

SOLUTIONS: Trial Exam 2013

MATHEMATICAL METHODS

Written Examination 2

SECTION 1: Multiple Choice

1. D 2. C 3. B 4. B 5. A 6. D 7. C 8. E 9. B 10. A 11. D
 12. B 13. D 14. E 15. D 16. E 17. C 18. A 19. E 20. A 21. D 22. C

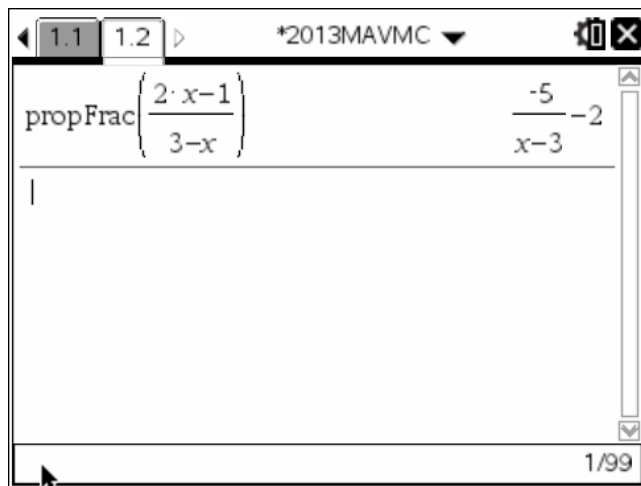
Question 1

$$f(x) = \frac{2x-1}{3-x} = \frac{-5}{x-3} - 2$$

The maximal domain is $R \setminus \{3\}$.

The range is $R \setminus \{-2\}$.

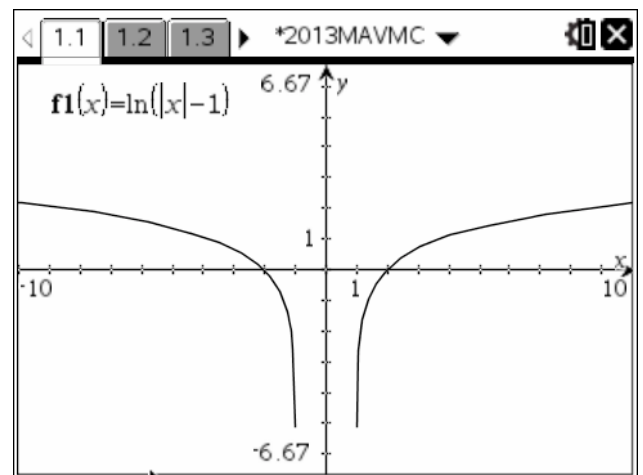
D



Question 2

The equations of the asymptotes are $x = -1$ and $x = 1$

C



Question 3

$$f: R^+ \rightarrow R, f(x) = \log_e(x), \text{ and } g: \left(\frac{1}{2}, \infty\right) \rightarrow R, g(x) = (2x-1)^2$$

$$f(g(x)) = \log_e(2x-1)^2$$

$$f(g(x)) = 2\log_e\left(2x - \frac{1}{2}\right)$$

Dilation by a factor of 2 from the x -axis, dilation by a factor of a $\frac{1}{2}$ from the y -axis and a translation

of a $\frac{1}{2}$ of a unit to the right. **B**

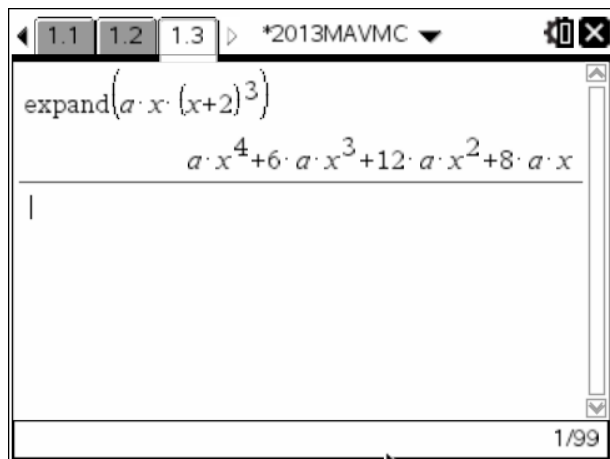
Question 4

$$f(x) = Ax(x+2)^3 = Ax^4 + 6Ax^3 + 12Ax^2 + 8Ax$$

$$f(x) = ax^4 + bx^3 - 24x^2 + cx$$

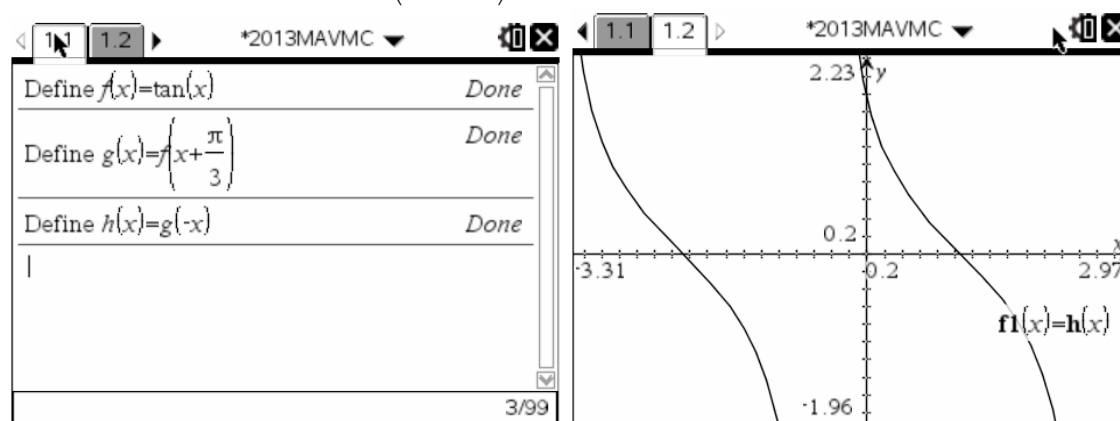
$$12A = -24, A = -2$$

$$b = 6A = -12$$

B**Question 5**

Translation of $\frac{\pi}{3}$ in the negative direction of the x axis $\rightarrow y = \tan\left(x + \frac{\pi}{3}\right)$,

Reflection in the y axis $\rightarrow y = \tan\left(-x + \frac{\pi}{3}\right)$

A**Question 6**

Amplitude of 2

Translation of c in the negative direction from the x axis

$$\text{Range is } [-2 - c, 2 - c] = [-(2 + c), (2 - c)]$$

D**Question 7**

$$f(x + h) \approx f(x) + hf'(x)$$

$$f(x + h) - f(x) \approx hf'(x)$$

$$h = 0.1, f(x) = \sqrt[3]{x}, f'(x) = \frac{1}{3x^{\frac{2}{3}}}, f'(8) = \frac{1}{3 \times 8^{\frac{2}{3}}} = \frac{1}{12}$$

$$hf'(x) = 0.1 \times \frac{1}{12} = \frac{1}{120} \quad \mathbf{C}$$

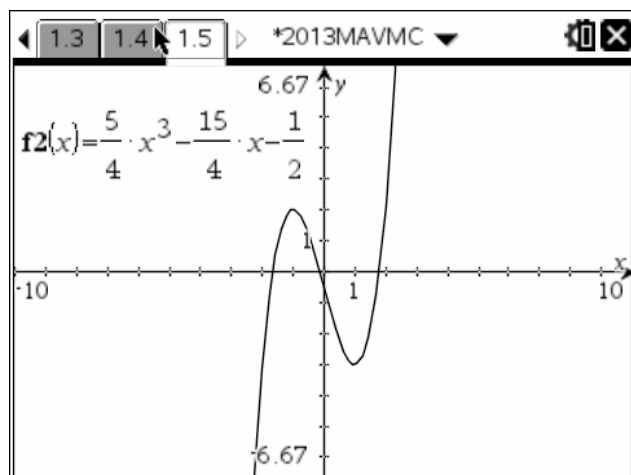
Question 8

$$g(x) + c = 0$$

Sketch a possible graph for g .

For one solution f needs to be translated down more than 2 units or translated up more than 3 units.

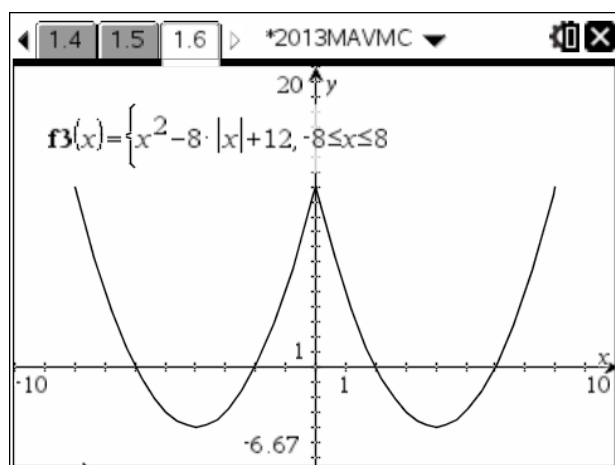
$$\{c : c < -2\} \cup \{c : c > 3\}$$

E**Question 9**

$$g : [-8, 8] \rightarrow R, \text{ where } g(x) = x^2 - 8|x| + 12$$

The graph is not differentiable at the endpoints or the sharp point.

$$x = -8, x = 0, x = 8$$

B**Question 10**

$$g(x) = e^{-x}$$

$$\text{Area} = 0.5(g(0) + g(0.5) + g(1) + g(1.5))$$

$$= 0.5 \left(1 + \frac{1}{e^{0.5}} + \frac{1}{e} + \frac{1}{e^{1.5}} \right) \quad \mathbf{A}$$

Question 11

$f : [a, b] \rightarrow \mathbb{R}$, where $f(x) = x - 1$

The average value will be zero if a and b are equally spaced either side of $x = 1$ as the area above the line $y = 0$ will equal the area below the line $y = 0$.

$$[-2, 2] \quad \mathbf{D}$$

Question 12

$$\int_1^3 (f(x)) dx = 5$$

$$\begin{aligned} 2 \int_1^3 (f(x) - 1) dx &= 2 \int_1^3 (f(x)) - 2[x]_1^3 \\ &= 10 - (6 - 2) = 6 \end{aligned} \quad \mathbf{B}$$

Question 13

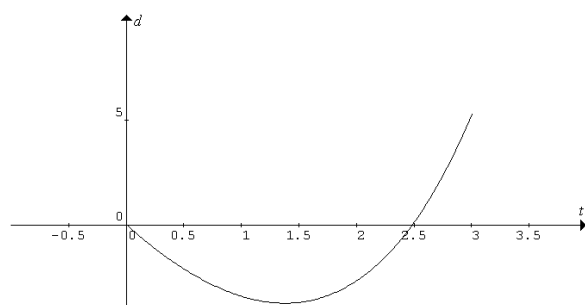
$$a = t^2 + 3$$

$$v = \frac{t^3}{3} + 3t + c$$

$$-5 = \frac{0}{3} + 3 \times 0 + c$$

$$v = \frac{t^3}{3} + 3t - 5$$

$$d = \frac{t^4}{12} + \frac{3t^2}{2} - 5t + c_1$$

**D**

Option A is acceleration against time

Option B is velocity against time

Option C is $y = 2t - 5$

Option E is $y = 2$

Question 14

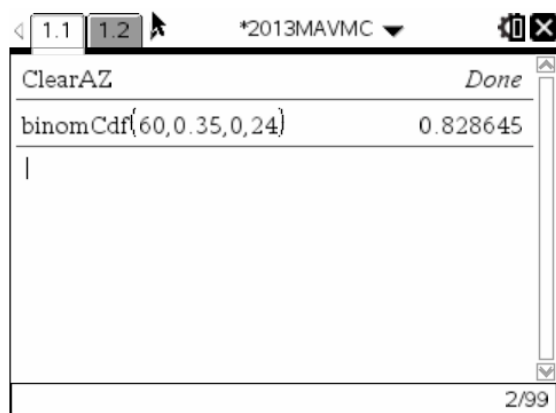
$$\begin{aligned} SD(X) &= \sqrt{E(X^2) - [E(X)]^2} \\ &= \sqrt{1.44 - a^2} \end{aligned} \quad \mathbf{E}$$

Question 15

Let X be the number who play a musical instrument out of 60

$$X \sim \text{Bi}(60, 0.35)$$

$$\Pr(X < 25) = 0.8286 \text{ correct to four decimal places}$$

D**Question 16**

$$\begin{bmatrix} 0.75 & 0.55 \\ 0.25 & 0.45 \end{bmatrix}^2 \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.657 \\ 0.343 \end{bmatrix}$$

0.657

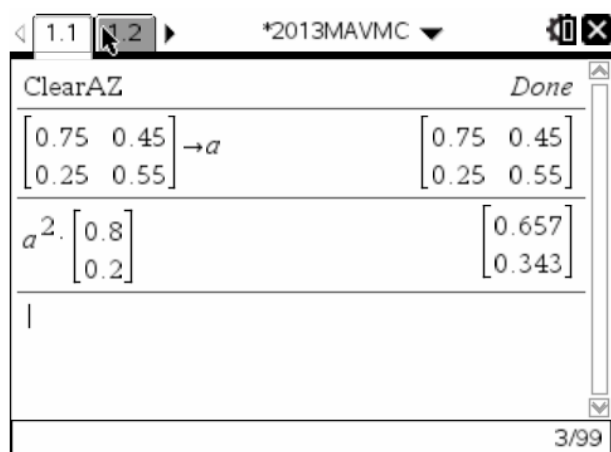
E**OR**

$$w \times w \times w + w \times w' \times w + w' \times w \times w + w' \times w' \times w$$

$$0.8 \times 0.75^2 + 0.8 \times 0.25 \times 0.45 + 0.2 \times 0.45 \times 0.75 + 0.2 \times 0.55 \times 0.45$$

$$= 0.45 + 0.09 + 0.0675 + 0.0495$$

$$= 0.657$$

E**Question 17**

$$\int_2^6 f(t) dt = -\int_6^2 f(t) dt$$

C**Question 18**

Gradient is $a \rightarrow a = \frac{2a}{2}$

$$\text{Area} = \frac{1}{2} \times 2 \times 2a$$

$$= 2a$$

$$2a = 1$$

$$a = \frac{1}{2}$$

A

OR

$$a \int_1^3 (x-1) dx = 1 \quad \text{Solve on the CAS or}$$

$$a \left[\frac{x^2}{2} - x \right]_1^3 = 1$$

$$a \left\{ \left[\frac{9}{2} - 3 \right] - \left[\frac{1}{2} - 1 \right] \right\} = 1$$

$$a \{4 - 2\} = 1$$

$$2a = 1$$

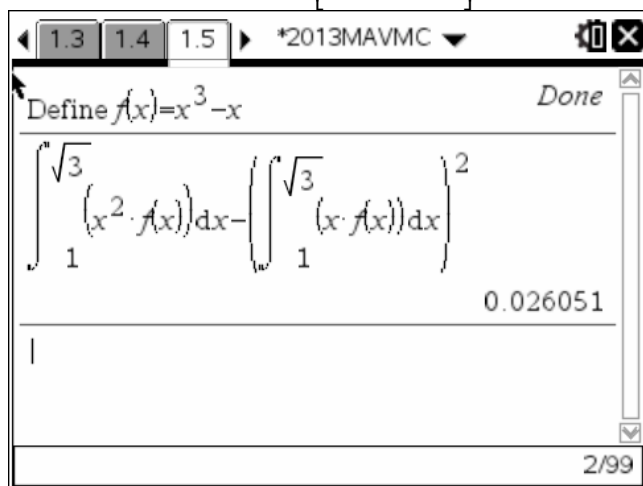
$$a = \frac{1}{2}$$

A

Question 19

Define $f(x) = x^3 - x$

$$\text{Var}(X) = \int_1^{\sqrt{3}} x^2 f(x) dx - \left[\int_1^{\sqrt{3}} x f(x) dx \right]^2$$



0.0261 correct to four decimal places

E

Question 20

$$\Pr(X < a_2) = 0.925$$

$$\Pr\left(Z < \frac{a_2 - 13.4}{3.2}\right) = 0.925$$

The left screenshot shows the 'Inverse Normal' dialog box with the following values:

- Area: 0.925
- μ : 13.4
- σ : 3.2

The right screenshot shows the calculator screen with the command `invNorm(0.925,13.4,3.2)` entered, resulting in the value 18.0065.

$$18.0065 - 13.4 = 4.6065$$

$$13.4 - 4.6065 = 8.7935$$

$$a_1 = 8.79; a_2 = 18.01$$

A

The calculator screen shows the following calculations:

<code>invNorm(0.925,13.4,3.2)</code>	18.0065
<code>18.006500707037-13.4</code>	4.6065
<code>13.4-4.606500707037</code>	8.7935

Question 21

$$1 - [\Pr(X=0) + \Pr(X=1)]$$

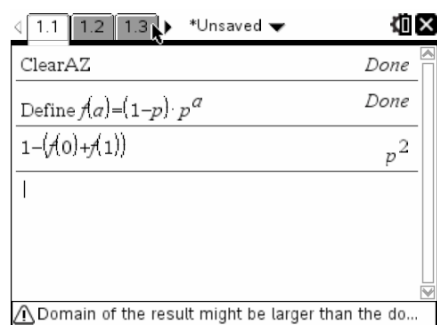
$$1 - [(1-p) \times 1 + (1-p) \times p]$$

$$1 - [(1-p)(1+p)]$$

$$1 - [1 - p^2]$$

$$p^2$$

D**OR**

**Question 22**

$$\frac{dV}{dt} = -750 \text{ cm}^3 / \text{min}$$

$$h = 3r$$

$$r = \frac{h}{3}$$

$$V = \frac{1}{3} \pi r^3 h$$

$$= \frac{1}{3} \pi \frac{h^3}{9}$$

$$= \frac{\pi h^3}{27}$$

$$\frac{dV}{dh} = \frac{\pi h^2}{9}$$

$$\begin{aligned}\frac{dh}{dt} &= \frac{dh}{dV} \times \frac{dV}{dt} \\ &= \frac{9}{\pi h^2} \times -750 \\ &= -\frac{6750}{\pi h^2}\end{aligned}\quad \text{C}$$

END OF SECTION 1 SOLUTIONS

SECTION 2: Extended Answer Solutions**Question 1**

a. $TSA = \pi rs + \pi r^2$

Curved surface area = $\pi rs = 100$

$$s = \sqrt{r^2 + h^2} \quad \mathbf{1M}$$

$$\pi r \sqrt{r^2 + h^2} = 100 \quad \mathbf{1M}$$

$$\sqrt{r^2 + h^2} = \frac{100}{\pi r}$$

$$r^2 + h^2 = \frac{10\,000}{\pi^2 r^2}$$

$$h^2 = \frac{10\,000}{\pi^2 r^2} - r^2$$

$$h^2 = \frac{10\,000 - \pi^2 r^4}{\pi^2 r^2}$$

$$h = \frac{\sqrt{10\,000 - \pi^2 r^4}}{\pi r} \quad \text{as required} \quad \mathbf{1M} \text{ Show that}$$

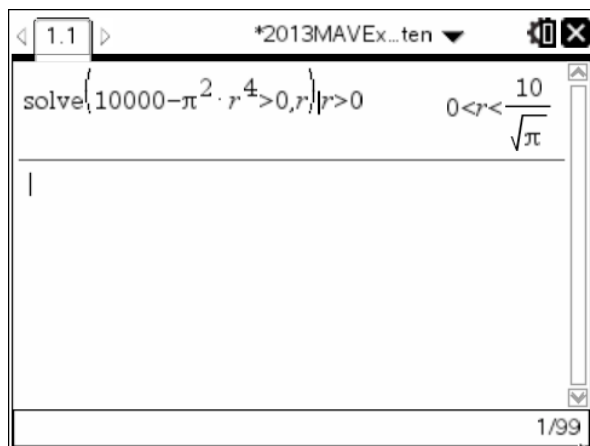
b. $V = \frac{1}{3} \pi r^2 h$

$$= \frac{1}{3} \pi r^2 \frac{\sqrt{10\,000 - \pi^2 r^4}}{\pi r}$$

$$V = \frac{r \sqrt{10\,000 - \pi^2 r^4}}{3} \quad \text{as required} \quad \mathbf{1M} \text{ Show that}$$

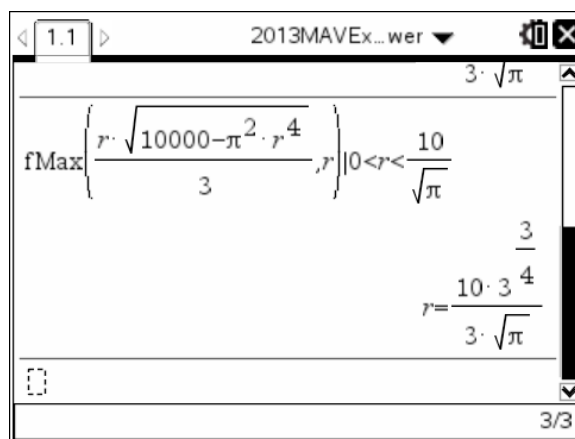
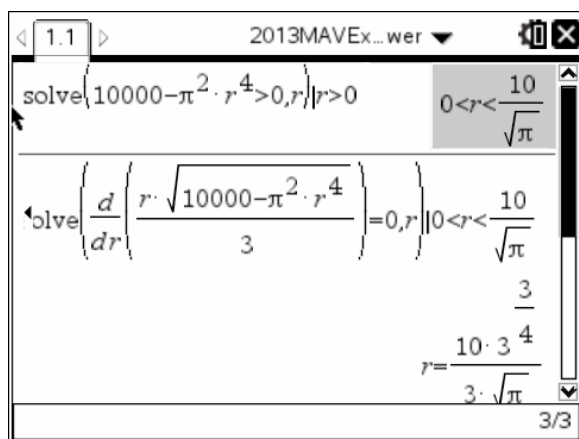
c. $10\,000 - \pi^2 r^4 > 0 \quad \mathbf{1M}$

$$0 < r < \frac{10}{\sqrt{\pi}} \quad \mathbf{1A}$$

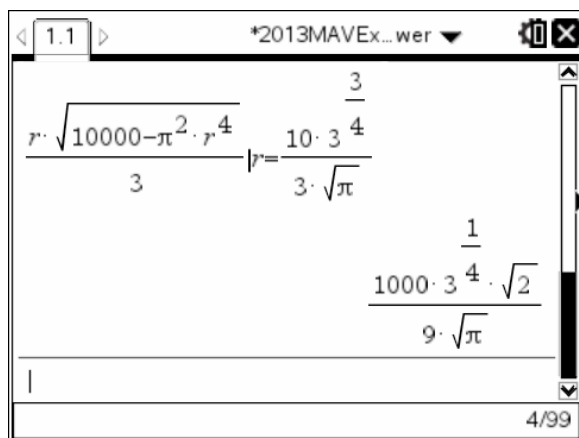


d. Solve $V'(r) = 0$ or find the maximum value

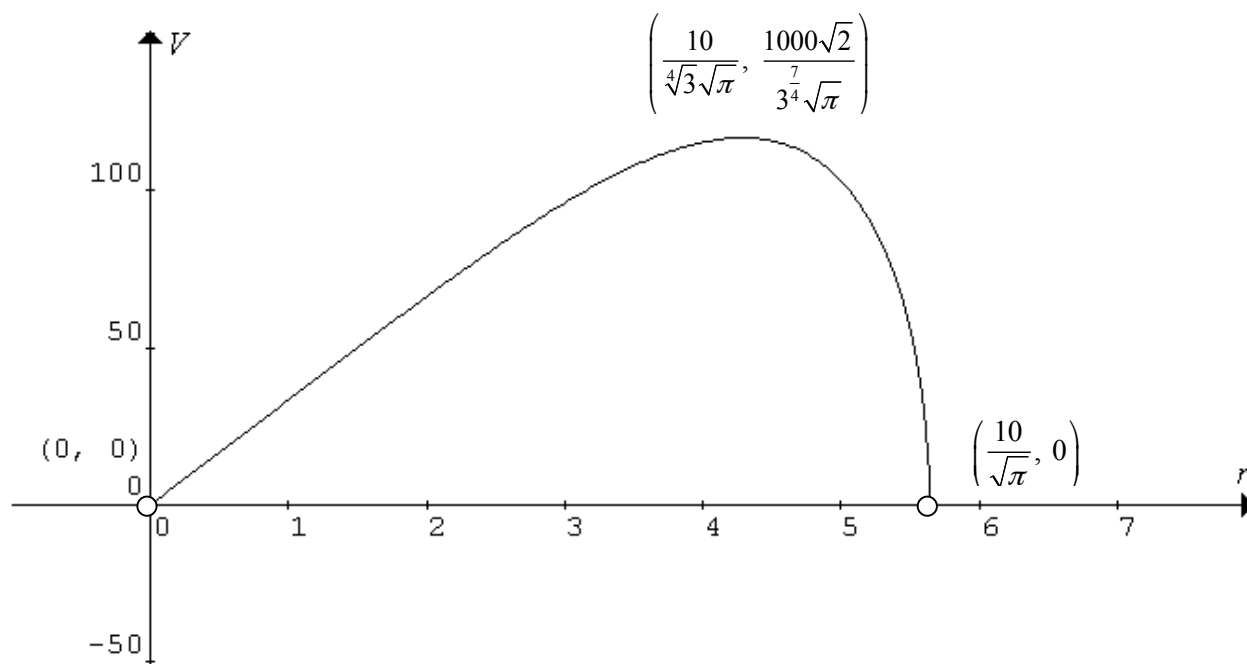
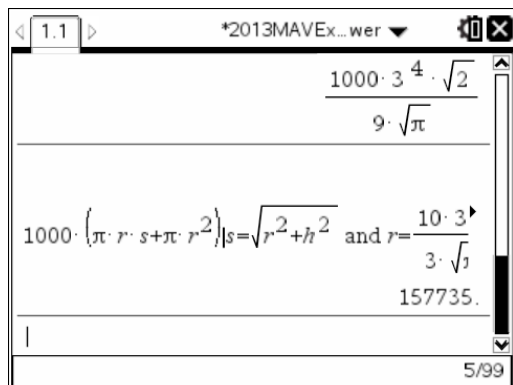
$$r = \frac{10}{\sqrt[4]{3}\sqrt{\pi}} \quad 1A$$



$$V(r_{\max}) = \frac{1000\sqrt{2}}{3^4\sqrt{\pi}} \quad 1A$$

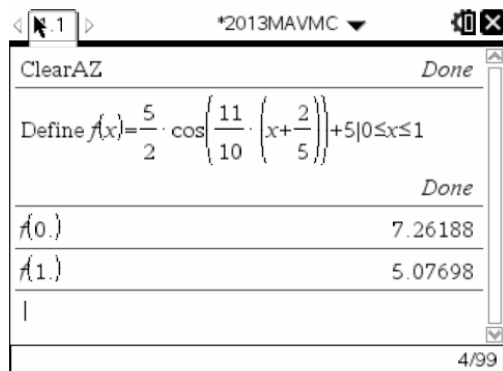


e.

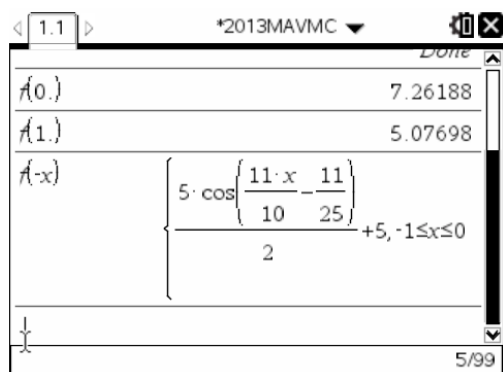
**Shape 1A****Coordinates 1A****Drawn to scale ½ A****Open circles ½ A****Round down****f. TSA for 1000 cones = $1000(\pi rs + \pi r^2)$ 1M****Substitute $\pi rs = 100$ and $r = \frac{10}{\sqrt[4]{3\sqrt{\pi}}}$ 1M****TSA for 1000 cones = $157\,735\text{ cm}^2$ 1A**

Question 2

a. i. $A = (0, 7.262), B = (1, 5.077)$

2x1A

ii. $g(x) = f(-x)$



$$g(x) = \frac{5}{2} \cos\left(\frac{11}{10}x - \frac{11}{25}\right) + 5 \quad \mathbf{1A}$$

or

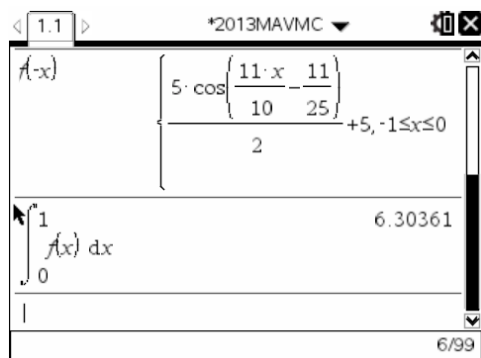
$$g(x) = \frac{5}{2} \cos\left(-\frac{11}{10}x + \frac{11}{25}\right) + 5 \quad \mathbf{1A}$$

iii. domain: $-1 \leq x \leq 0$ **1A**

b. i. Area $= \int_0^1 f(x) dx$

$$= 6.3036...$$

$$= 6.304 \text{ m}^2 \text{ correct to three decimal places} \quad \mathbf{1A}$$



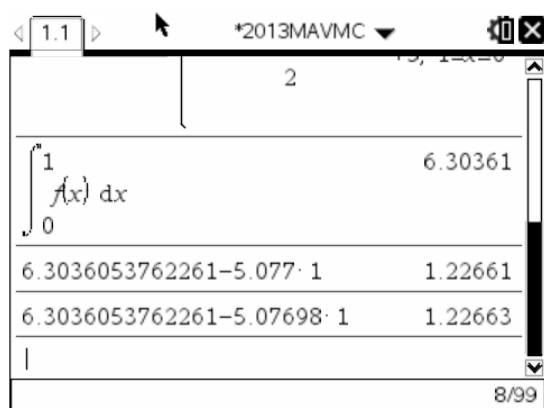
ii. By symmetry, Total area = $2 \times 6.3036... = 12.607 \text{ m}^2$ correct to three decimal places **1A**

iii. $a = 5.0769...$

1M

$$\text{Area: } \int_0^1 f(x) dx - (5.0769... \times 1) = 1.2266...$$

Area: 1.227 m^2 correct to three decimal places **1A**

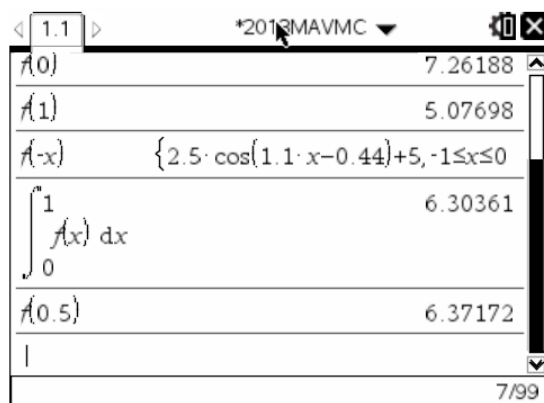


c. i. $C = (0.500, 6.372)$

1A

By symmetry, $F = (-0.500, 6.372)$

1A

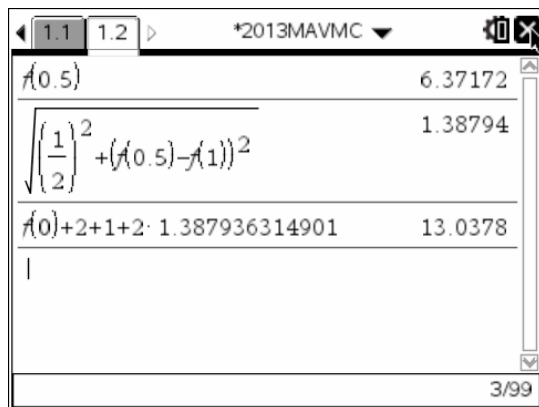


ii. $AO = 7.261\dots$, $BD = 2$, $FC = 1$, **1M**

$$CE = \sqrt{\left(\frac{1}{2} - 0\right)^2 + (6.371\dots - 5.076\dots)^2} = 1.387\dots = FE \quad \mathbf{1M}$$

Total length $= 7.216\dots + 2 + 1 + 2 \times 1.387\dots$

$= 13.04$ m (nearest cm) **1A**

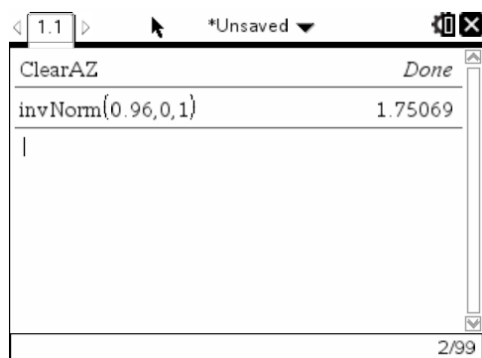
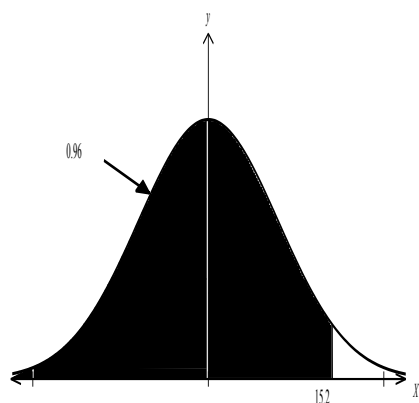


Question 3

a. i. % non-defective: $100 - (4 + 6) = 90\%$ **1A**

ii. Let X mm be the diameter of a cylinder.

$$\Pr(X < 15.2) = 0.96 \rightarrow \Pr\left(Z < z = \frac{15.2 - \mu}{\sigma}\right) = 0.96$$



$$\frac{15.2 - \mu}{\sigma} = 1.75069... \quad \text{eq 1} \quad \mathbf{1A}$$

$$\Pr(X < 14.7) = 1 - 0.94 \rightarrow \Pr\left(Z < z = \frac{14.7 - \mu}{\sigma}\right) = 0.06$$

1.1		*Unsaved	✖
ClearAZ		Done	
invNorm(0.96,0,1)	1.75069		
invNorm(0.06,0,1)	-1.55477		
		3/99	

$$\frac{14.7 - \mu}{\sigma} = -1.55477... \quad \text{eq 2}$$

1A

Solving the simultaneous for μ and σ :

1.1		*Unsaved	✖
ClearAZ		Done	
invNorm(0.96,0,1)	1.75069		
invNorm(0.06,0,1)	-1.55477		
solve $\left\{ \frac{15.2-x}{s} = 1.75069 \text{ and } \frac{14.7-x}{s} = -1.55477 \right\}$			
$x = 14.9352 \text{ and } s = 0.151265$			
		4/99	

$$\mu = 14.935, \sigma = 0.151$$

1M show that

$$95\% \text{ of cylinders: } 14.935 \pm 2 \times 0.151 = 14.633 \text{ mm}; 15.237 \text{ mm}$$

1A

$$\text{iii. } \Pr(14.8 < X < 15.0) = 0.48092...$$

$$= 0.48 \text{ correct to two decimal places}$$

1A

1.1		*Unsaved	✖	1.1		*Unsaved	✖
ClearAZ		Done		ClearAZ		Done	
invN	Lower Bound: 14.8	69		invNorm(0.96,0,1)	1.75069		
invN	Upper Bound: 15.0	77		invNorm(0.06,0,1)	-1.55477		
solve	$\mu: 14.935$	4		solve $\left\{ \frac{15.2-x}{s} = 1.75069 \text{ and } \frac{14.7-x}{s} = -1.55477 \right\}$			
	$\sigma: 0.151$	65		$x = 14.9352 \text{ and } s = 0.151265$			
OK Cancel				normCdf(14.8,15.,14.935,0.151) 0.48092			
		4/99					
				5/99			

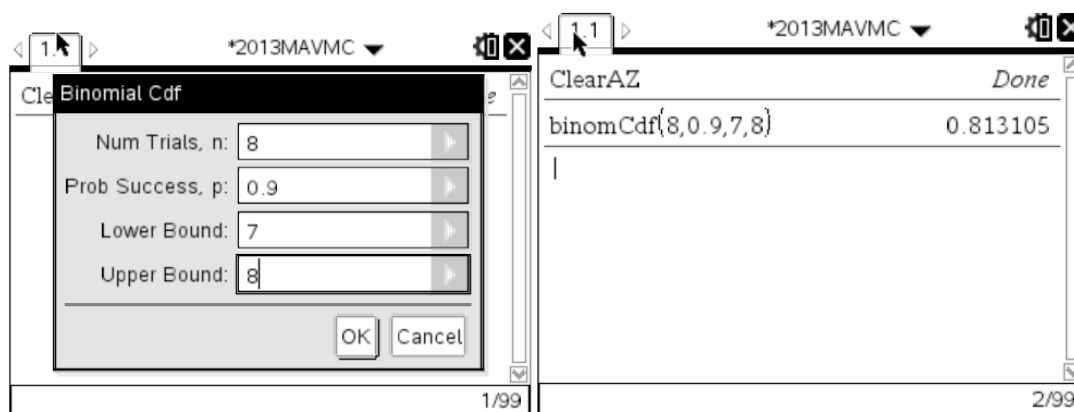
b. i. Y is number of non-defective cylinders out of 8: Binomial $n = 8, p = 0.9$

1M

$$\Pr(Y \geq 7) = \Pr(Y = 7) + \Pr(Y = 8)$$

$$= \binom{8}{7} \times 0.9^7 \times 0.1 + \binom{8}{8} \times 0.9^8 \times 0.1^0$$

$$= 0.8131 \text{ correct to four decimal places}$$

1A**OR**Binomial $n = 8, p = 0.1$ **1M**

$$\Pr(Y \leq 1) = \Pr(Y = 0) + \Pr(Y = 1)$$

$$= 0.8131 \text{ correct to four decimal places}$$

1Aii. Let $\$P$ be the profit per box.

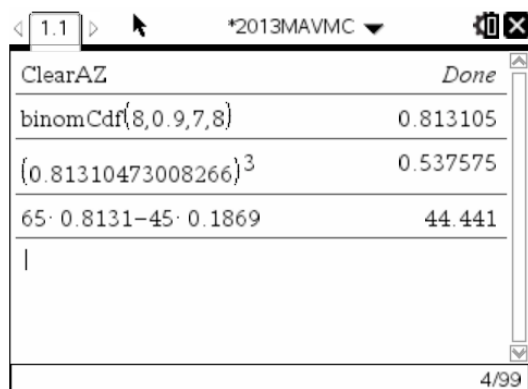
$$P = \{65, -45\}$$

1M

$$E(P) = 65 \times 0.8131 + -45 \times 0.1869$$

$$= 52.8515 - 8.4105$$

$$= \$44.44$$

1Aiii. $0.813105^3 = 0.5376$ correct to four decimal places**1A**

1.1	*2013MAVMC	
ClearAZ	Done	
binomCdf{8,0.9,7,8}	0.813105	
{0.81310473008266} ³	0.537575	
		3/99

c. i.

Transition matrix $\begin{bmatrix} 0.95 & 0.88 \\ 0.05 & 0.12 \end{bmatrix}$

1M

$$\% \text{ purchased after two years: } \begin{bmatrix} 0.95 & 0.88 \\ 0.05 & 0.12 \end{bmatrix}^2 \begin{bmatrix} \frac{5}{8} \\ \frac{3}{8} \end{bmatrix} = \begin{bmatrix} 0.944663 \\ 0.055338 \end{bmatrix}$$

1.1	*Unsaved	
Define $a =$	Done	
$\begin{bmatrix} 0.95 & 0.88 \\ 0.05 & 0.12 \end{bmatrix}$		
$a^2 \cdot \begin{bmatrix} \frac{5}{8} \\ \frac{3}{8} \end{bmatrix}$	$\begin{bmatrix} 0.944663 \\ 0.055338 \end{bmatrix}$	
		2/99

94% purchase from this company (market share)

1A

ii. Long run: $\begin{bmatrix} 0.95 & 0.88 \\ 0.05 & 0.12 \end{bmatrix}^{100} \begin{bmatrix} \frac{5}{8} \\ \frac{3}{8} \end{bmatrix} = \begin{bmatrix} 0.946237 \\ 0.053763 \end{bmatrix}$

OR $\frac{0.88}{0.88 + 0.05} = 0.946\dots$

Overall percentage of cylinders purchased from this company: 95%

1A

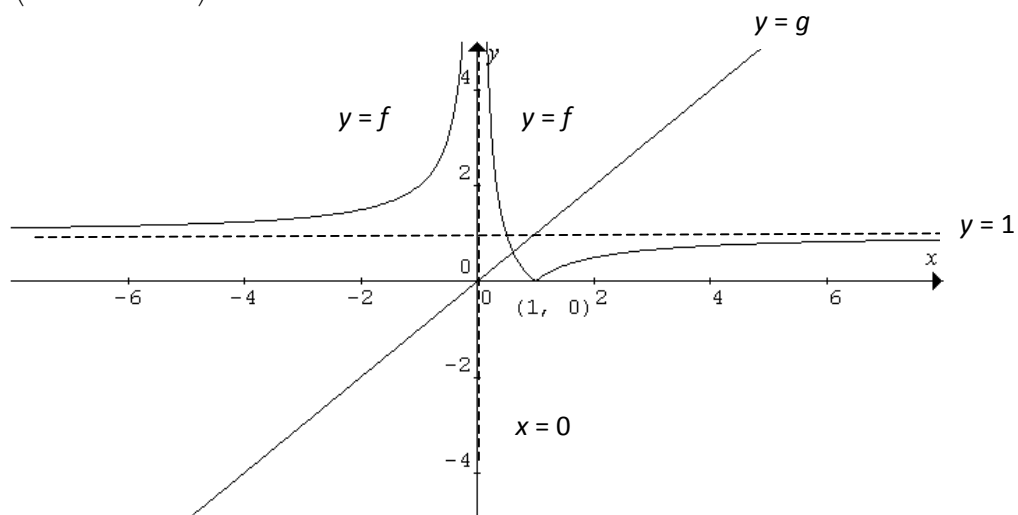
iii. The long term market share of the competitor is 5%

1A

Question 4

a. Solve $\left|1 - \frac{1}{x}\right| = x$ for x

$$\left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}\right)$$

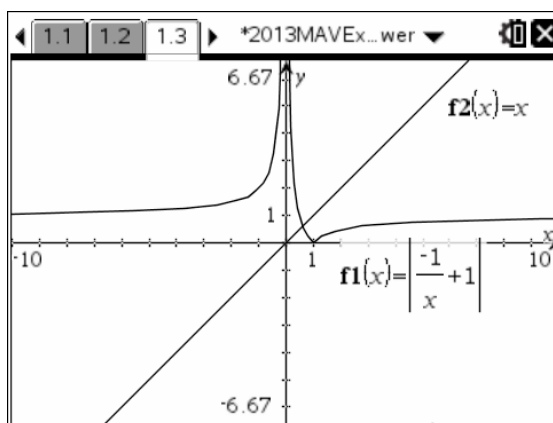
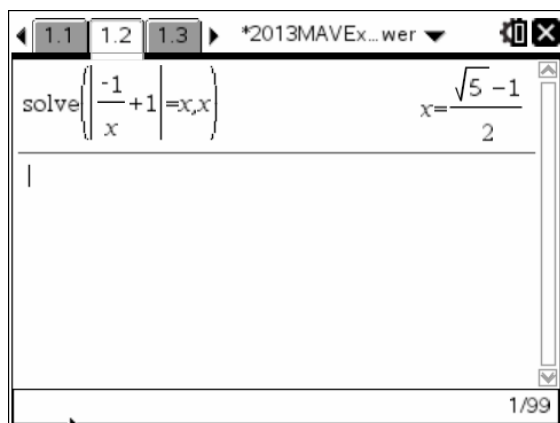


$y = g$ with $\left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}\right)$ and $(0, 0)$ 1A

Shape for $y = \left|1 - \frac{1}{x}\right|$ 1A

Sharp point at $(1, 0)$ 1A

Asymptotes 1A



b.

$$f(x) = \begin{cases} 1 - \frac{1}{x}, & x < 0 \cup x \geq 1 \\ -1 + \frac{1}{x}, & 0 < x < 1 \end{cases} \quad 4 \times \frac{1}{2} = 2\text{A Round down}$$

$$\text{c. } \int_0^{\frac{\sqrt{5}-1}{2}} (x) dx + \int_{\frac{\sqrt{5}-1}{2}}^1 \left(\frac{1}{x} - 1 \right) dx \quad 2 \times 1 = 2\text{A}$$

$$= \log_e \left(\frac{\sqrt{5}+1}{2} \right) + \frac{\sqrt{5}-3}{4} \quad 1\text{A}$$

d. The area will be same when $y = x + k$ crosses the right hand branch of the hyperbola when $x > 1$.
 x -intercept, for $y = x + k$ is $x = -k$. 1A

x coordinate of the point of intersection with $y = 1 - \frac{1}{x}$ and $x > 1$ and $k < 0$ is

$$x = \frac{\sqrt{k^2 - 2k - 3} - k + 1}{2} \quad 1\text{A}$$

$$\text{Solve } \int_1^{\frac{\sqrt{k^2 - 2k - 3} - k + 1}{2}} \left(1 - \frac{1}{x} \right) dx - \int_{-k}^{\frac{\sqrt{k^2 - 2k - 3} - k + 1}{2}} (x + k) dx = \log_e \left(\frac{\sqrt{5}+1}{2} \right) + \frac{\sqrt{5}-3}{4} \text{ for } k. \quad 1\text{A}$$

or

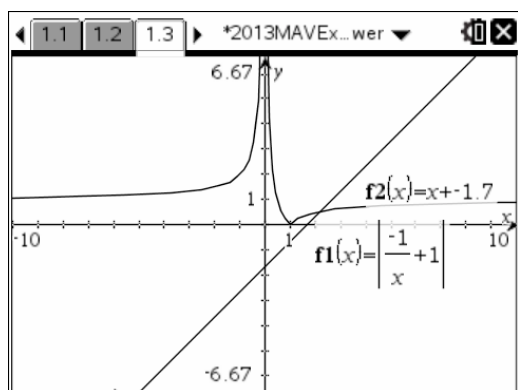
$$\text{Solve } \int_1^{\frac{\sqrt{k^2 - 2k - 3} - k + 1}{2}} \left(1 - \frac{1}{x} \right) dx - \int_{-k}^{\frac{\sqrt{k^2 - 2k - 3} - k + 1}{2}} (x + k) dx = 0.2902 \text{ for } k. \quad 1\text{A}$$

$$= -1.7 \text{ correct to one decimal place} \quad 1\text{A}$$

$\ln\left(\frac{\sqrt{5}+1}{2}\right) + \frac{\sqrt{5}-3}{4}$
 solve $\left\{x+k = \frac{-1}{x} + 1, x\right\} | x > 1 \text{ and } k < 0$
 $x = \frac{\sqrt{k^2 - 2k - 3} - k + 1}{2}$ and $\sqrt{k^2 - 2k - 3} - k + 1 > 1$

solve $\int \frac{\sqrt{k^2 - 2k - 3} - k + 1}{2} \left(\frac{-1}{x} + 1\right) dx - \int \frac{\sqrt{k^2 - 2k - 3} - k + 1}{2} dx$
 $k = -1.70429$

$\int \frac{\sqrt{k^2 - 2k - 3} - k + 1}{2} \left(\frac{-1}{x} + 1\right) dx = \ln\left(\frac{\sqrt{5}+1}{2}\right) + \frac{\sqrt{5}-3}{4}, k$
 $k = -1.70429$



e. There will be three solutions when $y = x + k$ crosses the left hand branch of the hyperbola twice.

x coordinates of the point of intersection with $y = 1 - \frac{1}{x}$ and $x < 0$ and $k > 0$ are

$$x = \frac{\pm\sqrt{k^2 - 2k - 3} - k + 1}{2}$$

1A from part d.

Solve $k^2 - 2k - 3 > 0$ for $k > 0$.

1M

$k > 3$

1A

END OF SECTION 2 SOLUTIONS