

Mathematical Methods Exam 1: Solutions

Question 1

a. $f(x) = \frac{2}{x-3} + 4$

Let $y = \frac{2}{x-3} + 4$

Inverse: swap x and y .

$$x = \frac{2}{y-3} + 4$$

$$(x-4)(y-3) = 2$$

$$y = \frac{2}{x-4} + 3$$

$$f^{-1}: R \setminus \{4\} \rightarrow R, \text{ where}$$

$$f^{-1}(x) = \frac{2}{x-4} + 3$$

b. Equating f to f^{-1} or f to x ,

$$x = \frac{2}{x-3} + 4$$

$$(\text{alternatively, } \frac{2}{x-3} + 4 = \frac{2}{x-4} + 3)$$

$$(x-4)(x-3) = 2$$

$$x^2 - 7x + 10 = 0$$

$$(x-5)(x-2) = 0$$

$$x = 5 \text{ or } x = 2$$

$$(5, 5) \text{ and } (2, 2)$$

Question 2

$$2 \sin^2(x) = 1$$

$$\sin^2(x) = \frac{1}{2}$$

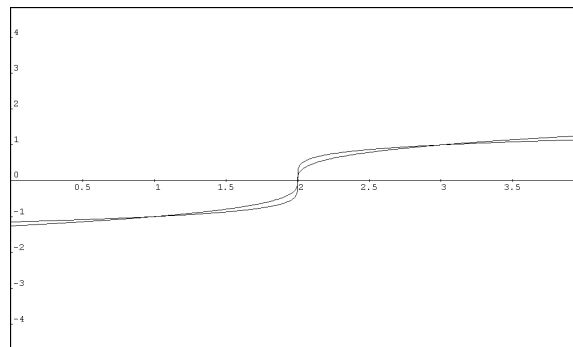
$$\sin(x) = \pm \frac{1}{\sqrt{2}}$$

$$x = \frac{-3\pi}{4}, \frac{-\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$$

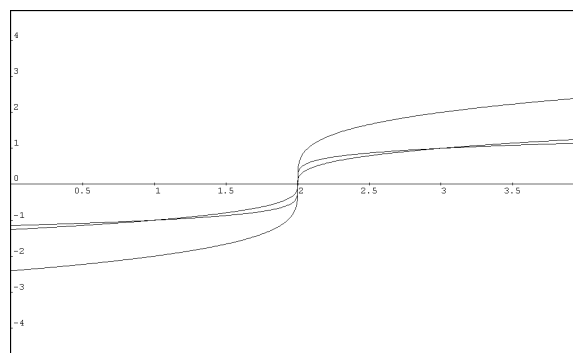
2A
(i.e. $4 \times \frac{1}{2}A$)

Question 3

- a. Correct Endpoints **1A**
Correct Intersections **1A**
Correct Shape **1A**



- b. Correct shape and open circles for endpoints **1A**



c. $2 \int_2^3 ((x-2)^{\frac{1}{5}} - (x-2)^{\frac{1}{3}}) dx$ **1M**

$$= 2 \left[\frac{5}{6}(x-2)^{\frac{6}{5}} - \frac{3}{4}(x-2)^{\frac{4}{3}} \right]_2^3$$
 1A

$$= 2 \left(\left(\frac{5}{6}(3-2)^{\frac{6}{5}} - \frac{3}{4}(3-2)^{\frac{4}{3}} \right) - \right.$$

$$\left. \left(\frac{5}{6}(2-2)^{\frac{6}{5}} - \frac{3}{4}(2-2)^{\frac{4}{3}} \right) \right)$$

$$= 2 \left(\frac{5}{6} - \frac{3}{4} \right)$$

$$= \frac{1}{6} \text{ square units} \quad \mathbf{1A}$$

Question 4

- a. Rule: $f(g(x)) = \log_e(|2x + 5|)$ **1A**
 Domain: $R^- \setminus \{-\frac{5}{2}\}$ or $(-\infty, 0) \setminus \{-\frac{5}{2}\}$ **1A**

or

$$f(g(x)): (-\infty, 0) \setminus \{-\frac{5}{2}\} \rightarrow R, \text{ where}$$

$$f(g(x)) = \log_e(|2x + 5|)$$

- b. $\frac{d}{dx}[f(g(x))] = \frac{2}{2x + 5}$ **1M**

$$\frac{d}{dx}[f(g(-4))] = \frac{2}{-8 + 5}$$

$$= \frac{-2}{3}$$

$$f(g(-4)) = \log_e(3)$$

$$y - \log_e(3) = \frac{3}{2}(x + 4)$$

$$y = \frac{3}{2}x + 6 + \log_e(3)$$
 1A

Question 5

- a. $h = 2r$

$$V = \pi r^2 h$$

$$= \frac{\pi h^3}{4}$$

- b. $\frac{dV}{dh} = \frac{3\pi h^2}{4}$ **1M**

$$\frac{dh}{dt} = \frac{dV}{dt} \frac{dh}{dv}$$

$$= 8 \times \frac{4}{3\pi h^2}$$

When $h = 2$ cm

$$\frac{dh}{dt} = \frac{8}{3\pi} \text{ cm/s}$$
 1A

Question 6

- a. $f(x) = \frac{2e^{3x}}{\sqrt{x+1}}$ **1M**

$$f'(x) = \frac{6e^{3x}\sqrt{x+1} - \frac{1}{2\sqrt{x+1}}2e^{3x}}{x+1}$$

$$= \frac{6e^{3x}(x+1) - e^{3x}}{(x+1)^{\frac{3}{2}}}$$
 1M

$$= \frac{e^{3x}(6x+5)}{(x+1)^{\frac{3}{2}}}$$
 1A

- b. $f'(x) = \frac{e^{3x}(6x+5)}{(x+1)^{\frac{3}{2}}} = 0$

$$6x + 5 = 0$$

$$x = -\frac{5}{6}$$
 1M

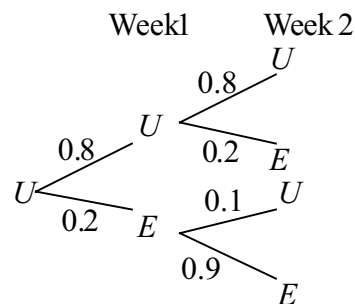
x	-0.9	$-\frac{5}{6}$	0
$f'(x)$	-ve	0	+ve

Local minimum stationary point at **1A**

$$\left(-\frac{5}{6}, 2\sqrt{6}e^{-\frac{5}{2}}\right)$$
 1A

Question 7

Let U denote the purchase of unleaded and E denote the purchase of ethanol blend.



$$\Pr(UE + EE)$$
 1M

$$= 0.8 \times 0.2 + 0.2 \times 0.9$$
 1M

$$= 0.16 + 0.18$$

$$= 0.34$$
 1A

Question 8

$$\begin{aligned}
 \text{a. } \int_0^{\pi} \cos\left(\frac{x}{2}\right) dx & \\
 &= \left[2 \sin\left(\frac{x}{2}\right) \right]_0^{\pi} && \mathbf{1M} \\
 &= 2 \sin\left(\frac{\pi}{2}\right) - 0 \\
 &= 2 && \mathbf{1A}
 \end{aligned}$$

$$\begin{aligned}
 \text{b. } k \int_0^{\pi} \cos\left(\frac{x}{2}\right) dx &= 1 \\
 2k &= 1 \\
 k &= \frac{1}{2} && \mathbf{1A}
 \end{aligned}$$

$$\begin{aligned}
 \text{c. } \frac{d}{dx} \left(x \sin\left(\frac{x}{2}\right) + 2 \cos\left(\frac{x}{2}\right) \right) & \\
 &= \sin\left(\frac{x}{2}\right) + \frac{1}{2} x \cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right) && \mathbf{1M} \\
 &= \frac{1}{2} x \cos\left(\frac{x}{2}\right) && \mathbf{1A}
 \end{aligned}$$

$$\begin{aligned}
 \text{d. } \int_0^{\pi} x \cos\left(\frac{x}{2}\right) dx &= 2 \left[x \sin\left(\frac{x}{2}\right) + 2 \cos\left(\frac{x}{2}\right) \right]_0^{\pi} \\
 &= 2 (\pi - 2) \\
 &= 2\pi - 4 && \mathbf{1A}
 \end{aligned}$$

$$\begin{aligned}
 \text{e. } E(X) &= \int_0^{\pi} \left(x \times \frac{1}{2} \cos\left(\frac{x}{2}\right) \right) dx \\
 &= \frac{1}{2} (2\pi - 4) \\
 &= \pi - 2 && \mathbf{1A}
 \end{aligned}$$