

SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 1



2010 Trial Examination

SOLUTIONS

Question 1

- a. Vectors $\mathbf{a} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = m\mathbf{i} + n\mathbf{j} - 2\mathbf{k}$ are perpendicular, therefore their scalar product must be zero.

$$(\mathbf{i} - \mathbf{j} + 4\mathbf{k}) \bullet (m\mathbf{i} + n\mathbf{j} - 2\mathbf{k}) = 0 \Rightarrow m - n - 8 = 0$$

$$m = n + 8 \quad (1) \quad \text{M1}$$

$$\vec{AB} = \vec{OB} - \vec{OA} = (m-1)\mathbf{i} + (n+1)\mathbf{j} - 6\mathbf{k}$$

$$|\vec{AB}| = \sqrt{(m-1)^2 + (n+1)^2 + 36} = 2\sqrt{14} \quad (2) \quad \text{M1}$$

Substituting (1) and (2) and then simplifying gives

$$n^2 + 8n + 15 = 0, \text{ so } n = -3 \text{ or } n = -5 \quad \text{A1}$$

$$\text{Therefore } m=5 \text{ or } m=3. \quad \text{A1}$$

So the possible values of m and n are $n = -5, m = 3$ or $n = -3, m = 5$.

- b. The angle between $\mathbf{a} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\vec{AB} = 4\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$ is determined by finding the angle between $\vec{AB}$ and $\vec{AO}$.

$$\vec{AO} \bullet \vec{AB} = -4 - 2 + 24 = 18 \quad \text{M1}$$

$$\cos \theta = \frac{18}{2\sqrt{14} \times \sqrt{18}} = \frac{3}{2\sqrt{7}} \Rightarrow p = 3, q = 2 \quad \text{A1}$$

Question 2

- a. If $z = i$ is a solution, then $p(i) = 0$

$$p(i) = i^3 - ci^2 + 3i^2 + 1 - i$$

$$-i + c - 3 + 1 - i = 0$$

M1

$$c = 2 + 2i$$

- b. Method 1

By dividing polynomials:

$$z^3 - (2 + 2i)z^2 + 3iz + 1 - i = (z - i)(z^2 - (2 + i)z + (1 + i)) \quad \text{A1}$$

The quadratic equation $z^2 - (2 + i)z + (1 + i) = 0$ can be solved by using the quadratic formula.

$$z = \frac{(2 + i) \pm \sqrt{(2 + i)^2 - 4(1 + i)}}{2} \quad \text{M1}$$

$$= \frac{2 + i \pm \sqrt{-1}}{2}$$

$$z = 1 \text{ or } z = 1 + i \quad \text{A2}$$

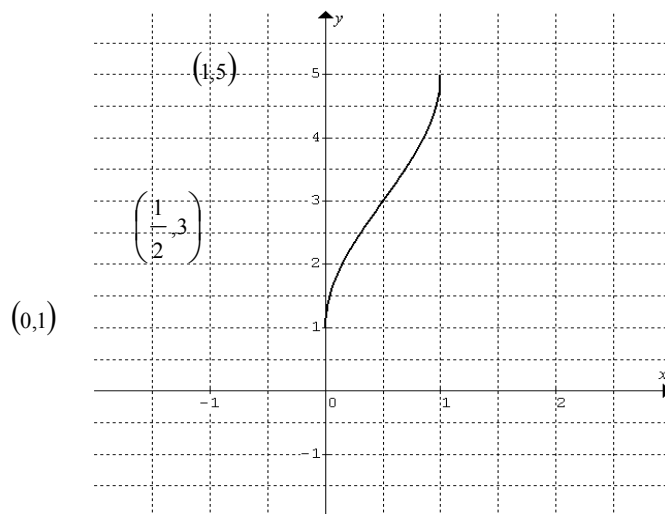
Method 2

By inspection, $p(1) = 1^3 - (2 + 2i) + 3i + 1 - i = 0$, therefore $z - 1$ is a factor of

$$z^3 - (2 + 2i)z^2 + 3iz + 1 - i. \quad \text{M2}$$

$(z - 1)(z - i) = z^2 - (1 + i)z + i$ is the quadratic factor and $z - (1 + i)$ is the third linear factor.

The required two solutions are $z = 1$ and $z = 1 + i$ A2

Question 3

Domain: $-1 \leq 2x - 1 \leq 1 \Rightarrow 0 \leq x \leq 1$

Correct graph A1
Correct domain and range A1

Range: $\left[\frac{4}{\pi} \times \frac{-\pi}{2} + 3, \frac{4}{\pi} \times \frac{\pi}{2} + 3 \right] = [1, 5]$

Inflection point $\left(\frac{1}{2}, 3 \right)$ A1

Question 4

If $\tan \alpha = \frac{1}{12}$, $\tan \beta = \frac{2}{5}$ and $\tan \gamma = \frac{1}{3}$, where α, β and γ are acute angles, show that

$$\alpha + \beta + \gamma = \frac{\pi}{4}.$$

$$\tan(\alpha + \beta + \gamma) = \frac{\tan(\alpha + \beta) + \tan \gamma}{1 - \tan(\alpha + \beta)\tan \gamma}$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{\frac{1}{12} + \frac{2}{5}}{1 - \frac{2}{60}} = \frac{\frac{29}{60}}{\frac{58}{60}} = \frac{29}{58} = \frac{1}{2} \quad \text{M1A1}$$

$$\tan(\alpha + \beta + \gamma) = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = 1 \quad \text{M1}$$

$$\alpha + \beta + \gamma = \tan^{-1}(1) = \frac{\pi}{4}$$

Note that since all α, β and γ are all less than $\frac{\pi}{4}$ (since their tangents are less than 1) it is impossible for $\alpha + \beta + \gamma$ to be in the third quadrant. We can therefore disregard the possibility that $\alpha + \beta + \gamma = \frac{5\pi}{4}$.

Question 5

a. $y^2 = xy - \log_e y$

$$2y \frac{dy}{dx} = y + x \frac{dy}{dx} - \frac{1}{y} \frac{dy}{dx} \quad \text{M1}$$

$$2y \frac{dy}{dx} - x \frac{dy}{dx} + \frac{1}{y} \frac{dy}{dx} = y$$

$$\frac{dy}{dx} \left(2y - x + \frac{1}{y} \right) = y \Rightarrow \frac{dy}{dx} = \frac{y^2}{2y^2 - xy + 1} \quad \text{A1}$$

b. $xy = y^2 + \log_e y \Rightarrow x = y + \frac{\log_e y}{y} \quad \text{A1}$

$$\frac{dx}{dy} = 1 + \frac{\frac{1}{y} \times y - \log_e y}{y^2} = \frac{y^2 + 1 - \log_e y}{y^2}. \quad \text{A1}$$

Alternative approach: differentiate implicitly with respect to y to get:

$$2y = x + y \frac{dx}{dy} - \frac{1}{y}$$

$$\Rightarrow \frac{dx}{dy} = \frac{2y^2 - xy + 1}{y^2}.$$

The result in part c then follows immediately from this.

c. Substituting $x = y + \frac{\log_e y}{y}$ into $\frac{dy}{dx} = \frac{y^2}{2y^2 - xy + 1}$ gives

$$\frac{dy}{dx} = \frac{y^2}{2y^2 - y^2 - \log_e y + 1} = \frac{y^2}{y^2 - \log_e y + 1} = \frac{1}{\frac{dx}{dy}} \quad \text{M1}$$

Question 6

$$V = \pi \int_2^4 \left(\frac{x+1}{\sqrt{x^2-1}} \right)^2 dx \quad \text{A1}$$

$$\begin{aligned} V &= \pi \int_2^4 \frac{(x+1)^2}{x^2-1} dx \\ &= \pi \int_2^4 \frac{x+1}{x-1} dx \\ &= \pi \int_2^4 \left(1 + \frac{2}{x-1} \right) dx \quad \text{M1} \\ &= \pi \left[x + 2 \ln|x-1| \right]_2^4 \\ &= \pi (4 + 2 \ln 3 - 2 - 2 \ln 1) \\ &= 2\pi (1 + \ln 3) \text{ units}^3 \quad \text{A1} \end{aligned}$$

Question 7

- a. Let $y = x \sin^2 x$. Using the product rule:

$$\frac{dy}{dx} = \sin^2 x + 2x \sin x \cos x \quad \text{A1}$$

b. $x \sin^2 x = \int (\sin^2 x + 2x \sin x \cos x) dx \quad \text{M1}$
 $x \sin^2 x = \int \sin^2 x dx + \int x \sin(2x) dx$

Using double angle formula $\sin^2 x = \frac{1 - \cos(2x)}{2}$, we have

$$\begin{aligned} \int x \sin(2x) dx &= x \sin^2 x - \int \frac{1 - \cos(2x)}{2} dx \\ &= x \sin^2 x - \frac{x}{2} + \frac{1}{4} \sin(2x) \quad \text{M1} \end{aligned}$$

$$\int_0^{\frac{3\pi}{2}} x \sin(2x) dx = \left[x \sin^2 x - \frac{x}{2} + \frac{1}{4} \sin(2x) \right]_0^{\frac{3\pi}{2}} = \frac{3\pi}{4} \quad \text{M1A1}$$

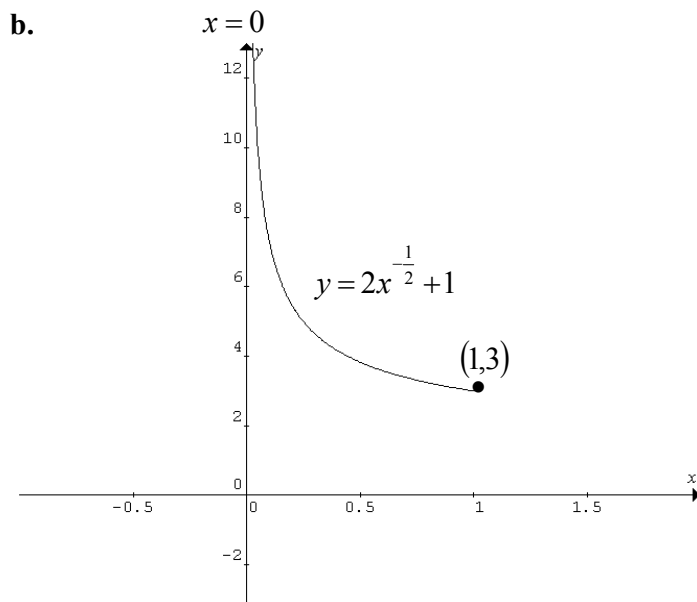
Question 8

a. From $\mathbf{r}(t) = e^{-2t}\mathbf{i} + (2e^t + 1)\mathbf{j}$, $t \geq 0$, we have $x = e^{-2t}$, $y = 2e^t + 1$

$$e^t = x^{-\frac{1}{2}}, \quad y = 2x^{-\frac{1}{2}} + 1 \quad \text{A1}$$

As $t \geq 0$, the domain is $x \in (0, 1]$ and the range is $y \in [3, +\infty)$ M2

(from their Cartesian equation)



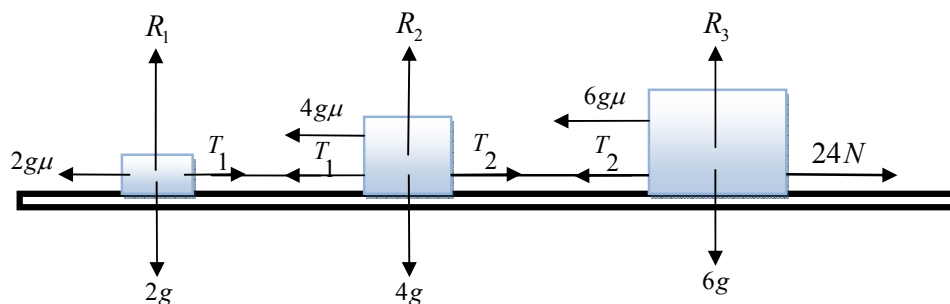
correct shape M1

asymptotic behavior and end point shown M1

(from their Cartesian equation)

Question 9

a.



All forces correct A2
(one mark off for any errors)

$$24 - \frac{1}{10} \times 12g = 12a \Rightarrow a = 2 - \frac{g}{10} \quad \text{A1}$$

b. $T_1 - 2g\mu = 2a \Rightarrow T_1 = 4N$ M1

$$T_2 - 6g\mu = 6a \Rightarrow T_2 = 12N \quad \text{M1}$$