a$$

$$\frac{9x^2}{3} = a$$

$$x^2 = \frac{ea}{9}$$

$$x = \frac{\sqrt{ae}}{3}$$

Question 9

Answer: D

Explanation :

Determine the translations of the arbitrary values for x and y : $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x+2 \\ -y \end{bmatrix}$.

Therefore, $\begin{matrix} u = x+2 \\ v = -y \end{matrix}$, or $\begin{matrix} x = u-2 \\ y = -v \end{matrix}$. This gives the new equation $-v = (u-2)^2 + 2$, which expands

to $-v = u^2 - 4u + 6$. Letting $\begin{matrix} x = u \\ v = y \end{matrix}$, and changing signs, gives the final equation $y = 4x - x^2 - 6$,

Question 10

Answer: B

Explanation:

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$r = 7, \frac{dV}{dr} = 4\pi \times 7^2$$

$$\frac{dV}{dr} = 196\pi \text{ cm}^3 / \text{cm}$$

Question 11

Answer: A

Explanation:

$$f(x) = 2x^3 + ax^2 + bx$$

$$f'(x) = 6x^2 + 2ax + b$$

$$x = -2, f'(x) = 0;$$

$$0 = 6(4) - 4a + b$$

$$4a - b = 24 \quad \dots\dots\dots(1)$$

Stationary point at (2, -4)

$$-4 = 16 + 4a - 2b$$

$$12 = 4a - 3b$$

$$6 = 2a - b \quad \dots\dots\dots(2)$$

$$b = 12 \text{ and } a = 9$$

Question 12

Answer: A

Explanation:

$$\frac{1}{3-0.5} \times \int_{0.5}^3 (\log_e x + 1) dx = \frac{1}{3-0.5} \times [x \log_e x]_{0.5}^3 \approx 1.457$$

Question 13

Answer: C

Explanation:

Since $f(x - \pi) = -f(x)$, the only answer which does not satisfy this functional equation is

$$f(x) = 3 \sin 2x, \text{ since } \begin{array}{l} f(x - \pi) = 3 \sin 2(x - \pi) \\ -f(x) = -3 \sin 2x \end{array}.$$

Question 14

Answer: E

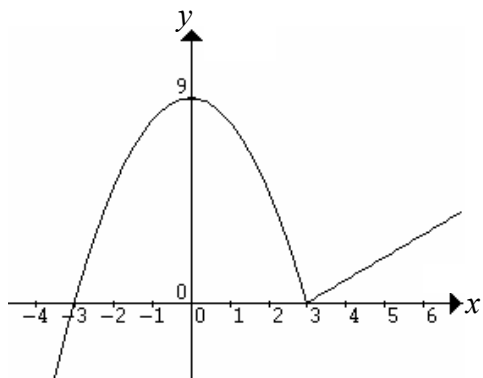
Explanation:

Since a unique tangent cannot be drawn at the sharp point

$x = 3$ the function is not differentiable but is continuous at

$x = 3$

Continuous but not differentiable at $x = 3$



Question 15

Answer: E

Explanation:

$$\begin{aligned}\Pr(X < 2) &= \int_0^2 \frac{1}{2} e^{-\frac{x}{2}} dx \\ &= \left[-e^{-\frac{x}{2}} \right]_0^2 \\ &= -\frac{1}{e} + 1 \\ &= 0.6321\end{aligned}$$

Question 16

Answer: D

Explanation:

$$n = 10, p = 0.2$$

$$\Pr(X = x) = \binom{10}{x} 0.2^x 0.8^{10-x}$$

$$\begin{aligned}\Pr(X = 5) &= \binom{10}{5} 0.2^5 0.8^5 \\ &= 0.0264\end{aligned}$$

Question 17

Answer: A

Explanation:

$$0.2 + 3k + 2a + 3a = 1$$

$$5a + 3k = 0.8 \dots\dots\dots(1)$$

$$E(X) = 2.6$$

$$2.6 = 0.2 + 6k + 6a + 12a$$

$$2.4 = 18a + 6k \dots\dots\dots(2)$$

$$8a = 0.8$$

$$a = 0.1 \text{ and } k = 0.1$$

Question 18

Answer: D

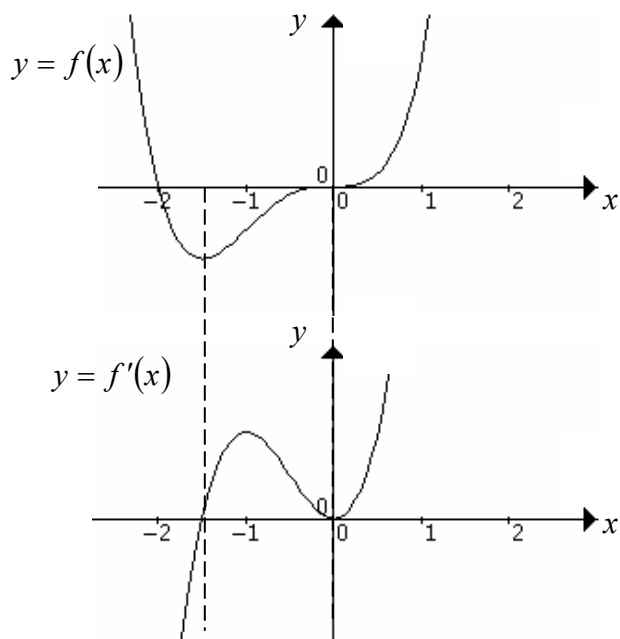
Explanation:

$$\begin{aligned} Pr(X \geq 20) &= Pr\left(z \geq \frac{20-16}{4}\right) \\ &= Pr(z \geq 1) \end{aligned}$$

Question 19

Answer: B

Explanation:



Question 20

Answer: A

Explanation:

$$A = 0.5(8.75 + 8 + 6.75 + 5 + 2.75) \\ = 15.625 \text{ units squared}$$

Question 21

Answer: C

Explanation:

$$A = \int_0^2 m(x^2 + 2) dx \\ 2 = m \left[\frac{x^3}{3} + 2x \right]_0^2 \\ 2 = \left[\frac{8}{3} + 4 \right] \\ 2 = \frac{20}{3} m \\ 20m = 6 \\ m = \frac{3}{10}$$

Question 22

Answer: B

Explanation:

The area between $x = -3$ and $x = 1$ is below the x -axis therefore must take absolute value of the

area or $\int_{-3}^1 f(x) dx$ and add the area of $\int_1^5 f(x) dx$.

$$-A = \int_{-3}^1 f(x) dx + \int_1^5 f(x) dx$$

SECTION 2 : Analysis questions**Question 1**

a. $f(x) = \log_e(5 - 2x)$ M1
 x - intercept : $0 = \log_e(5 - 2x)$

$$5 - 2x = e^0$$

$$2x = 4$$

$$x = 2$$

$$(2, 0)$$

y - intercept : $y = \log_e 5$ A1
 $(0, \log_e 5)$

b. For f to be defined:

$$5 - 2x > 0$$

$$-2x > -5$$

$$x < \frac{5}{2}$$

$\therefore D = \left(-\infty, \frac{5}{2}\right)$ A1

c.

$$f(x) = \log_e(5 - 2x)$$

$$f'(x) = \frac{-2}{5 - 2x}$$
 M1

From **part b** $5 - 2x > 0$ (ie always positive) A1

from $f'(x) = \frac{-2}{5 - 2x}$ the answer will always be negative since -2 will always be divided by a positive number.

d.**i.**

$$y = \log_e(5 - 2x)$$

$$\text{inverse: } x = \log_e(5 - 2y)$$

$$5 - 2y = e^x$$

$$2y = 5 - e^x$$

$$y = \frac{1}{2}(5 - e^x)$$

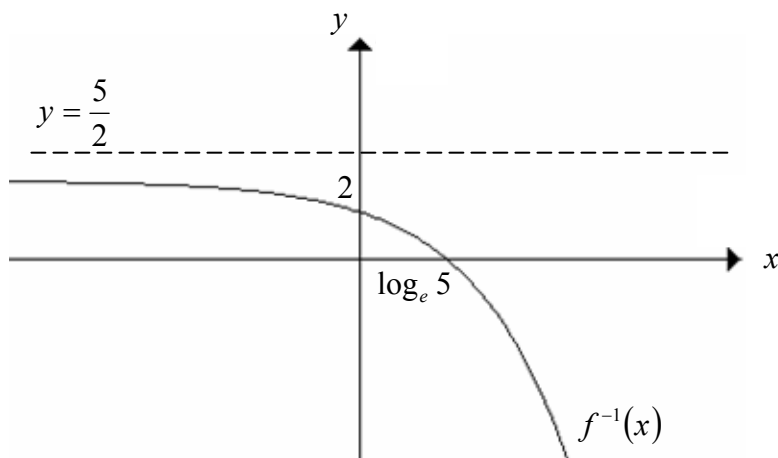
$$\therefore f^{-1}(x) = \frac{1}{2}(5 - e^x)$$

M1

A1

$$\text{ii. } \text{dom } f^{-1} = \text{ran } f = \mathbb{R}$$

A1

e.

M1 correct shape of graph

M1 labelling axes intercepts

f.

$$A = \int_0^{\log_e 5} f(x) dx$$

M1

$$= \int_0^{\log_e 5} \frac{1}{2}(5 - e^x) dx$$

$$= \frac{1}{2} [5x - e^x]_0^{\log_e 5}$$

$$= \frac{5}{2} \log_e 5 - 2 \text{ square units}$$

A1

$$\text{g. Average of } f^{-1}(x) = \frac{1}{\log_e 5} \times \left(\frac{5}{2} \log_e 5 - 2 \right) \approx 1.257$$

M1

A1

Total 14 marks

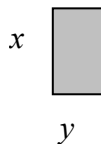
Question 2**a.**

$$\text{Area window} = 24$$

$$yx + yx = 24$$

$$2xy = 24$$

$$y = \frac{12}{x}$$



M1

A1

b.

$$L = 6 + 2y$$

$$L = 6 + 2\left(\frac{12}{x}\right)$$

$$L = 6 + \left(\frac{24}{x}\right)$$

M1

c. $H = 4 + x$

A1

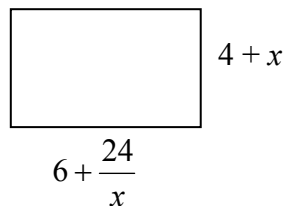
d. $A = \text{area shop front} - \text{Area window}$

$$A = LH - 24$$

$$= \left(6 + \frac{24}{x}\right)(x + 4) - 24$$

$$= \left(24 + 6x + \frac{96}{x} + 24\right) - 24$$

$$= 24 + 6x + \frac{96}{x}, \quad x > 0$$



M1

A1

e.

$$\frac{dA}{dx} = 6 - \frac{96}{x^2}$$

M1

f. For minimum value let $\frac{dA}{dx} = 0$

M1

$$0 = 6 - \frac{96}{x^2}$$

$$0 = 6x^2 - 96$$

$$x^2 = 16$$

$$x = 4$$

minimum value of $x = 4$

$$\text{minimum Area} = 24 + 6x + \frac{96}{x} = 24 + 24 + \frac{96}{4} = 72 \text{ m}^2$$

A1

g.

To verify $x = 4$ is a minimum

If the stationary points of a function, $A = 24 + 6x + \frac{96}{x}$ are at $x = 4$ and $\frac{dA}{dx} = 6 - \frac{96}{x^2}$

For $x = 4$

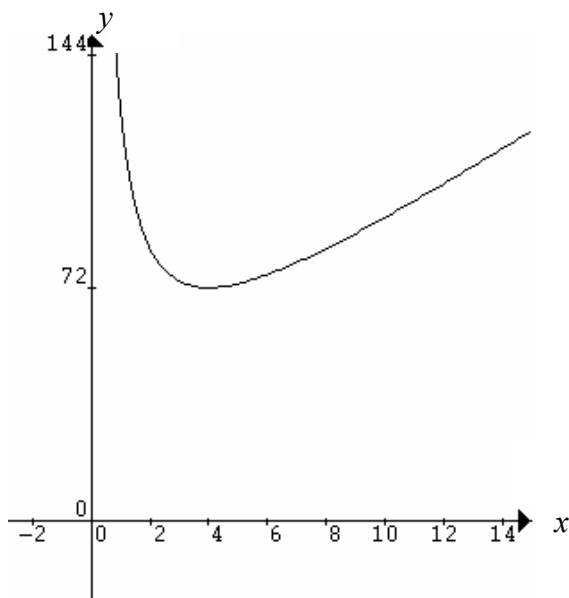
	$x < 4$	$x = 4$	$x > 4$
x value	$x = 2$	$x = 4$	$x = 5$
$\frac{dy}{dx}$	$\frac{dy}{dx} = -18$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 3.33$
	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} > 0$
			

M1

A1

Give co-ordinate of stationary point (4, 72)

h.



M1 correct shape graph
M1 labelling stationary point

i. $y = \frac{12}{x} = \frac{12}{4} = 3$

A1

Total 14 marks

Question 3**a.**

$$\text{i. } \Pr(X = 5) = 0.3 \quad (\text{from the table}) \quad \text{A1}$$

$$\begin{aligned} \text{ii. } \Pr(\text{less than half of their Music questions correctly}) \\ &= \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) \quad \text{M1} \\ &= 0.06 + 0.04 + 0.3 \\ &= 0.4 \quad \text{A1} \end{aligned}$$

$$\begin{aligned} \text{b. } p + 3q + q + p + p + p &= 1 \quad \text{M1} \\ 4p + 4q &= 1 \\ 4p + p &= 1 \quad \text{since } p = 4q \\ 5q &= 1 \end{aligned}$$

$$p = \frac{1}{5} = 0.2 \quad \text{M1}$$

$$\therefore \Pr(X = 3) = p = 0.2 \quad \text{A1}$$

$$\begin{aligned} \text{c. } \therefore \Pr(\text{Total of 10 for Maths and Music}) &= 0.2 \times 0.3 \\ &= 0.06 \quad \text{A1} \end{aligned}$$

d. Let Y = the number of times the celebrity team competes on the next 6 shows
 Y is a Binomial distribution where $n = 6$ and $p = 0.85$

$$\begin{aligned} \Pr(Y = y) &= \binom{6}{y} 0.85^y (0.15)^{6-y} \\ &= 0.3771 \quad \text{A1} \end{aligned}$$

e.

$$0.9^n > 0.5 \quad \text{M1}$$

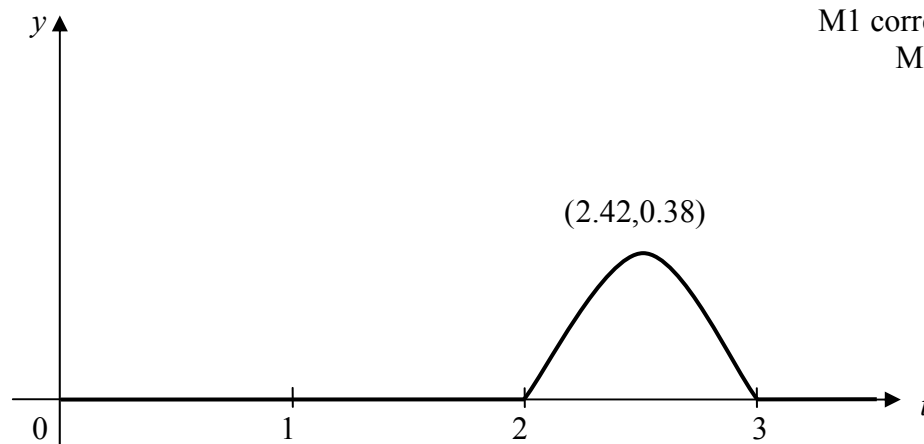
$$\log_e 0.85^n \geq \log_e 0.5$$

$$n \leq \frac{\log_e 0.5}{\log_e 0.85}$$

$$n \leq 4.26$$

$\therefore$ A probability of greater than 0.5 of playing on the next 4 shows. A1

f.



M1 correct shape of graph
M1 stationary point.

g.

$$\begin{aligned}\Pr(T < 150) &= \int_2^{\frac{150}{60}} (t^3 - 9t^2 + 26t - 24) dt \\ &= \left[\frac{t^4}{4} - 3t^3 + 13t^2 - 24t \right]_2^{2.5} \\ &= 0.141\end{aligned}$$

A1

h.

$$\begin{aligned}E(T) &= \int_2^3 t f(t) dt \\ &= \int_2^3 t(t^3 - 9t^2 + 26t - 24) dt \\ &= \int_2^3 (t^4 - 9t^3 + 26t^2 - 24t) dt \\ &= 0.62 \text{ hrs} \\ &= 37 \text{ minutes}\end{aligned}$$

A1

i.

$$T = \begin{bmatrix} 0.8 & 0.25 \\ 0.2 & 0.75 \end{bmatrix}$$

A1

j.

Since $s_0 = \begin{bmatrix} 5000 \\ 5000 \end{bmatrix}$,

it follows that: $s_1 = T \times s_0 = \begin{bmatrix} 5250 \\ 4750 \end{bmatrix}$, $s_2 = T \times s_1 = \begin{bmatrix} 5387.5 \\ 4612.5 \end{bmatrix}$, $s_3 = T \times s_2 \approx \begin{bmatrix} 5463 \\ 4537 \end{bmatrix}$. M1

Therefore, quiz = 5463 viewers and club = 4537 viewers. A1

k.

Steady state means that: $\begin{bmatrix} 0.8 & 0.25 \\ 0.2 & 0.75 \end{bmatrix} \times \begin{bmatrix} Q \\ C \end{bmatrix} = \begin{bmatrix} Q \\ C \end{bmatrix}$, so $\begin{matrix} 0.8Q + 0.25C = Q \\ 0.2Q + 0.75C = C \end{matrix}$ M1

Therefore: $\begin{matrix} 0.8Q + 0.25C = Q \\ Q + C = 1 \end{matrix}$ M1

It follows that: $\begin{matrix} Q = 0.556 \\ C = 0.444 \end{matrix}$, so the Quiz show will have 55.6% of the viewers, and the

Clubbing show will have 44.4% of the viewers. A1

Total 20 marks

Question 4

a. y- intercept : $y = 4 - \frac{1}{2}e^{\frac{x}{2}} - \frac{1}{2}e^{-\frac{x}{2}}$

$$y = 4 - \frac{1}{2}e^0 - \frac{1}{2}e^0$$

$$y = 3$$

$$\Rightarrow b = 3$$

M1

b. x-intercept :

Solve $4 - \frac{1}{2}e^{\frac{x}{2}} - \frac{1}{2}e^{-\frac{x}{2}} = 0$, to get $a = \pm 4.127$, but use only the positive value ($a = 4.127$)

A1

c.

i.

$$\begin{aligned} A &= 2[f(1) + f(2) + f(3)] \\ &= 2(2.872 + 2.457 + 1.648) \\ &= 13.95 \text{ m}^2 \end{aligned}$$

A1

ii. cost = Area $\times$ 35
= \$485.25

A1

d. $y = ax^2 + bx + c$
 $c = 3$

(0,3) $y = 0 + 3$

(0, 3) $y = ax^2 + bx + 3$

(4, 0) $0 = 16a - 4b + 3$ (1)

(-4, 0) $0 = 16a - 4b + 3$ (2)

(2) + (3) $32a + 6 = 0$

$$a = \frac{-3}{16}$$

$$b = 0$$

$$y = -\frac{3}{16}x^2 + 3$$

M1 for finding value of a
M1 for finding value of b
M1 for finding value of a

e.

$$\begin{aligned} A &= 2 \int_0^4 \left(3 - \frac{3}{16} x^2 \right) dx \\ &= 2 \left[3x - \frac{x^3}{16} \right]_0^4 \\ &= 16m^2 \end{aligned}$$

A1

f.

$$\begin{aligned} \text{Average Height} &= \frac{1}{2(4.127)} \times \int_{-4.127}^{4.217} \left(4 - \frac{1}{2} e^{\frac{x}{2}} - \frac{1}{2} e^{-\frac{x}{2}} \right) dx \\ &\approx 2.122 \text{ m}^2 . \end{aligned}$$

M1

A1

Total 10 marks