



The Mathematical Association of Victoria MATHEMATICAL METHODS (CAS)

Trial written examination 2

2007

Reading time: 15 minutes

Writing time: 2 hours

Student's Name:

QUESTION AND ANSWER BOOK

Structure of book

<i>Section</i>	<i>Number of questions</i>	<i>Number of questions to be answered</i>	<i>Number of marks</i>
1	22	22	22
2	4	4	58
			Total 80

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

These questions have been written and published to assist students in their preparations for the 2007 Mathematical Methods (CAS) Examination 2. The questions and associated answers and solutions do not necessarily reflect the views of the Victorian Curriculum and Assessment Authority. The Association gratefully acknowledges the permission of the Authority to reproduce the formula sheet.

This Trial Examination is licensed to the purchasing school or educational organisation with permission for copying within that school or educational organisation. No part of this publication may be reproduced, transmitted or distributed, in any form or by any means, outside purchasing schools or educational organisations or by individual purchasers, without permission.

*Published by The Mathematical Association of Victoria
"Cliveden", 61 Blyth Street, Brunswick, 3056
Phone: (03) 9380 2399 Fax: (03) 9389 0399
E-mail: office@mav.vic.edu.au Website: <http://www.mav.vic.edu.au>*

Working space

MULTIPLE CHOICE ANSWER SHEET

Student Name:

Circle the letter that corresponds to each correct answer

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E

Working space

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Question 1

If $R \rightarrow R$, where $f(x) = 2x + 1$ and $g: R \setminus \{0\} \rightarrow R$, where $g(x) = \frac{1}{x^2}$ then the range of $f(g(x))$ is

- A. $R \setminus \{1\}$
- B. $(1, \infty)$
- C. $(0, \infty)$
- D. $R \setminus \{-\frac{1}{2}\}$
- E. $R \setminus \{0\}$

Question 2

If $h: (1, 2] \rightarrow R$, where $h(x) = (x - 1)^2(x + 2)$ and $f: [-1, 2) \rightarrow R$, where $f(x) = 1 - x$ then $g = hf$ is defined by

- A. $g: R \rightarrow R$, where $h(x) = -(x - 1)^3(x + 2)$
- B. $g: (1, 2) \rightarrow R$, where $h(x) = -(x - 1)^3(x + 2)$
- C. $g: (1, 2] \rightarrow R$, where $h(x) = -(x - 1)^3(x + 2)$
- D. $g: (1, 2) \rightarrow R$, where $h(x) = -(x - 1)^2(x + 2)(x + 1)$
- E. $g: [-1, 2] \rightarrow R$, where $h(x) = -(x - 1)^3(x + 2)$

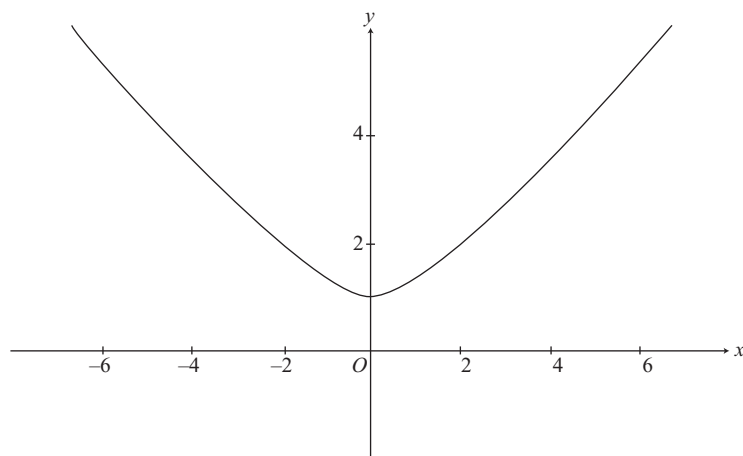
Question 3

The graph of $y = f(x)$ undergoes the following transformations:

- a translation of 5 units up, then
- a reflection in the y -axis, then
- a dilation by a scale factor of 3 from the x -axis, then
- a dilation by a scale factor of 2 from the y -axis.

The transformation $T: R^2 \rightarrow R^2$, which maps the curve with equation $y = f(x)$ onto its image could be

- A. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix}\right)$
- B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix}$
- C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -5 \end{bmatrix}$
- D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix}$
- E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix}$

Question 4

The graph shown above passes through the point $(2, 2)$. A possible equation for this graph could be

- A. $y = x^{\frac{4}{3}} + 1$
- B. $y = x^{\frac{2}{3}} + 1$
- C. $y = x^{\frac{5}{3}} + 1$
- D. $y = x^2 + 1$
- E. $y = \left(\frac{x}{2}\right)^{\frac{4}{3}} + 1$

Question 5

A system of simultaneous linear equations is represented by the matrix equation

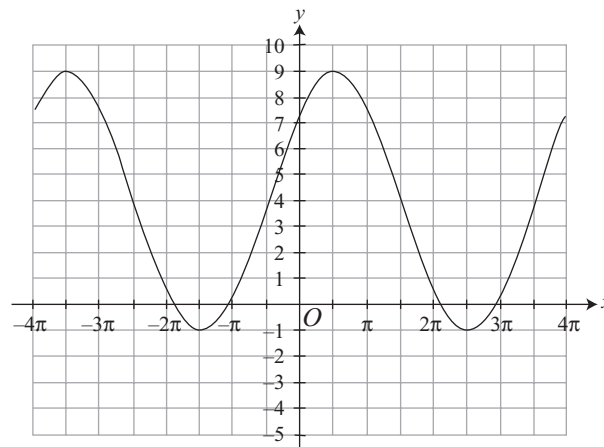
$$\begin{bmatrix} 0 & -k & 1 \\ 1 & -1 & k \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \text{ where } k \text{ is a real constant.}$$

The system of equations has a unique solution for

- A. $k \in \mathbb{R}$
- B. $k \in \mathbb{R} \setminus \{0\}$
- C. $k \in \mathbb{R} \setminus \{\pm 1\}$
- D. $k \in \mathbb{R} \setminus \{2\}$
- E. $k \in \mathbb{R} \setminus \{\pm\sqrt{2}\}$

Question 6

The diagram below shows the graph of a circular function.

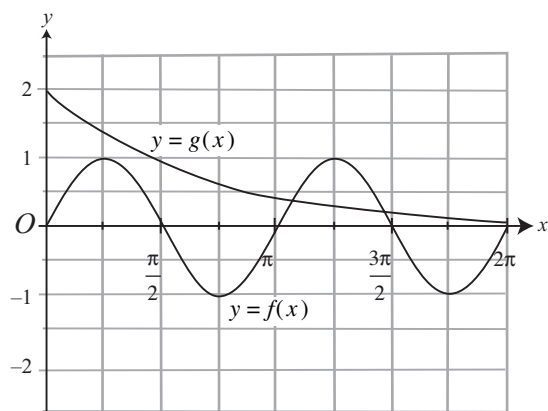


The **amplitude**, **period** and **rule**, respectively, of this graph are

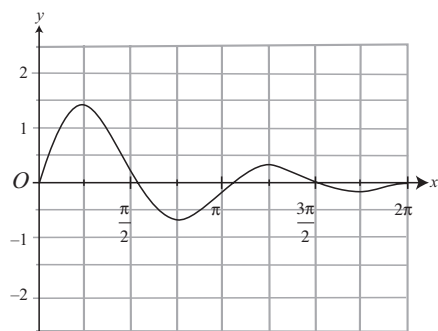
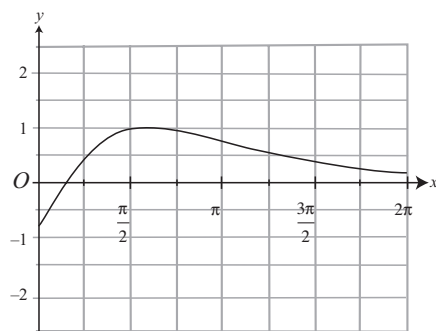
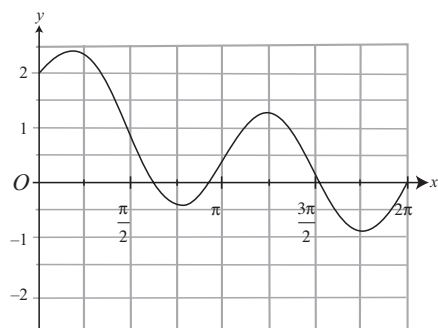
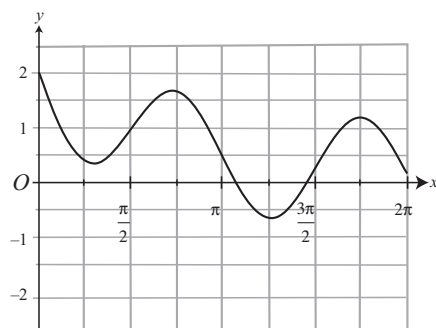
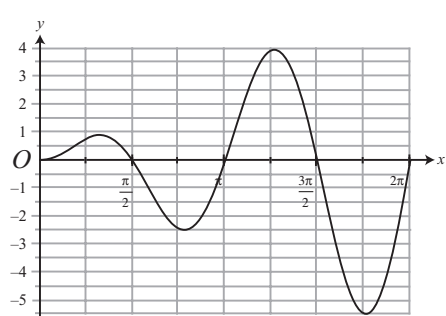
- A. $10, 8\pi, y = 10\cos\left(\frac{x}{2} - \frac{\pi}{2}\right) + 4$
- B. $5, 4\pi, y = 5\cos\left(\frac{x}{2} - \frac{\pi}{4}\right) + 4$
- C. $5, 4\pi, y = 5\cos\left(2x - \frac{\pi}{2}\right) + 4$
- D. $10, 4\pi, y = 10\cos\left(2x + \frac{\pi}{2}\right) + 4$
- E. $5, 8\pi, y = 5\cos\left(\frac{x}{4} - \frac{\pi}{2}\right) + 4$

Question 7

The diagram below shows the graphs of two functions, f and g .



Which one of the following could be the graph of the product function $y = (fg)(x)$?

A.**B.****C.****D.****E.**

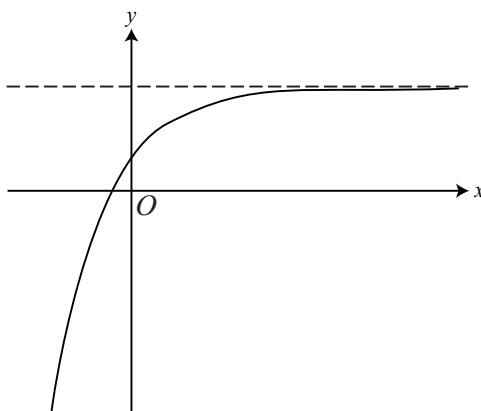
Question 8

Let $R \rightarrow R$, where $f(x) = e^{2(x-1)} + 3$. If the graph of g is the graph of f dilated by a factor of 3 from the x -axis then the graph of the inverse function g^{-1} has

- A. a vertical asymptote with equation $x = 9$.
- B. a horizontal asymptote with equation $y = 3$.
- C. a vertical asymptote with equation $x = 1$.
- D. a horizontal asymptote with equation $y = 1$.
- E. a vertical asymptote with equation $x = 3$.

Question 9

Part of the graph of an exponential function with rule $y = f(x)$ is shown below.



If $a > 0$, $b < 0$ and $k > 0$, the rule for f could be

- A. $y = ae^{bx} + k$
- B. $y = ae^{b(x-k)}$
- C. $y = -ae^{bx} + k$
- D. $y = ae^{-bx} + k$
- E. $y = ae^{bx} - k$

Question 10

A function f satisfies the functional equation $-af(|x|) = f\left(\frac{1}{|x|^a}\right)$, where $a \in R$ and $x \in R \setminus \{0\}$.

A possible rule for f is

- A. $f(x) = e^x$
- B. $f(x) = \sqrt{x}$
- C. $f(x) = \sin(x)$
- D. $f(x) = \log_e(x)$
- E. $f(x) = x^2$

Question 11

The equation of the tangent to the graph with equation $y = 2(x-3)^6 + 4$ at its turning point is

- A. $x = 3$
- B. $y = 3$
- C. $x = -3$
- D. $y = 4$
- E. $x = 4$

Question 12

The average rate of change of $f(x) = \sqrt{2-x}$ from $x = -2$ to $x = 1$ is

- A. -1
- B. $-\frac{1}{2}$
- C. $-\frac{1}{3}$
- D. $-\frac{1}{4}$
- E. $\frac{1}{3}$

Question 13

If $f(x) = \log_e |x-1|$, then the derivative of f is

- A. $\frac{1}{x-1}$, for $x > 1$
- B. $\frac{1}{x-1}$, for $x < 1$
- C. $\frac{1}{x-1}$, for $x > 0$
- D. $\frac{1}{x-1}$, for $x \in \mathbb{R} \setminus \{1\}$
- E. $\frac{1}{x-1}$, for $x \in \mathbb{R} \setminus \{-1\}$

Question 14

The area bounded by the curve with equation $y = \log_e(x)$, the x -axis and the line $x = 4$ is approximated using left-end rectangles of width 1 unit. This area, in units², is

- A. $\log_e 6$
- B. $\log_e 24$
- C. $8\log_e(2) - 4$
- D. 1.55
- E. 1.79

Question 15

The area, correct to three decimal places, bounded by the y -axis and the curves with equations $f(x) = \cos(2x) + 2$ and $g(x) = e^x$ is

- A. 3.357
- B. 0.879
- C. 0.878
- D. 0.882
- E. -0.882

Question 16

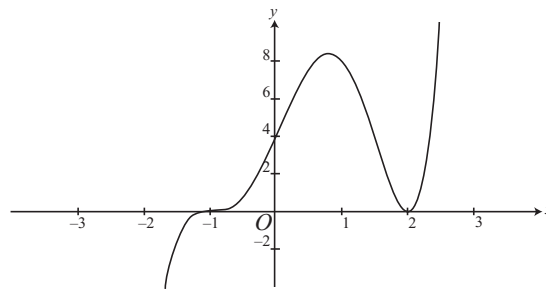
For a particular city, the time from sunrise to sunset, S hours, x days after the winter solstice, is modelled by the function $S(x) = -5 \cos\left(\frac{\pi}{183}x\right) + 12$, for $0 \leq x \leq 183$.

The average value of S , over the interval $[0, n]$, is 8 hours. The value of n is closest to

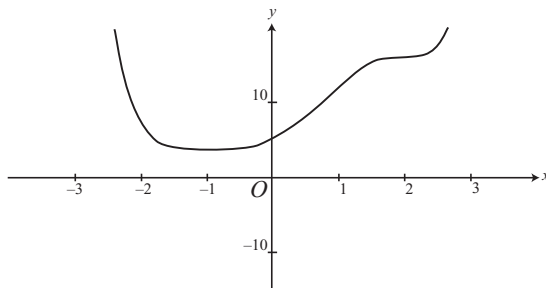
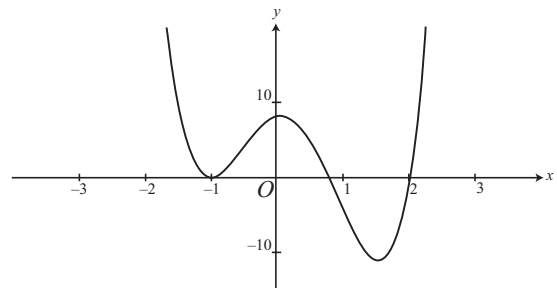
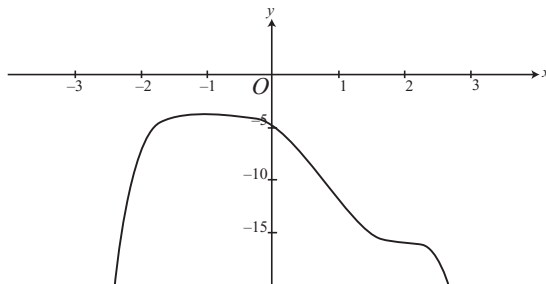
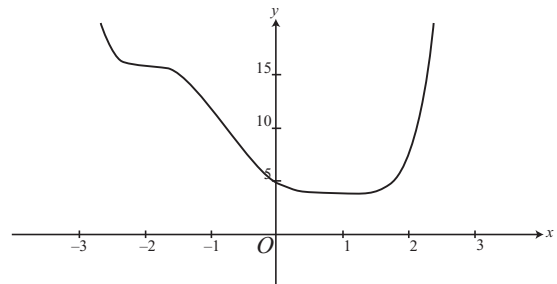
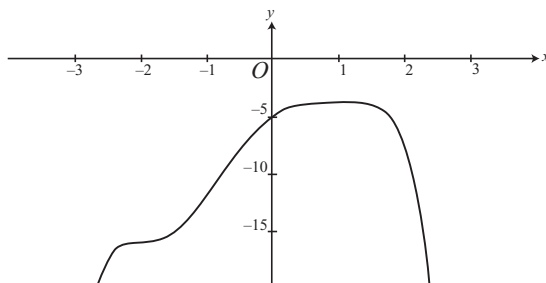
- A. 28 days
- B. 30 days
- C. 60 days
- D. 66 days
- E. 99 days

Question 17

Part of the graph of the derivative of f is shown below.



Part of the graph of f could be

A.**B.****C.****D.****E.**

Question 18

Miriam and her husband have brown eyes, but both carry the recessive gene for blue eyes. It is known that couples in these circumstances will have a child with blue eyes in 1 out of every 4 births. If they have three children altogether, the probability that at least one will have blue eyes is

- A. $\frac{27}{64}$
- B. $\frac{37}{64}$
- C. $\frac{1}{4}$
- D. $\frac{3}{4}$
- E. $\frac{9}{64}$

Question 19

The probability distribution for a discrete random variable, X , is defined by the probability function

$$p(x) = \frac{x^2 + 1}{10}, \text{ for } x \in \{-1, 0, 1, 2\}.$$

The expectation of X , $E(X)$, is

- A. 1.5
- B. 1.4
- C. 1.1
- D. 1
- E. 0.8

Question 20

The probability distribution for a continuous random variable, T , is defined by the probability density function

$$f(t) = \begin{cases} \sin(2t) & 0 \leq t \leq \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

The mode of the distribution is

- A. 0
- B. 1
- C. π
- D. $\frac{\pi}{2}$
- E. $\frac{\pi}{4}$

Question 21

The speed of vehicles travelling along a particular section of a rural freeway is modelled as a normally distributed random variable with a mean of 96 km/h and a standard deviation of 12 km/h.

According to this model, the percentage of vehicles that are exceeding the speed limit of 110 km/h is closest to

- A. 5%
- B. 10%
- C. 12%
- D. 14%
- E. 20%

Question 22

Consider a Markov chain where the n th state is given by $S_n = T^n \times S_0$.

Which one of the following could represent the transition matrix, T , and the initial state matrix, S_0 , respectively?

- A. $\begin{bmatrix} 0.35 & 0.65 \\ 0.55 & 0.45 \end{bmatrix}$ and $\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$
- B. $\begin{bmatrix} 0.25 & 0.55 \\ 0.85 & 0.45 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- C. $\begin{bmatrix} 0.35 & 0.55 \\ 0.65 & 0.45 \end{bmatrix}$ and $\begin{bmatrix} 27 \\ 33 \end{bmatrix}$
- D. $\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$ and $\begin{bmatrix} 0.35 & 0.65 \\ 0.55 & 0.45 \end{bmatrix}$
- E. $\begin{bmatrix} 27 \\ 33 \end{bmatrix}$ and $\begin{bmatrix} 0.35 & 0.55 \\ 0.65 & 0.45 \end{bmatrix}$

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

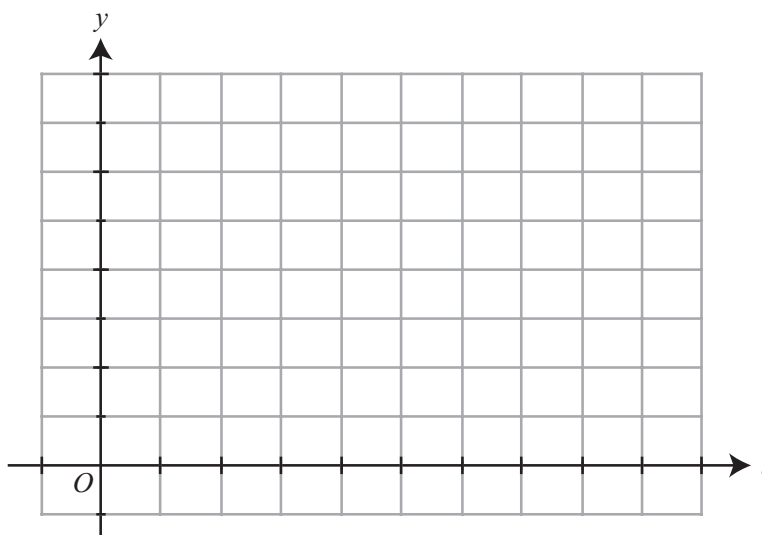
The incubation period, T days, for a new strain of influenza virus, commonly known as blue flu, is modelled by the probability density function

$$v(t) = \begin{cases} -\frac{1}{108}(t^3 - 15t^2 + 63t - 81) & 3 \leq t \leq k \\ 0 & \text{otherwise} \end{cases}$$

- a. Show that $k = 9$.

1 mark

- b. Sketch the graph of $y = v(t)$ on the axes below. Label axes intercepts and any stationary points with their **exact** coordinates.



3 marks

- c. Find the **median** incubation period for this virus, in days, correct to two decimal places.

2 marks

A new drug, Fluaway, has been developed to treat the symptoms of blue flu. The time taken for a standard dose of Fluaway to take effect is modelled by a normally distributed random variable with a mean of 5 hours.

For 90% of patients, a standard dose of Fluaway takes effect within 6 hours and 15 minutes of being administered.

- d. Find the standard deviation, in hours, correct to two decimal places.

2 marks

Vivienne is a medical practitioner at an infectious disease clinic.

- e. Vivienne randomly selects ten patients with blue flu and administers a standard dose of Fluaway to each of them. Find the probability that, in at least two of the patients, the drug will take more than 6 hours and 15 minutes (that is, 6.25 hours) to take effect. Give the answer correct to three decimal places.

2 marks

- f. Vivienne randomly selects k patients with blue flu and administers a standard dose of Fluaway to each of them. What is the minimum value of k required to ensure that there is at least a 95% chance that at least one patient will take more than 6 hours and 15 minutes to respond to the drug?

2 marks

- g. An alternative drug, Blugone, has been released. At the clinic, Vivienne's decision as to which of the two drugs to prescribe to patients on a particular day depends only on which drug was prescribed the previous day. If Fluaway was prescribed yesterday, there is a probability of 0.2 that Blugone will be prescribed today.

The transition matrix for the probabilities of Vivienne prescribing Fluaway or Blugone is $\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$.

- i. What is the probability that Blugone will be prescribed tomorrow, given that it was prescribed today?

- ii. On Monday of this week she prescribed Fluaway to her blue flu patients. What is the probability that she will prescribe Blugone on Friday of the same week?

- iii. In the long term, what proportion of days does she prescribe Blugone?

1 + 2 + 1 = 4 marks

Total 16 marks

Question 2

The shape of the vertical cross-section of an irrigation channel can be modelled by the following function:

$$f : [-5, 5] \rightarrow \mathbb{R}, \text{ where } f(x) = A(x^{\frac{4}{3}} - 5) \text{ and } A \text{ is a real constant.}$$

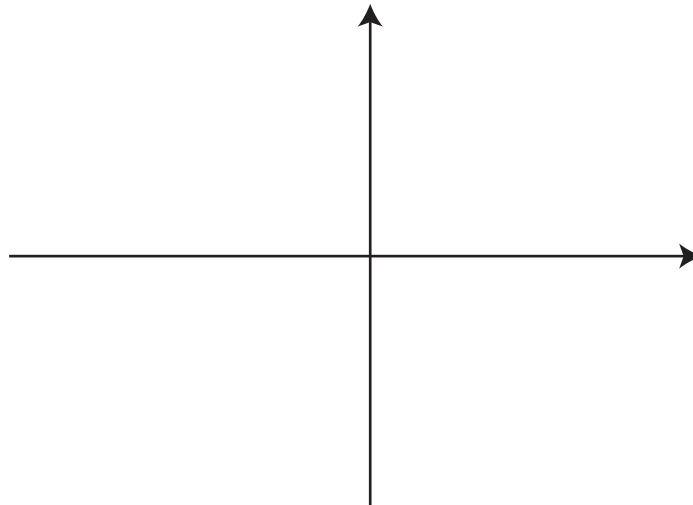
The x -axis represents normal ground level. f is the vertical distance, in metres, of the edge of the channel above ground level. x is the horizontal distance, in metres, of the edge of the channel from the centre of the channel. Give exact answers unless otherwise stated.

- a.** The lowest point of the channel is represented by the turning point of the graph of f which has coordinates $(0, -2)$.

Show that the value of A is $\frac{2}{5}$.

1 mark

- b.** Sketch the graph of f on the axes below, clearly labelling the endpoints and axial intercepts with their coordinates.



5 marks

- c.** What is the maximum possible depth, in metres, of the water in the channel?

1 mark

On a particular day, the maximum depth of the water in the channel is 2 m.

- d. i.** Find the area, in m^2 , bounded by the x -axis and the graph of f .

- ii.** Hence, if the length of the channel is 1 km, find the volume, in m^3 , of water in the channel.

2 + 1 = 3 marks

Due to weeds at the bottom of the channel, a more accurate estimate of the vertical cross-sectional area of water in the channel, on the day given in **part d**, can be found by using a triangle with vertices $(0, -2)$, $(-5^{\frac{3}{4}}, 0)$ and $(5^{\frac{3}{4}}, 0)$. Assume the length of the channel is 1 km.

- e. i.** What is the estimated volume, in m^3 correct to two decimal places, of water in the channel using this method?

- ii.** The depth of water in the channel starts to decrease by 2 cm/h because of evaporation.

Show that the rate at which the volume of water is decreasing in the channel is $20 \times 5^{\frac{3}{4}}h \text{ m}^3/\text{h}$, where h m is the maximum depth of water in the channel. Assume no water is entering the channel.

2 + 4 = 6 marks

Total 16 marks

Question 3

Let $g : R \rightarrow R$, where $g(x) = A \times B^x$, and A and $B \in Z^+$.

- a. Find values for A and B if the graph of g passes through the points $(2, 12)$ and $(4, 48)$.

2 marks

The graph of g_2 is the graph of g dilated by a factor of $\frac{1}{3}$ from the y -axis, followed by a translation of $\frac{4}{3}$ units parallel to the x -axis and 1 unit parallel to the y -axis.

- b. Write down the rule for g_2 and state the range.

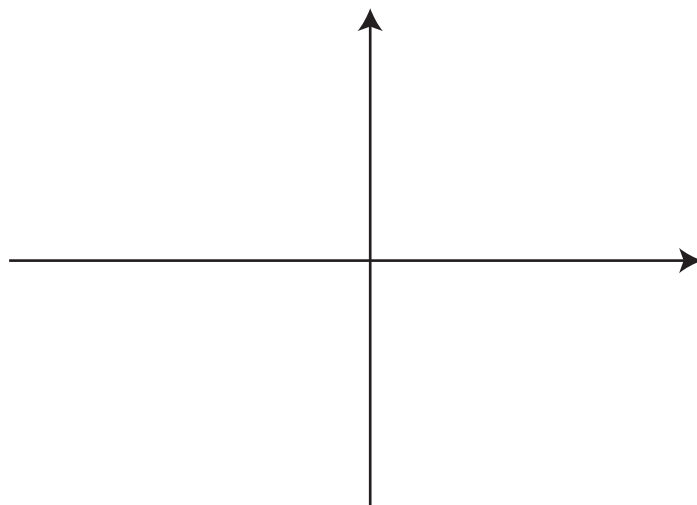
3 marks

Let $g_3 : R \rightarrow R$, where $g_3(x) = 3 \times 2^{(2x+1)} - 1$.

- c. i. Show that $g_3^{-1}(x) = \frac{1}{2} \log_2\left(\frac{x+1}{6}\right)$.

- ii. State the transformations that have occurred to the graph of $y = \log_2(x)$ to get the graph of g_3^{-1} .

- iii. Sketch the graph of g_3^{-1} on the set of axes below, labelling any asymptotes with their equations and intercepts with their coordinates.



- iv. Find the area, in units² correct to one decimal place, bounded by the x -axis and the graphs of $|g_3^{-1}|$ and g_3 .

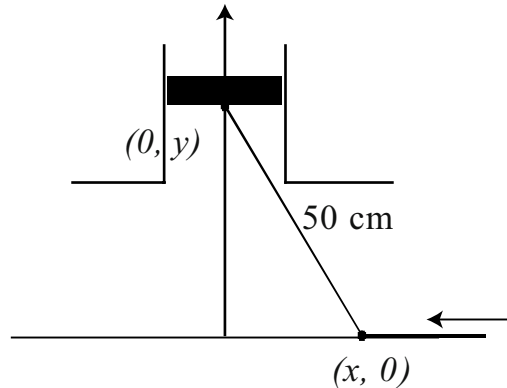
$2 + 2 + 2 + 2 = 8$ marks

Total 13 marks

Question 4

A machine uses a 50 cm long moveable rod to move a piston on a vertical shaft. The coordinates of the ends of the rod are $(x, 0)$ and $(0, y)$, as shown in the diagram.

The position, x cm, of the end of the rod on the x -axis, at time t seconds, is given by $x(t) = 30 \cos\left(\frac{\pi}{8}t\right)$, $t \geq 0$.



- a. Show that the distance y cm in terms of x is given by $y = \sqrt{2500 - x^2}$.

1 mark

- b. Consider y in terms of t .

- i. Show that a rule for $y(t)$, the distance y cm at time t s, is $y(t) = \sqrt{2500 - 900 \cos^2\left(\frac{\pi}{8}t\right)}$

- ii. Hence find the average value of y in the first 8 seconds of the motion of the rod. (Answer correct to three decimal places.)

1 + 2 = 3 marks

- c. The piston is at its highest position on the y -axis when $x = 0$. The piston is at its lowest position on the y -axis when $x = 30$ or $x = -30$. Find the distance y cm, from the origin to the highest and lowest positions of the end of the rod on the y -axis.

2 marks

- d. For $t \in [0, 16]$, solve the equations $x(t) = 15$ and $x(t) = -15$. Hence find, in the first 16 seconds of the motion of the rod, the total length of time for which $-15 \leq x \leq 15$.

2 marks

- e. Show that the speed of the end of the rod on the **x -axis**, when the x -axis endpoint is $(15, 0)$, is $\frac{15\sqrt{3}\pi}{8}$ cm/s.

2 marks

- f. Find the **exact** maximum speed of the end of the rod on the **x -axis**.

3 marks

Total 13 marks

Mathematical Methods (CAS) Exam 2: SOLUTIONS

Section 1: Multiple Choice Questions

1. B 2. B 3. A 4. E 5. E 6. B 7. A 8. A 9. C
 10. D 11. D 12. C 13. D 14. A 15. D 16. D
 17. A 18. B 19. D 20. E 21. C 22. C

Question 1

$$f(g(x)) = \frac{2}{x^2} + 1. \text{ The range is } (1, \infty).$$

Question 2

$$d_g = d_h \cap d_f \text{ which is } (1, 2).$$

$$\begin{aligned} g(x) &= (x-1)^2(x+2)(1-x) \\ &= -(x-1)^3(x+2) \end{aligned}$$

Question 3

Translation 5 units up:

$$y \rightarrow y + 5$$

Reflection in the y -axis

$$x \rightarrow -x$$

Dilation by a factor of 3 from the x -axis

$$y \rightarrow 3(y + 5)$$

Dilation by a factor of 2 from the y -axis

$$x \rightarrow 2x$$

Combining these:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2x \\ 3y+5 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix}\right)$$

Question 4

The graph has the shape of the graph with rule $y = (ax)^{\frac{4}{3}} + 1$ or $y = (bx)^2 + 1$, where a and b are real constants. When $x = 2$, $y = 2$.

Thus, $y = x^2 + 1$ and $y = x^{\frac{4}{3}} + 1$ are

not possible. $y = \left(\frac{x}{2}\right)^{\frac{4}{3}} + 1$.

Question 5

Solving the matrix equation gives

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3k+4}{k^2-2} + 5 \\ \frac{3k+4}{k^2-2} \\ -\frac{4k+6}{k^2-2} \end{bmatrix}$$

B

The image shows a CAS calculator screen with the matrix equation $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3k+4}{k^2-2} + 5 \\ \frac{3k+4}{k^2-2} \\ -\frac{4k+6}{k^2-2} \end{bmatrix}$ displayed. The bottom of the screen shows the command `ref([0,-k,1,3][1,1,0,5][1,-`.

B

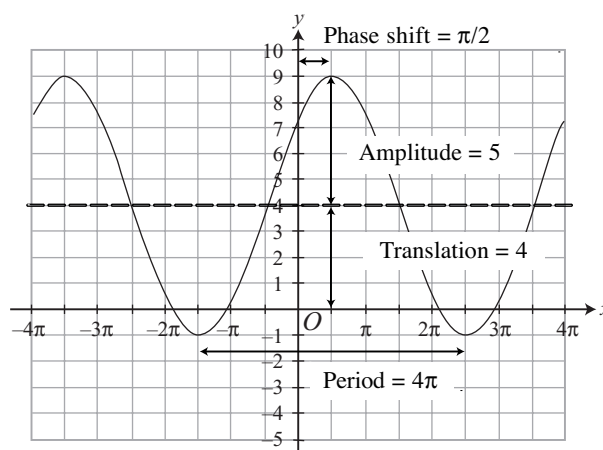
There will be no solution when

$$k^2 - 2 = 0, \text{ i.e. } k = \pm\sqrt{2}.$$

A unique solution for $k \in \mathbb{R} \setminus \{\pm\sqrt{2}\}$

E

Question 6



A

Amplitude = 5

Period = 4π

Rule is of the form $y = a\cos(n(x - \epsilon)) + k$

$$a = 5, \epsilon = \frac{\pi}{2}, k = 4$$

$$\text{Period} = \frac{2\pi}{n}, \text{ therefore } n = \frac{2\pi}{4\pi} = \frac{1}{2}$$

$$\text{Rule is } y = 5\cos\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right) + 4$$

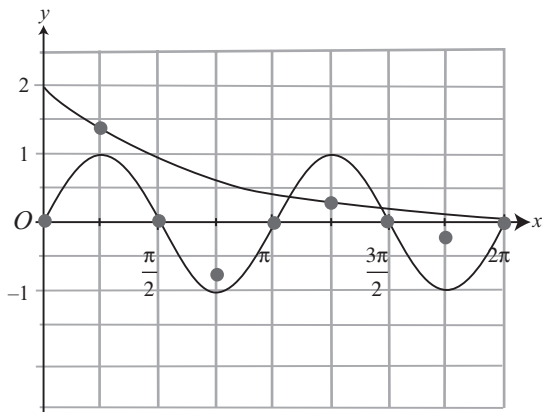
$$\text{or } y = 5\cos\left(\frac{x}{2} - \frac{\pi}{4}\right) + 4$$

E

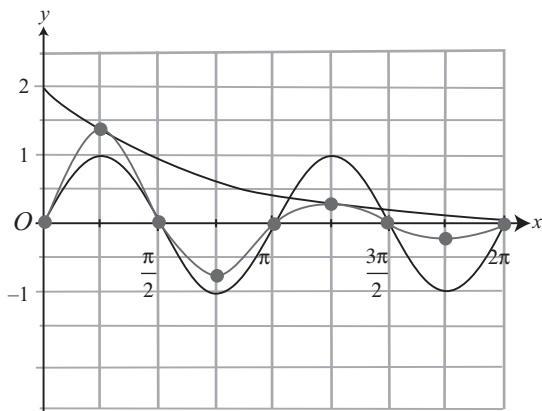
B

Question 7

When graphing product of functions, the key points occur when either function has the value of ± 1 or 0, as illustrated in the diagram below. Mark these key points.



Then “join the dots”.



Question 8

$$g(x) = 3(e^{2(x-1)} + 3)$$

$$= 3e^{2(x-1)} + 9$$

The graph of g has an asymptote at $y = 9$. Thus the graph of the inverse of g has an asymptote at $x = 9$.

Question 9

The graph of $y = e^x$ has been reflected in both x and y axes, and then translated up.

Reject option B because it does not show the translation up. None of the other options indicate the possibility of a translation parallel to the x -axis.

Ignoring the magnitude of any dilations (because there is no scale on the axes), the transformed graph will be of the form $y = -e^{-x} + k$. Since $b < 0$, require $y = -ae^{bx} + k$.

C

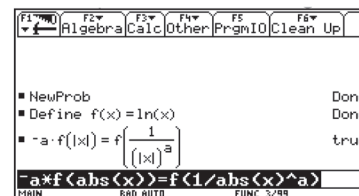
Question 10

This is a property of logarithms.

$$-a \log_e(|x|) = \log_e(|x|^{-a}) = \log_e\left(\frac{1}{|x|^a}\right)$$

This may be confirmed using CAS.

D



Question 11

A

The coordinates of the turning point of $y = 2(x - 3)^6 + 4$ are (3, 4). The tangent is $y = 4$.

D

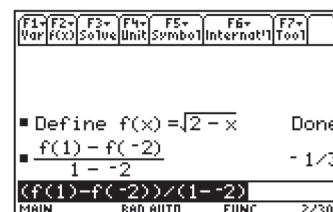
Question 12

$$\text{The average rate of change} = \frac{f(1) - f(-2)}{1 - (-2)}$$

$$= \frac{\sqrt{1} - \sqrt{4}}{3}$$

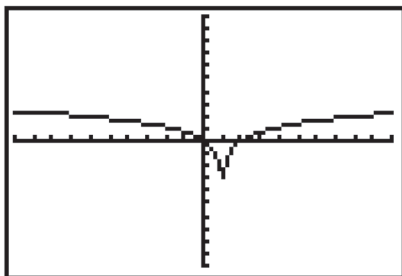
$$= -\frac{1}{3}$$

C

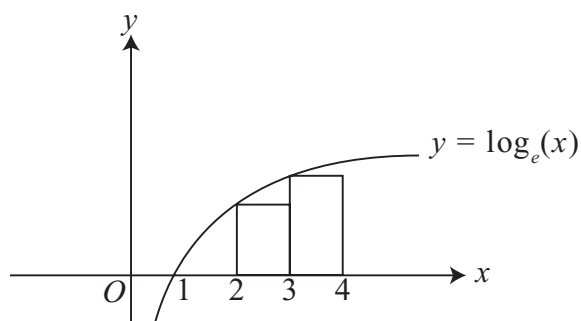


Question 13

The graph of $y = \log_e |x - 1|$ is shown below.



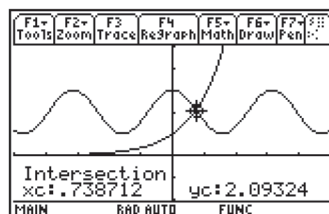
$$\frac{d}{dx} \log_e |x - 1| = \frac{1}{x - 1} \text{ for } x \in \mathbb{R} \setminus \{1\}$$

Question 14

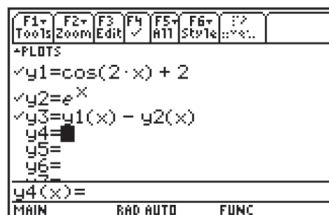
The area of the rectangles = $\log_e(2) + \log_e(3)$
 $= \log_e(6)$

A**Question 15**

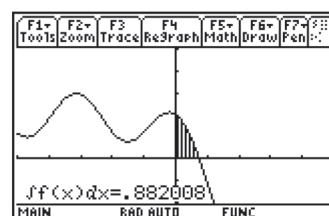
Sketch both curves and find the intersection.



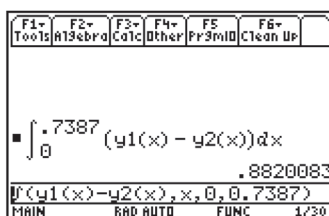
Graph $f(x) - g(x)$

D

Find $\int_0^{0.7387} (f(x) - g(x)) dx$



OR

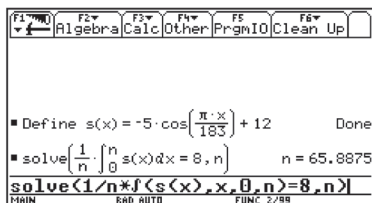
**D**

Question 16

$$\text{Average value} = \frac{1}{b-a} \int_a^b s(x) dx$$

$$\text{Therefore, } \frac{1}{n} \int_0^n s(x) dx = 8$$

Solving for n gives the result $n \approx 66$, days.

**Question 17**

f has a minimum turning point at $x = -1$ and a stationary point of inflection at $x = 2$.

Question 18

Let X denote the number of children with

blue eyes. $X \sim Bi\left(n=3, p=\frac{1}{4}\right)$

$$\Pr(X \geq 1) = 1 - \Pr(X = 0)$$

$$= 1 - \left(\frac{3}{4}\right)^3$$

$$= 1 - \left(\frac{27}{64}\right)$$

$$= \frac{37}{64}$$

Question 19

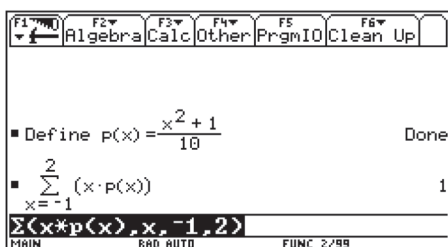
$$p(x) = \frac{x^2 + 1}{10}, \text{ for } x \in \{-1, 0, 1, 2\}$$

$$p(-1) = 0.2, p(0) = 0.1, p(1) = 0.2, p(2) = 0.5$$

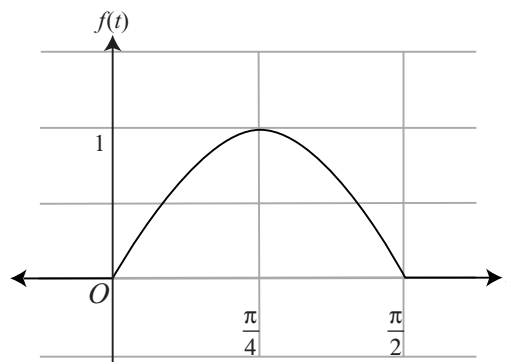
$$E(X) = \sum x p(x)$$

$$= -1 \times 0.2 + 0 \times 0.1 + 1 \times 0.2 + 2 \times 0.5$$

$$= 1$$

**Question 20**

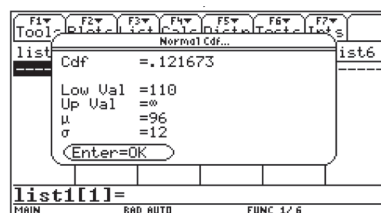
The mode is the value of t that gives the maximum value of $f(t)$.



By symmetry, this occurs at $\frac{\pi}{4}$.

Question 21

$$T \sim N(\mu = 96, \sigma^2 = 12^2)$$



$$\Pr(X > 110) = 0.1217$$

Approximately 12% exceeded the limit.

Question 22

The transition matrix must be of the form,

$$\begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix} \text{ where } 0 \leq a \leq 1 \text{ and } 0 \leq b \leq 1.$$

An element of the initial state matrix may be a cardinal number or a proportion: that is,

$$S_0 = \begin{bmatrix} p \\ q \end{bmatrix}, \text{ where } p + q = n(\xi), p, q \in \mathbb{Z}^+ \cup \{0\},$$

$$\text{or } S_0 = \begin{bmatrix} r \\ 1-r \end{bmatrix}, \text{ where } 0 \leq r \leq 1.$$

$$\begin{bmatrix} 0.35 & 0.55 \\ 0.65 & 0.45 \end{bmatrix} \text{ and } \begin{bmatrix} 27 \\ 33 \end{bmatrix} \text{ are the only pair of}$$

matrices that meet these criteria.

Mathematical Methods (CAS) Exam 2: SOLUTIONS

Section 2: Extended answers

Question 1

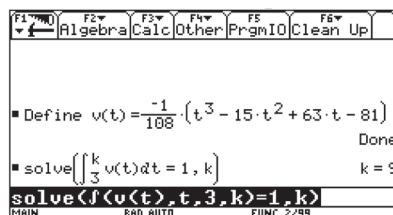
- a. For a probability density function,

$$\int_{-\infty}^{\infty} v(t) dt = 1. \text{ Therefore,}$$

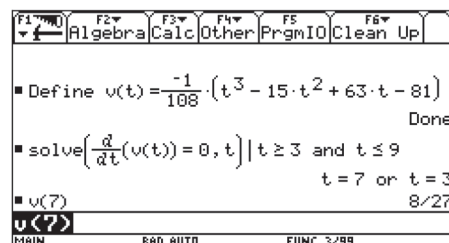
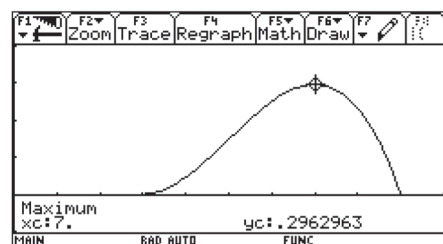
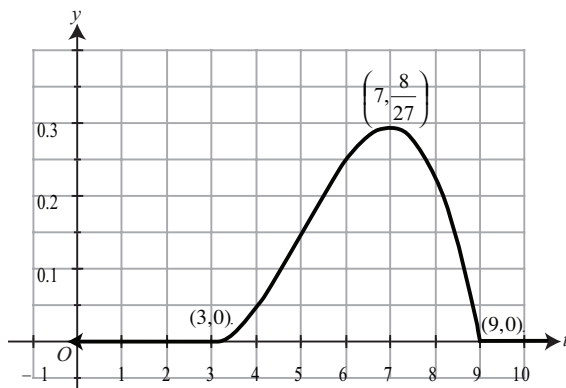
$$0 + \int_3^k -\frac{1}{108}(t^3 - 15t^2 + 63t - 81) dt = 1 \quad 1M$$

(dt must be shown to get the mark)

Solving this equation for k gives the result that $k = 9$, as required.



- b.

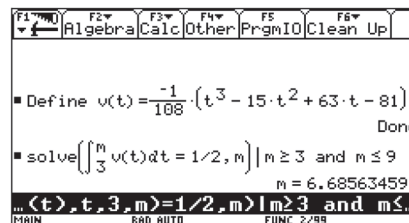


Correct shape and skew 1A
 Correct domain and axes intercepts 1A
 Correct exact stationary point 1A

- c. For the median, $\int_3^m v(t) dt = \frac{1}{2}$. 1M

Solving for m gives $m = 6.69$
 (correct to two decimal places)

The median incubation period is
 6.69 days. 1A

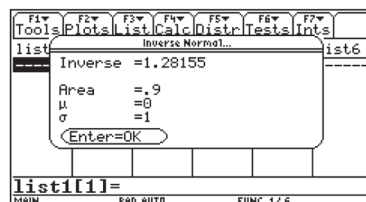


- d. $X \sim N(5, \sigma^2)$, $Z \sim N(0, 1)$

6 hours and 15 minutes = $6\frac{1}{4}$ hours

$$\Pr(Z < z) = 0.9$$

$$z = 1.28155$$



$$z = \frac{x - \mu}{\sigma}$$

$$1.28155 = \frac{6\frac{1}{4} - 5}{\sigma}$$

$$\sigma = 0.98$$

The standard deviation is 0.98 hours 1A

- e. Let Y be the number of patients for whom the response time to the drug is more than 6 hours 15 minutes.

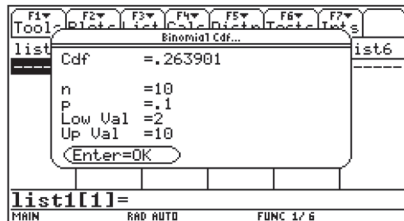
$$Y \sim \text{Bi}(10, 0.1)$$

1M

$$\Pr(Y \geq 2) = \sum_{n=2}^{10} {}^{10}C_n (0.9)^n (0.1)^{10-n}$$

$$= 0.264$$

1A



- f. $Y \sim \text{Bi}(k, 0.1)$

Consider the case where there is 95% chance.

$$\Pr(Y \geq 1) = 0.95$$

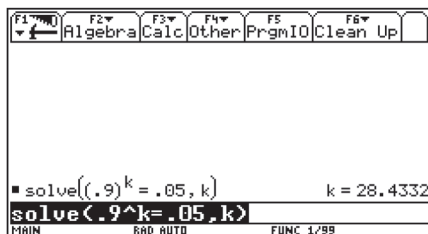
$$1 - \Pr(Y = 0) = 0.95$$

1M

$$1 - 0.9^k = 0.95$$

$$0.9^k = 0.05$$

$$k \approx 28.43 \quad (\text{using numerical solve/ solver})$$



Round up because $k \in \mathbb{Z}^+$ and the probability must be at least 95%.

The least value of k is 29 patients.

1A

- g.

	Fluaway Yesterday	Blugone Yesterday
Fluaway Today	0.8	0.3
Blugone Today	0.2	0.7

i. $\Pr(B_{\text{tomorrow}} | B_{\text{today}}) = 0.7$

1A

ii. $S_4 = T^4 \times S_0$

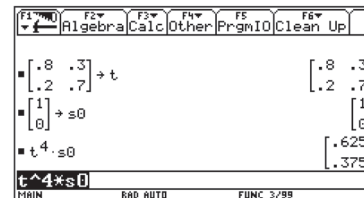
1M

$$= \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}^4 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.625 \\ 0.375 \end{bmatrix}$$

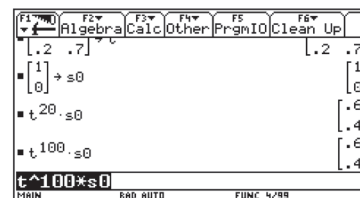
$$\Pr(B_{\text{Friday}} | F_{\text{Monday}}) = 0.375$$

1A



- iii. Consider, say, the 20th and 100th states.

$$T^{20} \times S_0 = T^{100} \times S_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$



In the long term, the proportion of days where Blugone is prescribed is 0.4

1A

Alternatively, if $T = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix}$,

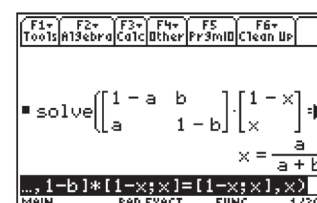
then as $n \rightarrow \infty$, $S_n \rightarrow \begin{bmatrix} 1 - \frac{a}{a+b} \\ \frac{a}{a+b} \end{bmatrix}$

$$\frac{a}{a+b} = \frac{0.2}{0.2+0.3} = \frac{2}{5} = 0.4$$

This formula arises from the fact that if the steady state, $\begin{bmatrix} 1-x \\ x \end{bmatrix}$, has been reached then the next state will also be $\begin{bmatrix} 1-x \\ x \end{bmatrix}$.

Solving the matrix equation

$$T \begin{bmatrix} 1-x \\ x \end{bmatrix} = \begin{bmatrix} 1-x \\ x \end{bmatrix} \text{ gives } x = \frac{a}{a+b}.$$



Question 2

a. $f(x) = A(x^{\frac{4}{3}} - 5)$

$-2 = A(0 - 5)$ **1M**

$A = \frac{2}{5}$

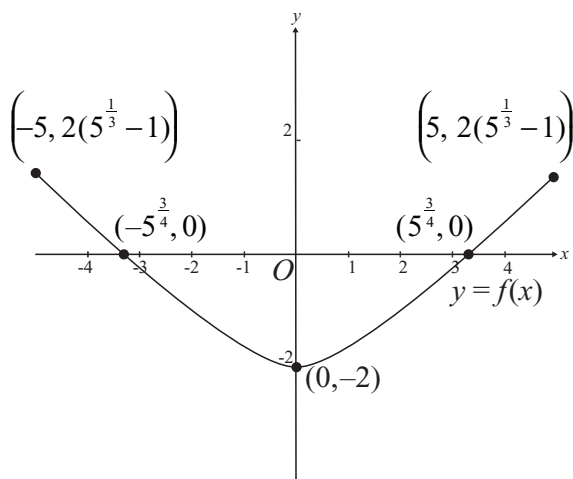
b. endpoints $(-5, 2(5^{\frac{1}{3}} - 1))$ and $(5, 2(5^{\frac{1}{3}} - 1))$ **2A**

turning point $(0, -2)$ and correct shape **1A**

x-intercepts $(-5^{\frac{3}{4}}, 0)$ and $(5^{\frac{3}{4}}, 0)$ **2A**

TI-84 Plus calculator screen showing the definition of the function $f(x) = \frac{2}{5}(x^{\frac{4}{3}} - 5)$ and the calculation of $f(-5)$ and $f(5)$.

TI-84 Plus calculator screen showing the calculation of the x-intercepts by solving $f(x) = 0$.



c. Max depth $= 2 + 2(5^{\frac{1}{3}} - 1)$
 $= 2(5^{\frac{1}{3}})$ **1A**

d. i. $A = -2 \times \frac{2}{5} \int_0^{5^{\frac{3}{4}}} (x^{\frac{4}{3}} - 5) dx$ **1M**
 $= \frac{16(5^{\frac{3}{4}})}{7} \text{ m}^2$ **1A**

TI-84 Plus calculator screen showing the integration of the function $f(x)$ from 0 to $5^{\frac{3}{4}}$.

ii. $V = 1000A$
 $= \frac{16000(5^{\frac{3}{4}})}{7} \text{ m}^3$ **1A**

e. i. $V = 1000A$ **1M**
 $= 1000(\frac{1}{2} \text{ base} \times \text{height})$
 $= 2000 \times 5^{\frac{3}{4}}$
 $= 6687.40 \text{ m}^3 \text{ correct to two dp}$ **1A**

ii. $\frac{dV}{dt} = \frac{dh}{dt} \times \frac{dV}{dh}$ **1A**
 $\frac{dV}{dt} = -0.02 \times \frac{dV}{dh}$
 $V = 1000 \frac{1}{2} bh$

Using similar triangles:

$\frac{\frac{1}{2}b}{h} = \frac{5^{\frac{3}{4}}}{2}$
 $\frac{1}{2}b = \frac{5^{\frac{3}{4}}}{2} h$ **1M**

$V = 500 \times 5^{\frac{3}{4}} h^2$ **1A**

$\frac{dV}{dh} = 1000 \times 5^{\frac{3}{4}} h$
 $\frac{dV}{dt} = -0.02 \times 1000 \times 5^{\frac{3}{4}} h$ **1M**
 $= -20 \times 5^{\frac{3}{4}} h \text{ m}^3/\text{h}$

The rate at which water is decreasing

is $20 \times 5^{\frac{3}{4}} h \text{ m}^3/\text{h}$

Question 3

a. $12 = A \times B^2 \quad \dots(1)$

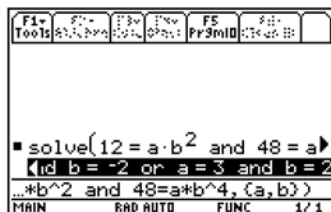
$48 = A \times B^4 \quad \dots(2)$

$B = 2, \text{ as } B > 0$

1A

$A = 3$

1A



b. Dilation by a factor of $\frac{1}{3}$ from the y -axis:

$y_1 = 3 \times 2^{3x} \quad 1A$

Translation of $\frac{4}{3}$ units parallel to the x -axis:

$y_2 = 3 \times 2^{(3x-4)} = 3 \times 2^{\left(3\left(x-\frac{4}{3}\right)\right)}$

Translation of 1 unit parallel to the y -axis:

$g_2(x) = 3 \times 2^{(3x-4)} + 1 = 3 \times 2^{\left(3\left(x-\frac{4}{3}\right)\right)} + 1 \quad 1A$

Range $(1, \infty)$ 1A

c. i. $g_3(x) = 3 \times 2^{(2x+1)} - 1$

Let $y = 3 \times 2^{(2x+1)} - 1$

Inverse: Swap x and y

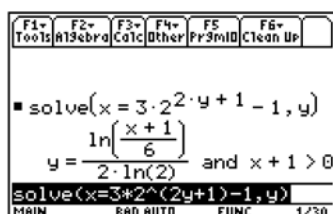
$x = 3 \times 2^{(2y+1)} - 1 \quad 1M$

$2^{2y+1} = \frac{x+1}{3}$

$2y+1 = \log_2\left(\frac{x+1}{3}\right)$

$g_3^{-1}(x) = \frac{1}{2} \log_2\left(\frac{x+1}{3}\right) - \frac{1}{2} \quad 1M$

$= \frac{1}{2} \log_2\left(\frac{x+1}{6}\right)$



ii. Dilation by a factor of $\frac{1}{2}$ from the x -axis

Dilation by a factor of 6 from the y -axis.

1A

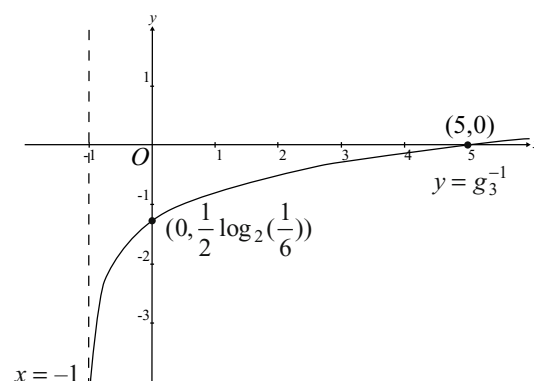
Translation of 1 unit to the left 1A

iii. Asymptote: $x = -1$ and shape 1A

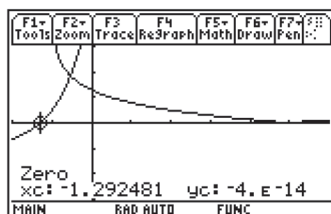
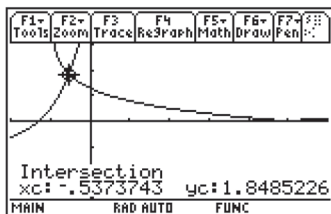
Coordinates of the x -intercept: $(5, 0)$

Coordinates of the y -intercept:

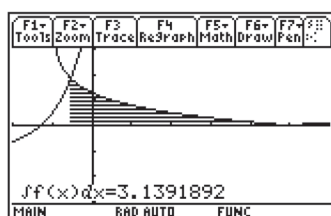
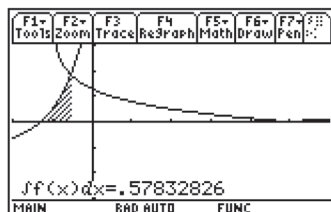
$\left(0, \frac{1}{2} \log_2\left(\frac{1}{6}\right)\right) \quad 1A$



- iv. Sketch both graphs and find the intersection and x-intercepts.

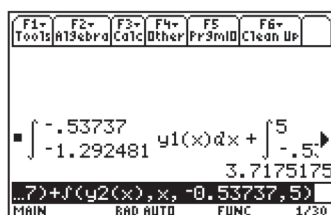


$$A = \int_{-1.292481}^{-0.53737} g_3(x) dx + \int_{-0.53737}^5 |g_3^{-1}(x)| dx \quad 1M$$



$$A \approx 0.578328 + 3.139189 \\ = 3.7 \text{ units}^2 \text{ correct to one decimal place}$$

OR



Question 4

- a. By Pythagoras' theorem

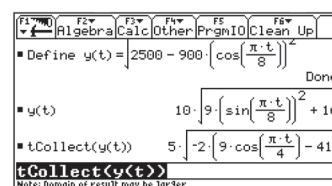
$$x^2 + y^2 = 50^2$$

$$y = \sqrt{2500 - x^2} \text{ (reject negative solution)} \quad 1A$$

- b. i. $y = \sqrt{2500 - x^2}$ and $x(t) = 30 \cos\left(\frac{\pi}{8}t\right)$

$$y(t) = \sqrt{2500 - 900 \cos^2\left(\frac{\pi}{8}t\right)}$$

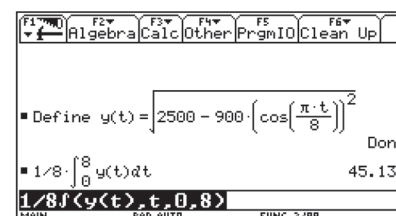
Other equivalent forms are possible. 1A



- ii. Average value = $\frac{1}{8} \int_0^8 y(t) dt$ 1M

$$= 45.139$$

(3 decimal places) 1A



- c. Highest position when $x = 0$. Endpoint will be $(0, 50)$. $y = 50$ cm 1A

Lowest position when $x = \pm 30$. The corresponding y-axis value is

$$y = \sqrt{2500 - (30)^2} = 40 \text{ or}$$

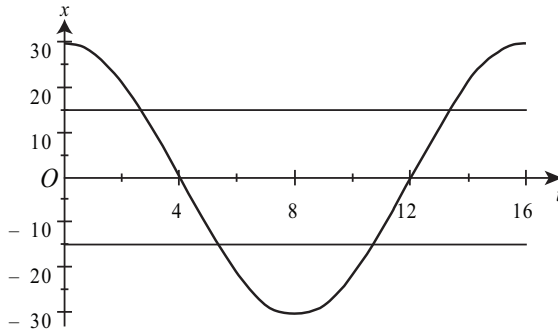
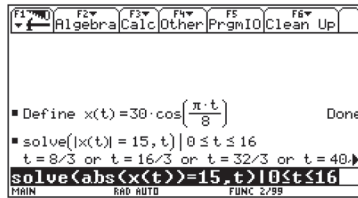
$$y = \sqrt{2500 - (-30)^2} = 40$$

$$y = 40 \text{ cm} \quad 1A$$

- d. Solve for t , $x(t) = \pm 15$ or $|x(t)| = 15$ such that $t \in [0, 16]$

$$t = \frac{8}{3}, \frac{16}{3}, \frac{32}{3}, \frac{40}{3}$$

1A



Time for which $-15 \leq x \leq 15$ is given by

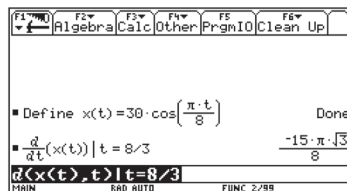
$$\left(\frac{16}{3} - \frac{8}{3}\right) + \left(\frac{40}{3} - \frac{32}{3}\right) = \frac{16}{3} \text{ seconds}$$

1A

- e. When $x = 15$, $t = \frac{8}{3}, \frac{40}{3}, \dots$ (from part d.)

$$x'\left(\frac{8}{3}\right) = -\frac{15\sqrt{3}\pi}{8}$$

1M



Speed is the magnitude of $x'(t)$

$$\text{Speed} = \frac{15\sqrt{3}\pi}{8} \text{ cm/s}$$

1A

f. $x'(t) = -\frac{15\pi}{4} \sin\left(\frac{\pi}{8}t\right)$

The maximum speed occurs where

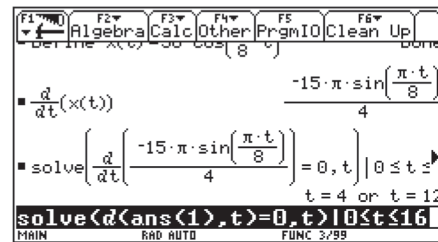
$$\frac{d}{dt}(x'(t)) = 0$$

1M

$$\frac{d}{dt}\left(-\frac{15\pi}{4} \sin\left(\frac{\pi}{8}t\right)\right) = 0$$

$$t = 4, 12, \dots$$

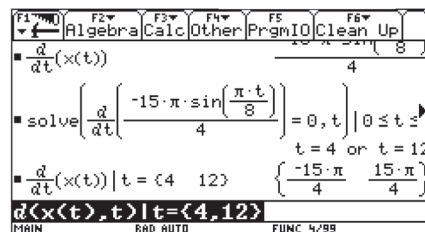
1M



(This is also evident from the graph of x v. t , which show the maximum gradient at $t = 4, 12, \dots$)

$$x'(4) = -\frac{15\pi}{4} \sin\left(\frac{\pi}{2}\right) = -\frac{15\pi}{4}$$

$$x'(12) = -\frac{15\pi}{4} \sin\left(\frac{3\pi}{2}\right) = \frac{15\pi}{4}$$



The speed is the magnitude of $x'(t)$.

Hence the maximum speed is $\frac{15\pi}{4}$ cm/s. 1A

MATHEMATICAL METHODS AND MATHEMATICAL METHODS (CAS)

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.

This formula sheet is provided for your reference.

Mathematical Methods and Mathematical Methods CAS

Formulas

Mensuration

area of a trapezium: $\frac{1}{2}(a+b)h$

curved surface area of a cylinder: $2\pi rh$

volume of a cylinder: $\pi r^2 h$

volume of a cone: $\frac{1}{3}\pi r^2 h$

volume of a pyramid: $\frac{1}{3}Ah$

volume of a sphere: $\frac{4}{3}\pi r^3$

area of a triangle: $\frac{1}{2}bc \sin A$

Calculus

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$

$$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin(ax)) = a \cos(ax)$$

$$\frac{d}{dx}(\cos(ax)) = -a \sin(ax)$$

$$\frac{d}{dx}(\tan(ax)) = \frac{a}{\cos^2(ax)} = a \sec^2(ax)$$

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + c$$

$$\int \frac{1}{x} dx = \log_e |x| + c$$

$$\int \sin(ax) dx = -\frac{1}{a} \cos(ax) + c$$

$$\int \cos(ax) dx = \frac{1}{a} \sin(ax) + c$$

product rule: $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$$\Pr(A) = 1 - \Pr(A')$$

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

mean: $\mu = E(X)$

variance: $\text{var}(X) = \sigma^2 = E((X - \mu)^2) = E(X^2) - \mu^2$

probability distribution		mean	variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x p(x)$	$\sigma^2 = \sum (x - \mu)^2 p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x) dx$	$\mu = \int_{-\infty}^{\infty} x f(x) dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$

END OF FORMULA SHEET

Version 2 – March 2006

© VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY 2006
REPRODUCED WITH PERMISSION MATHEMATICAL ASSOCIATION OF VICTORIA 2006