



Meet the Assessors

Mathematical Methods 2016

Paper 1

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Mathematical Methods MA082

Written Examination Paper 1 2016.

General comments.

* The number of candidates who sat for the 2016 examination was 16 135. Though 2016 heralded a new/revised study design, the format and structure of the 2016 examination for the most part, was relatively similar to previous years. Completion of past papers (available on the VCAA website) will enhance examination technique. The advice given in previous Examination Reports continues to contain valuable information. Especially pleasing was the increase in the number of students who attained all 40 marks available.

* Reports from Chief Assessor are now available on VCAA web site.

* Important note: markers will mark precisely what is provided by the candidate. They make no assumptions about what the candidate “might have meant”.



* Areas of some strength included:

- Application of product and quotient rules (Q.1)
- The ability to sketch a hyperbola (Q. 3a)
- Conditional probability (Q. 7).
- Highlighting the key components of a question is a good strategy and can be used to check the soundness or validity of the final answer. It may also assist in reducing transcription mistakes.



Areas of weakness included:

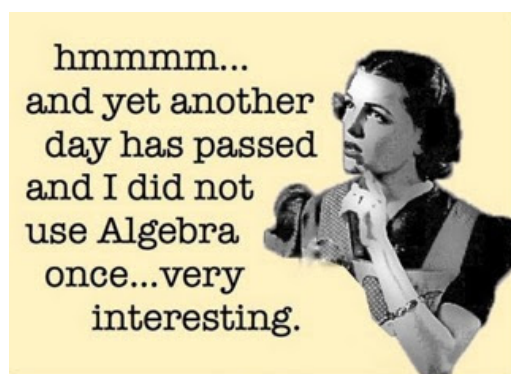
- Differentiation involving a fractional power (Q. 2)
- Application of binomial probability (Q. 4)
- “Show” questions (Q. 5.a.iii, Q.8b.ii)
- average rate of change and average value of function (Q.6)
- Inability to answer a question explicitly

- Using inappropriate notation such as $\log e^x$ or $\log ex$ or $\log(e^x)$ for $\log_e(x)$. Note that using the bracket is good insurance against ambiguity. $\ln(x)$ is fine.
- Inability to simplify arithmetic & algebraic expressions fully:
Answers such as $\frac{15}{32} + \frac{3}{24}$, $\frac{1+0.5}{2}$ and $\frac{9}{6}$ are considered incomplete.
- Misuse of brackets and omission of essential brackets.
- Square brackets mis-used to represent co-ordinates.
- Poor algebra and poor arithmetic.
- Omission of dx .
- Omission of units (Q. 2b).



- Probabilities outside $[0, 1]$
- Solution of non-routine equations
- And for some, failure to be familiar with the details on the list of Miscellaneous Formulae.

In summary, Mathematics is a precise language with its own notation and this must be respected by all users of that language, just as musical notation is standard. If the note is F# or B $\flat$, then that is what it is.



- * Not the case?
- * Transcription was a problem, both from the question paper and also from within a candidate's script.
- * Those candidates who were "short & sweet" generally fared better.

Question 1

a. Let $y = \frac{\cos(x)}{x^2 + 2}$.

Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{-\sin(x)(x^2 + 2) - 2x \cos(x)}{(x^2 + 2)^2} \quad \boxed{\text{STOP!}}$$

Marks	0	1	2	Mean
%	11	33	56	1.4

Points to note:

- * The question fairly well done. Issues included
 - * lack of appropriate brackets,
 - * inappropriate cancellation of between numerator and denominator,
 - * a “lost” denominator,
 - * further engagement.
- * One common problem was the expansion of $(x^2 + 2)^2$ as $x^4 + 4$.
- * As part of coping with ¥12 Mathematics, students need to appreciate what “tidying up” needs to be done such as cancellation of common factors and what is time-wasting.
- * There were also problems with candidates who did not put a bracket around $x^2 + 2$ when completing quotient/product rule or, worse still, omitting a bracket in one line and reinstating it in the next line. Markers are able to read only what is before them and so such working would be interpreted as two compensating mistakes and not awarded full credit.
- * The final answer is taken to be what is intended to be read by the marker and further engagement involving a mistake is penalised.

b. Let $f(x) = x^2 e^{5x}$

i. Find $f'(1)$.

$$f'(x) = 2x e^{5x} + 5x^2 e^{5x}$$

$$f'(1) = 2 \times e^5 + 5 \times e^5 \\ = 7e^5$$

Marks	0	1	2	Mean
%	13	18	69	1.6

Points to note.

* This type of double-barrelled question has appeared in every Paper 1 from 2008, except in 2015. Too often, the substitution of 1 for x was missed (as in previous years). Re-reading questions should be encouraged by teachers.

* What was a little surprising was that some candidates who followed a correct process in 1.a., failed to realise that the product rule was required and used their own “version”. An incorrect “merger” of the product and chain rule, resulting in an answer of $10xe^{5x}$, was a common error.

* Some failed to simplify $5e^2 + 2e^2$ to $7e^2$.

Question 2

Let $f: \left(-\infty, \frac{1}{2}\right] \rightarrow \mathbb{R}$, where $f(x) = \sqrt{1-2x}$.

a. Find $f'(x)$.

$$f(x) = (1-2x)^{\frac{1}{2}} \\ f'(x) = \frac{1}{2} \times (-2) \times (1-2x)^{-\frac{1}{2}} \\ = -(1-2x)^{-\frac{1}{2}} \text{ OR } -\frac{1}{\sqrt{1-2x}}$$

Marks	0	1	Mean
%	31	69	0.68

Points to note.

* One answer is sufficient!

* This requires great care given the number of steps involved.

* Either the derivative of $1-2x$ or the “-“ disappeared, sometimes both.

* Some indices given, $\frac{3}{2}$ or $-\frac{3}{2}$, suggest that anti-differentiation was involved or even double differentiation!

b. Find the angle θ from the positive direction of the x -axis to the tangent to the graph of f at $x = -1$, measured in the anticlockwise direction.

$$f'(-1) = -\frac{1}{\sqrt{3}} = \tan(\theta)$$

$$\theta = \frac{5\pi}{6}$$

Marks	0	1	2	Mean
%	42	39	20	0.77

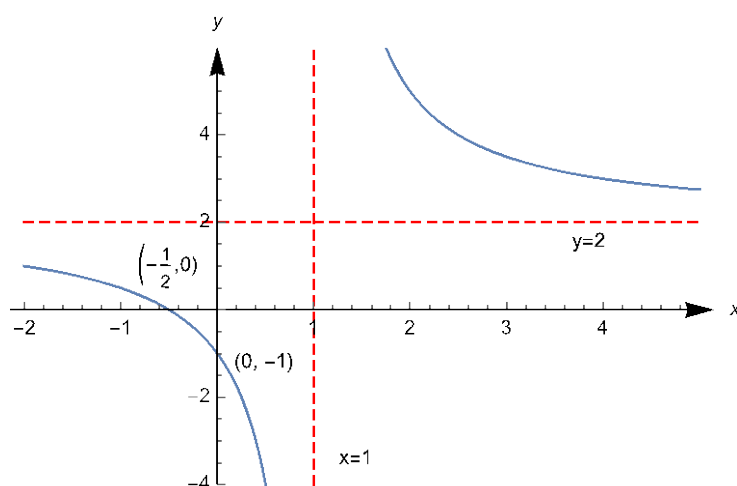
Points to note:

- * 150° was acceptable but don't encourage it!
- * Substitution of -1 was well done but there were problems finding the correct angle or even the correct quadrant. Few diagrams were seen.
- * After correctly evaluating the derivative as $-\frac{1}{\sqrt{3}}$, the negative sign sometimes disappeared.
- * 2017 candidates may need to be reminded that they need to know exact values of the trig ratios involving the angles 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$ and be able to find their counterparts in Q₂, Q₃, Q₄.
- * Many students incorrectly assumed gradient = $f'(-1)$ or wasted time finding the equation of the tangent.

Question 3

a. Let $f: \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}$, where $f(x) = 2 + \frac{3}{x-1}$.

Sketch the graph of f . Label the axis intercepts with their coordinates and label any asymptotes with the appropriate equation.



Marks	0	1	2	3	Mean
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%	9	10	24	57	2.3
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Points to note.

- * Some very good, carefully prepared graphs were seen. Most had asymptotes in the right places and asymptotic behaviour was generally good. Sometimes the “first quadrant” section of graphs “hugged” the axes too much. This was evidenced when some graphs showed a lack of rotational symmetry with respect to the intersections of the asymptotes.
- * Others showed carelessness with regard to asymptotic behaviour, with graphs bending away from the asymptote.

b. Find the area enclosed by the graph of f , the lines $x = 2$ and $x = 4$, and the x -axis.

$$\begin{aligned}
 \int_2^4 \left(2 + \frac{3}{x-1}\right) dx &= [2x + 3\log_e(x-1)]_2^4 \\
 &= [8 + 3\log_e(3)] - [4 + 3\log_e(1)] \\
 &= 4 + 3\log_e(3) \quad \text{or} \quad 4 + \log_e(27)
 \end{aligned}$$

Marks	0	1	2	Mean
%	26	29	47	1.2

Points to note.

- * This was a very straightforward application of integration. It is worrying that some candidates did not involve a “log” in the integration process.
- * There was also some poor algebraic/arithmetical manipulation. $3^3 = 9$ is worrying as is $3\log_e(3) = \log_e(9)$.
- * dx was not always seen.
- * some students tried sums of areas thus overcomplicating the problem.

Question 4

A paddock contains 10 tagged sheep and 20 untagged sheep. Four times each day, one sheep is selected at random from the paddock, placed in an observation area and studied, and then

returned to the paddock.

- a. What is the probability that the number of tagged sheep selected on a given day is zero?

Let X be the number of tagged sheep.

$$X \sim \text{Bi}(4, \frac{1}{3})$$
$$\Pr(X = 0) = (\frac{2}{3})^4$$
$$= \frac{16}{81}$$

Marks	0	1	Mean
%	42	58	0.6

Point to note.

* Well done except for those who missed “four times”.

- b. What is the probability that at least one tagged sheep is selected on a given day?

$$\Pr(X \geq 1) = 1 - \Pr(X = 0)$$
$$= \frac{65}{81}$$

Marks	0	1	Mean
%	42	58	0.6

Points to note.

* Quite well done. The correct answer of 65/81 was acceptable as was $1 -$ (the previous answer simplified), provided that both answers were in the interval $(0, 1)$. Some did not realise that the events in a. and b. were complementary and took “the scenic route” such as wasting their time in finding the sum of four probabilities; others made arithmetic errors.

- c. What is the probability that no tagged sheep are selected on each of six consecutive days? **Express your answer in the form $(\frac{a}{b})^c$, where a , b and c are positive integers.**

Let Y be the number of days when no tagged sheep are selected.

$$Y \sim \text{Bi}(6, \frac{16}{81})$$

$$\Pr(Y = 0) = (\frac{16}{81})^6 \text{ or } (\frac{2}{3})^{24}$$

Marks	0	1	Mean
%	40	60	0.6

Points to note.

- * Reasonably well done. Regrettably some gave $(1 - \text{part a.})^6$. Those who obtained an incorrect value for a. were treated leniently here.
- * Unsurprisingly, answers were given that were correct but not given in the correct format. Re-read the question!
- * The majority of students made the correct connection to part (a) and the exponent 6. The most common incorrect answer was $(\frac{2}{3})^6$.

Question 5

Let $f: (0, \infty) \rightarrow R$, where $f(x) = \log_e(x)$ and $g: R \rightarrow R$, where

$$g(x) = x^2 + 1.$$

a. i. Find the rule for h , where $h(x) = f(g(x))$.

$$h(x) = f(g(x)) = f(x^2 + 1) = \log_e(x^2 + 1)$$

Marks	0	1	Mean
%	5	95	0.9

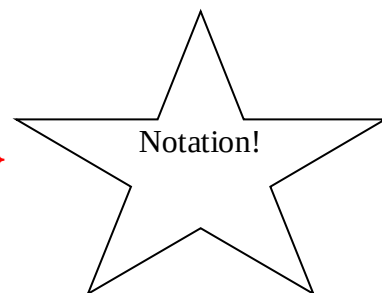
Point to note.

- * As the data indicate, this question was extremely well done. The few errors were more a result of poor notational work (e.g. $\log_e x^2 + 1$), rather than a lack of understanding in determining the rule of this composite function.

ii. State the domain and range of h .

$$\text{Domain} = R; \text{ range} = [0, \infty) \text{ or } R^+ \cup \{0\}$$

Marks	0	1	2	Mean
%	55	30	15	0.6



Points to note.

- * A mixed bunch with the range causing more problems.
- * Sometimes a single answer was given but no indication was provided to say whether the answer was domain or range. 2017 candidates will need reminders that they must be **explicit** in expressing their answers.

iii. **Show that** $h(x) + h(-x) = f((g(x))^2)$.

Here is a good sample solution.

$$\begin{aligned}
 \text{LHS} &= h(x) + h(-x) \\
 &= \log_e(x^2 + 1) + \log_e((-x)^2 + 1) \\
 &= \log_e(x^2 + 1) + \log_e(x^2 + 1) \\
 &= 2 \log_e(x^2 + 1) \\
 \text{RHS} &= f((g(x))^2) \\
 &= f(x^2 + 1)^2 \\
 &= \log_e((x^2 + 1)^2) \\
 &= 2 \log_e(x^2 + 1) \\
 &= \text{LHS as required to be shown.}
 \end{aligned}$$

Marks	0	1	2	Mean
%	27	42	31	1

Points to note.

* Many responses were very messy with students unsure of how to present their argument. The best approach was to take one side of the given equation and step by step clearly demonstrate how the expression is manipulated to be the equivalent of the other side of the given equation. In the sample answer above, both sides were **operated upon separately** to then arrive at the conclusion that one side was in fact equivalent to the other.

* Poor notational work was evident, in particular $\log_e(-x^2 + 1) \neq \log_e((-x)^2 + 1)$.

iv. Find the coordinates of the stationary point of h and state its nature.

$$h'(x) = \frac{2x}{x^2 + 1} = 0 \text{ when } x = 0.$$

$$h(0) = 0$$

$\therefore$ stationary point is $(0, 0)$ and it is a minimum.

[Explanation for students: Since the minimum value of $x^2 + 1$ is 1 and $\log_e(1) = 0$, the stationary point will be a minimum. Or, look at sign changes for $h'(x)$ as x passes through 0.]

Marks	0	1	2	Mean
%	45	28	27	0.82

Points to note.

* Other correct responses could include mention of the symmetry of the graph of g , taken with the log of an increasing function for both positive and negative x , use of a sketch graph or a reference to the range in a.ii.

* There was a lot of poor differentiation. $\frac{dy}{dx}$ was often given as $\frac{1}{x^2 + 1}$ which candidates equated to 0 and proceeded to “solve” for x . Unreal!

* Of those obtained the correct derivative, the “nature” was described as a turning point [despite the wording of the question] or sometimes a point of inflexion.

* Some candidates did not answer the question explicitly.

b. Let $k: (-\infty, 0] \rightarrow \mathbb{R}$, where $k(x) = \log_e(x^2 + 1)$.

i. Find the rule for k^{-1} .

Noting that k is a one:one function and letting $y = k(x)$,

$$x = \log_e(y^2 + 1)$$

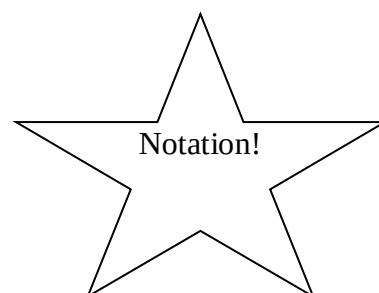
$$e^x = y^2 + 1$$

$$y^2 = e^x - 1$$

$$y = \pm \sqrt{e^x - 1}$$

But given the domain as $(-\infty, 0]$,

$$k^{-1}(x) = -\sqrt{e^x - 1}.$$

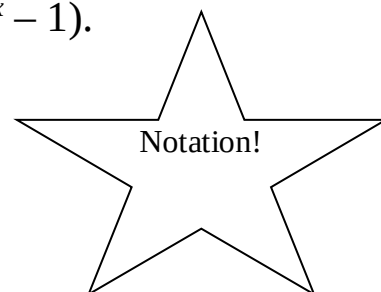


Marks	0	1	2	Mean
%	18	66	16	0.98

Points to note.

* Many knew that they had to swap x and “ y ”. Since y was an introduced term, the final answer needed to be expressed in terms of symbols given in the question, viz $k^{-1}(x) = \dots$.

* Most candidates missed the fact that $k^{-1}(x) = -\sqrt[3]{(e^x - 1)}$.



ii. State the domain and range of k^{-1} .

Domain = $[0, \infty)$ or $R^+ \cup \{0\}$.

Range = $(-\infty, 0]$ or $R^- \cup \{0\}$.

Marks	0	1	2	Mean
%	36	40	24	0.88

Points to note.

* Some gave the answers in the reverse order without specifying which was which. Markers had to assume that the candidates were giving answers in the order in which they were asked for.

* Domain and range were found by quite a few who were unable to complete 5b.i.

* Most students did utilise the fact that $Range_{k^{-1}} = Domain_k$ but found stating the domain of the inverse function more difficult.

Question 6. Let $f : [-\pi, \pi] \rightarrow R$, where $f(x) = 2 \sin(2x) - 1$.

a. Calculate the average rate of change of f between $x = -\frac{\pi}{3}$

and $x = \frac{\pi}{6}$.

$$f\left(\frac{\pi}{6}\right) = 2 \sin\left(\frac{\pi}{3}\right) - 1 = \sqrt{3} - 1$$

$$f\left(-\frac{\pi}{3}\right) = 2 \sin\left(-\frac{2\pi}{3}\right) - 1 = -\sqrt{3} - 1$$

$$\text{Average rate of change} = \frac{\sqrt{3} - 1 - (-\sqrt{3} - 1)}{\frac{\pi}{6} - (-\frac{\pi}{3})}$$

$$= \frac{4\sqrt{3}}{\pi}$$

Marks	0	1	2	Mean
%	37	31	32	0.94

Points to note.

- * There was some confusion between a. and b. and which was which.
- * Difficulties included those who added the x-values to get the “run” (or lost the “-“ sign), inability to evaluate sine of $\frac{\pi}{3}$ and $-\frac{2\pi}{3}$ and those thought that the derivative should be evaluated at these two values.
- * Some failed to simplify their final expression by cancelling common factors or left the answer as fraction/fraction. This is never acceptable.
- * Care needs to be taken with legibility. At times, it became difficult to distinguish between π [which was probably intended] and x .
- * Some good solutions were seen.

b. Calculate the average value of f over the interval

$$-\frac{\pi}{3} \leq x \leq \frac{\pi}{6}.$$

$$\text{Average Value} = \frac{1}{\frac{\pi}{6} - (-\frac{\pi}{3})} \int_{-\frac{\pi}{3}}^{\frac{\pi}{6}} (2\sin(2x) - 1) dx$$

$$= \frac{2}{\pi} \left[-\cos(2x) - x \right]_{-\frac{\pi}{3}}^{\frac{\pi}{6}}$$

$$= -\frac{2}{\pi} - 1.$$

Marks	0	1	2	3	Mean
%	31	23	30	16	1.3

Points to note.

- * Some good solutions were seen.

- * Difficulties arose from omission of $\frac{2}{\pi}$ from before the integral and poor algebra. For example $2\sin(2x) - 1$ was “factorised” to $2(\sin(2x) - 1)$ by taking a 2 out of the integral.
- * dx was frequently AWL.
- * Some found the algebra in the tidying up too difficult.
- * Of those who completed the question, no-one [very few] challenged the negative answer. Good!

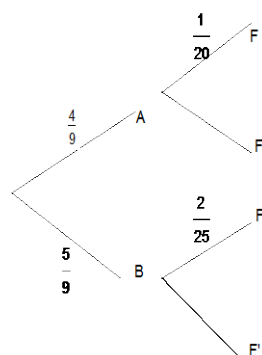
Question 7

A company produces motors for refrigerators. There are two assembly lines, Line A and Line B.

5% of the motors assembled on Line A are faulty and 8% of the motors assembled on Line B are faulty. In one hour, 40 motors are produced from Line A and 50 motors are produced from Line B. At the end of an hour, one motor is selected at random from all the motors that have been produced during that hour.

- a. What is the probability that the selected motor is faulty?**

Express your answer in the form $\frac{1}{b}$, where b is a positive integer.



From tree diagram: $\Pr(F) = \frac{1}{20} \times \frac{4}{9} + \frac{2}{25} \times \frac{5}{9} = \frac{1}{15}$.

Or, using obvious symbolism: A, B, F , we require

$$\begin{aligned}
 \Pr(F) &= \Pr(F \cap A) + \Pr(F \cap B) \\
 &= \Pr(F|A) \cdot \Pr(A) + \Pr(F|B) \cdot \Pr(B) \\
 &= \frac{1}{20} \times \frac{4}{9} + \frac{2}{25} \times \frac{5}{9} \\
 &= \frac{1}{45} + \frac{2}{45} = \frac{3}{45} = \boxed{\frac{1}{15}}
 \end{aligned}$$

Marks	0	1	2	Mean
%	39	15	46	1.1

Points to note.

- * This was a straightforward question that required the law of total probability or a tree diagram or $\frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$.
- * The format for the answer was intended as a guide to remind candidates to check the answer but this was not always appreciated. For example, answers such as $\frac{2}{30}$ appeared. Re-read the question!
- * Many attempts did not use $\frac{4}{9}$ and $\frac{5}{9}$.
- * Probabilities out of the interval $[0, 1]$ were observed.

b. The selected motor is found to be faulty.

What is the probability that it was assembled on Line A?

Express your answer in the form $\frac{1}{c}$, where c is a positive integer.

$$\begin{aligned}
 \Pr(A|F) &= \frac{\Pr(A \cap F)}{\Pr(F)} \\
 &= \frac{1}{45} / \frac{1}{15}
 \end{aligned}$$

$$= \boxed{\frac{1}{3}}$$

Or, in 1h, of the 90 motors produced, 2 faulty ones come from Line A & 4 faulty ones come from Line B.

$$\text{So } \Pr(A|F) = \frac{2}{2+4}$$

$$= \boxed{\frac{1}{3}}$$

Marks	0	1	Mean
%	68	32	0.32

Points to note.

In general the conditional probability was recognised, but not so the reduced sample space. Often the instruction regarding the format of the final answer was overlooked.

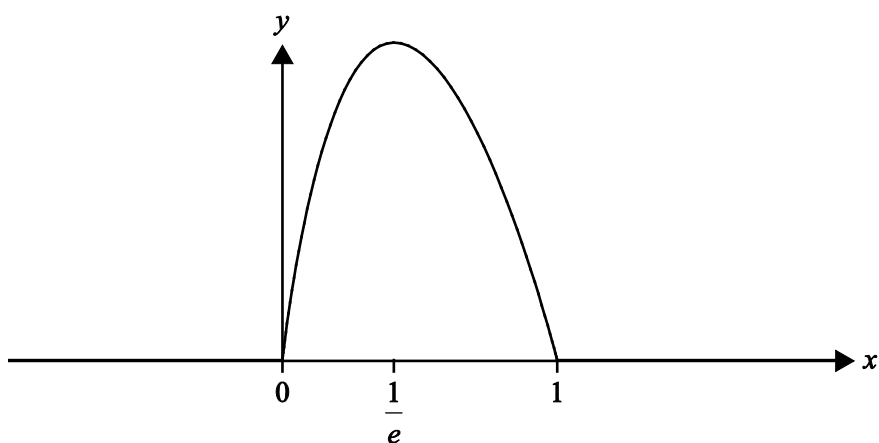
* Again, probabilities out of the interval $[0, 1]$ were observed, despite the hint in the question.

Question 8

Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} -4x \log_e(x) & 0 < x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Part of the graph of f is shown below. The graph has a turning point at $x = \frac{1}{e}$.



- a. **Show by differentiation** that $\frac{x^k}{k^2}(k \log_e(x) - 1)$ is an antiderivative of $x^{k-1} \log_e(x)$, where k is a positive real number.

[HINT: to use differentiation, we must demonstrate that the second expression can be obtained as the derivative of the first.]

$$\text{Let } y = \frac{x^k}{k^2} (k \log_e(x) - 1)$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{kx^{k-1}}{k^2} (k \log_e(x) - 1) + \frac{x^k}{k^2} \times \frac{k}{x} \\ &= x^{k-1} \times \log_e(x) - \frac{x^{k-1}}{k} + \frac{kx^{k-1}}{k^2} \\ &= x^{k-1} \times \log_e(x). \end{aligned}$$

Hence, $\frac{x^k}{k^2} (k \log_e(x) - 1)$ is an antiderivative of $x^{k-1} \times \log_e(x)$.

Marks	0	1	2	Mean
%	62	21	17	0.55

Point to note.

* Question 8 was quite difficult. Responses from many candidates demonstrated more cooking than we would see on a Jamie Oliver TV show. Some managed to get (cook) the right “answer” by differentiating the wrong expression.

* Others who knew what they were doing still had to fudge to obtain the required expression.

* What is really worrying was the inclusion of k as a variable in a quotient rule expression while some candidates were differentiating.

* Since this was a “show” question, the answer should have been given explicitly.

b. i. Calculate $\Pr(X > \frac{1}{e})$

$$\Pr(X > \frac{1}{e}) = \int_{\frac{1}{e}}^1 -4x \log_e(x) dx$$

$$= -4 \left[\frac{x^2}{4} \times (2\log_e(x) - 1) \right]_1^{\frac{1}{e}} \text{ using Part a. with } k = 2.$$

$$= 1 - \frac{3}{e^2}$$

Marks	0	1	2	Mean
%	81	7	12	0.31

Points to watch.

* This proved to be very difficult & was attempted by a smaller number of candidates. Having said that, some very good solutions were observed.

* Some answers given were negative, which is disturbing.

ii. Hence, explain whether the median of X is greater than or less than $\frac{1}{e}$, given that $e > \frac{5}{2}$.

$$e > \frac{5}{2} \text{ and so } e^2 > \frac{25}{4}.$$

$$\frac{1}{e^2} < \frac{4}{25}$$

$$\frac{3}{e^2} < \frac{12}{25}$$

$$-\frac{3}{e^2} > -\frac{12}{25}$$

$$1 - \frac{3}{e^2} > \frac{13}{25}$$

Whence $\Pr(X > \frac{1}{e}) > \frac{13}{25}$ and so $\Pr(X > \frac{1}{e}) > \frac{1}{2}$.

$\therefore$ Median is greater than $\frac{1}{e}$.

Marks	0	1	2	Mean
%	87	6	7	0.2

Points to watch.

- * Some very good solutions were seen. Some substituted correctly but drew the wrong conclusion that the median was less than $\frac{1}{e}$. A diagram should have helped.
- * “Hence” means that the previous answer must be employed.
- * Answers from 8bi that were outside the correct interval $[0, 1]$ could not be considered.
- * Finally, if a candidate were to try to calculate the median by integration, s(he) would have had to deal with the fact that f was not defined at $x = 0$. Hmmm!

Below is the link to the survey and I ask you to please complete it.

<https://www.surveymonkey.com/r/MAVVCE2017>

“Here endeth the first lesson of the 2017 paper!”

rodwatson1@hotmail.com